

## *Arc Length, Area, and the Arcsine Function*

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*Mathematics Magazine*, March 1983, Volume 56, Number 2, pp. 104–110.

Geometric interpretations of algebraic quantities provide an essential motivation for elementary calculus. The “calculus with analytic geometry” titles of many introductory texts allude to the reliance on pictorial images in the development of the derivative and the definite integral. While geometric considerations dominate most presentation of the trigonometric functions and their derivatives, there is a near universal switch to algebraic methods for the introduction and study of the inverse trigonometric functions (see [6], [3], and so on). This note shows how the geometric ideas used in the definitions of the definite integral, the trigonometric functions, and their derivatives may be continued in the discussion of the corresponding inverse functions. Reference will be made to the relation of these geometric ideas to the development of the theory of elliptic functions and to Euler’s method of finding algebraic addition theorems for circular, hyperbolic, and lemniscate sines.

### **“Typical calculus” versus the arcsine as arc length**

Taking [6] and [3] as our models, we note that in the typical modern textbook, after the definite integral has been defined, the applications include the area between two curves and the arc length formula. Since few integration techniques are available, the arc length problems are restricted to “nice” curves  $y = f(x)$  such that the integral  $\int_a^b \sqrt{1 + f'(x)^2} dx$  is particularly simple and sometimes an author’s apology is offered for the lack of interesting applications (see [3], p. 429).

The introduction of the trigonometric functions follows a review of radian measure as arc length measured from the point  $(1, 0)$  on the unit circle  $x^2 + y^2 = 1$ . The sine and cosine of a real number  $\theta$  are defined as the coordinates of the point  $(x, y)$  on the unit circle  $\theta$  radians from  $(1, 0)$  (see FIGURE 1). Then the properties of  $\sin \theta$  and  $\cos \theta$  are derived from the symmetries of the circle and the other trigonometric functions are defined in terms of the sine and cosine. The derivatives of the sine and the cosine are found as consequences of  $\lim_{\theta \rightarrow 0} (\sin \theta) / \theta = 1$ . This limit is established by equating arc length along the edge of the unit circle with the area of the sector determined by the arc (in Figure 2,  $\theta = 2 \cdot \text{area } AOB$ ) and then squeezing this area between two triangular regions.

After studying the calculus of the six trigonometric functions (“ $f(x)$ ”), the corresponding inverse functions (“ $f^{-1}(x)$ ”) are sought by reversing the graphs (“ $y = f(x)$ ” becomes “ $x = f^{-1}(y)$ ”), making arbitrary choices for the “principal values” (see [6], pp. 295–6), and then calculating  $D_x(f^{-1}(x))$  from the identity  $f(f^{-1}(x)) = x$ .

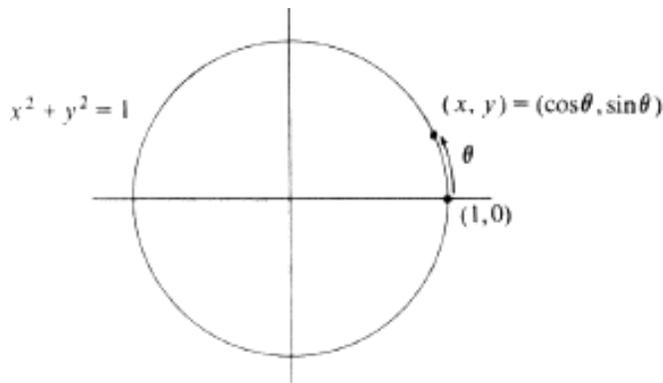


FIGURE 1

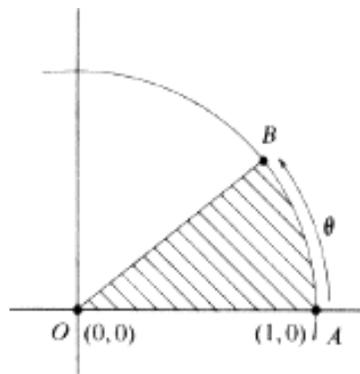


FIGURE 2

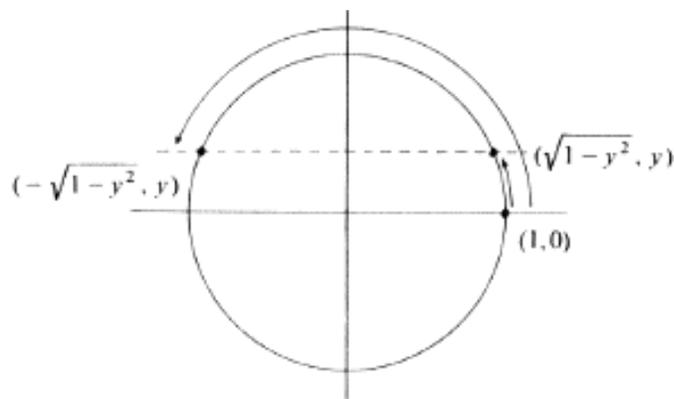


FIGURE 3

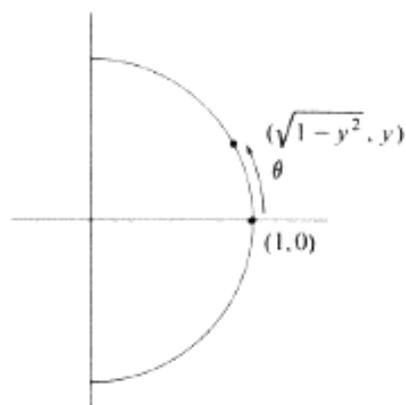


FIGURE 4

In contrast to exploiting the definitions and properties of inverse functions, the arcsine function can be approached in a more geometric way. Since  $\sin \theta$  was defined as the  $y$ -coordinate of the point on the unit circle an arc length of  $\theta$  away from  $(1, 0)$ , a geometric attempt to invert this function would ask “given that the second coordinate of the point is  $y$ , from what arc length did it come?” There are two “small” answers to this question (see Figure 3) and then infinitely more separated from each other by multiples of  $2\pi$ . The “principal value” of the inverse sine function may be introduced naturally as the smallest distance from the starting point,  $(1, 0)$ , that will work, and this arc length may be called the “arc sine of  $y$ ” and written “ $\arcsin(y)$ ” or the “inverse sine of  $y$ ” and written “ $\sin^{-1}(y)$ .” We will use the “arcsine” notation for the remainder of our discussion. Thus the arcsine function has  $\arcsin(y) = \theta$  where  $-\pi/2 \leq \theta \leq \pi/2$  and  $\sin \theta = y$ .

Since  $\arcsin(y)$  is an arc length, the arc length formula  $\int_a^b \sqrt{1 + f'(t)^2} dt$  can be applied to  $f(t) = \sqrt{1 - t^2}$  from  $t = 0$  to  $t = y$  (see FIGURE 4) to find that

$$\begin{aligned} \arcsin(y) &= \int_0^y \sqrt{1 + f'(t)^2} dt \\ &= \int_0^y \sqrt{1 + \frac{t^2}{1 - t^2}} dt \end{aligned}$$

$$= \int_0^y \frac{1}{\sqrt{1-t^2}} dt \quad (1)$$

and then, by the Fundamental Theorem of Calculus, it follows that

$$D_y(\arcsin(y)) = \frac{1}{\sqrt{1-y^2}}.$$

Since  $\arcsin(1) = \pi/2$ , we also have a simple example of an improper integral:

$$\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \lim_{y \rightarrow 1} \int_0^y \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2}.$$

For the inverse tangent, a similar argument yields an arctangent function  $\arctan(w) = \theta$  with principal value  $-\pi/2 < \theta < \pi/2$  such that  $\arctan(w)$  is the distance along the unit circle from  $(1, 0)$  to the point  $(x, y)$  with  $y/x = w$ . Since  $x = \sqrt{1-y^2}$ , we can solve  $y/x = w$  to find  $y = w/\sqrt{1+w^2}$ . From the arc length formula (1) we have

$$\arctan(w) = \int_0^{w/\sqrt{1+w^2}} \frac{1}{\sqrt{1-t^2}} dt.$$

Making the change of variable  $t = u/\sqrt{1+u^2}$ , so that  $t = 0$  when  $u = 0$  and  $t = w/\sqrt{1+w^2}$  when  $u = w$ , we find that

$$\begin{aligned} \arctan(w) &= \int_0^w \sqrt{1+u^2} \frac{1}{\sqrt{1+u^2}(1+u^2)} du \\ &= \int_0^w \frac{1}{1+u^2} du, \end{aligned}$$

and so

$$D_w(\arctan(w)) = \frac{1}{1+w^2}.$$

For the inverse secant of  $w$  (where  $|w| \geq 1$ ), if we seek the smallest positive arc length from  $(1, 0)$  to the point with first coordinate  $1/w$ , we obtain an arcsecant function  $\operatorname{arcsec}(w) = \theta$  with principal value  $0 \leq \theta < \pi/2$  and  $\pi/2 < \theta \leq \pi$ . For  $w > 1$ , the arc length  $\theta$  is  $\pi/2 - \phi$  (see FIGURE 5) so we have

$$\operatorname{arcsec}(w) = \frac{\pi}{2} - \arcsin(1/w)$$

and so

$$\begin{aligned} D_w(\operatorname{arcsec}(w)) &= -\frac{1}{\sqrt{1-(1/w)^2}} \cdot \frac{-1}{w^2} \\ &= \frac{1}{w\sqrt{w^2-1}} \text{ for } w > 1. \end{aligned}$$

For  $w < -1$ , we have by symmetry that

$$\operatorname{arcsec}(w) = \pi - \operatorname{arcsec}(-w)$$

and so

$$D_w(\operatorname{arcsec}(w)) = -D_w(\operatorname{arcsec}(-w))$$

and thus

$$D_w(\operatorname{arcsec}(w)) = \frac{1}{|w|\sqrt{w^2 - 1}} \quad \text{for } |w| > 1.$$

(We omit discussion of the formula

$$\operatorname{arcsec}(w) = \int_0^{\sqrt{w^2-1}/w} \frac{1}{\sqrt{1-t^2}} dt \quad \text{for } w \geq 1$$

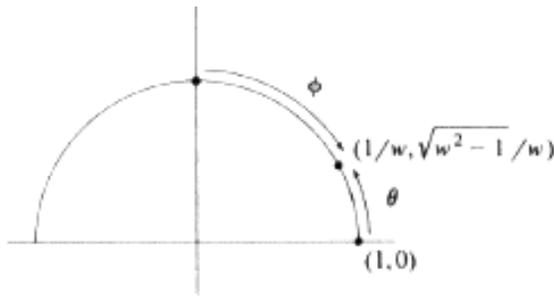


FIGURE 5

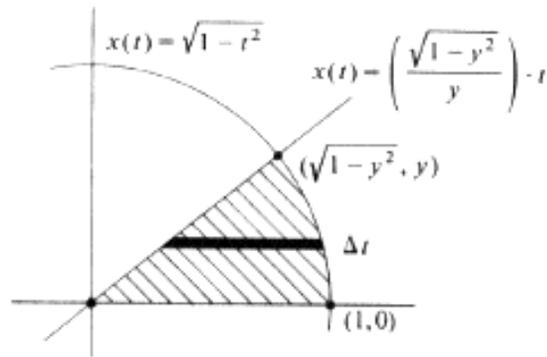


FIGURE 6

as it results in a subtle improper integral calculation.)

The formulas for the inverse cosine, cotangent, and cosecant can be obtained by similar arguments.

### Area and the arcsine

Let us now study the arcsine as an area. Let  $0 < y < 1$  be given and let  $A$  denote the area bounded by the line  $x(t) = (\sqrt{1-y^2}/y) \cdot t$ , the circle  $x(t) = \sqrt{1-t^2}$ , and the  $x$ -axis  $t = 0$  (see FIGURE 6). Since arc length along the edge of the unit circle equals twice the area of the sector determined by the arc, we have that

$$\begin{aligned} \arcsin(y) &= 2 \cdot A \\ &= 2 \cdot \int_0^y \left( \sqrt{1-t^2} - \frac{\sqrt{1-y^2}}{y} \cdot t \right) dt \\ &= 2 \cdot \int_0^y \sqrt{1-t^2} dt - y\sqrt{1-y^2}. \end{aligned} \tag{2}$$

Differentiating (2), we have

$$\begin{aligned} D_y(\arcsin(y)) &= \sqrt{1-y^2} + \frac{y^2}{\sqrt{1-y^2}} \\ &= \frac{1}{\sqrt{1-y^2}}, \end{aligned}$$

as before.

Alternatively, our integral expression (2) for  $\arcsin(y)$  provides a motivating example for the integration by parts formula  $\int u \cdot dv = u \cdot v - \int v \cdot du$ . Letting  $u = \sqrt{1 - t^2}$  and  $dv = dt$ , we have

$$\begin{aligned} \int \sqrt{1 - t^2} dt &= t\sqrt{1 - t^2} - \int t \frac{-t}{\sqrt{1 - t^2}} dt \\ &= t\sqrt{1 - t^2} - \int \frac{(\sqrt{1 - t^2})^2 - 1}{\sqrt{1 - t^2}} dt \\ &= t\sqrt{1 - t^2} - \int \sqrt{1 - t^2} dt + \int \frac{1}{\sqrt{1 - t^2}} dt \end{aligned}$$

so

$$2 \int \sqrt{1 - t^2} dt = t\sqrt{1 - t^2} + \int \frac{1}{\sqrt{1 - t^2}} dt.$$

But then

$$\begin{aligned} \arcsin(y) &= 2 \cdot \int_0^y \sqrt{1 - t^2} dt - y\sqrt{1 - y^2} \\ &= y\sqrt{1 - y^2} + \int_0^y \frac{1}{\sqrt{1 - t^2}} dt - y\sqrt{1 - y^2} \\ &= \int_0^y \frac{1}{\sqrt{1 - t^2}} dt, \end{aligned}$$

as before.

This development of the arcsine as the area bounded by the  $x$ -axis, the circle  $x(t) = \sqrt{1 - t^2}$  from  $t = 0$  to  $t = y$ , and the line from the origin to the point  $(\sqrt{1 - y^2}, y)$  may be adapted to the hyperbola  $x(t) = \sqrt{1 + t^2}$  to find the “inverse hyperbolic sine”  $\sinh^{-1}(y) = \ln|\sqrt{1 + y^2} + y|$ . However, unlike the circle, there is no simple relation between arc length and area for the hyperbola (see[4], §6 and 7; for a representation of arc length on the hyperbola, see [1], §60, and the corresponding geometric construction is §61). On the other hand, a similar procedure to develop a “lemniscate sine” using the lemniscate instead of the circle must be carried out in terms of arc length (see [4], §8).

### Rationalization of the arcsine

We now turn to the problem of expressing the integrand in (1) as a rational function; that is, we wish to express  $1 - t^2$  as the square of a rational function. If, in the two-variable representation of Pythagorean triples,

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2, \quad (3)$$

we let  $a = 1$  and  $b = u$  then

$$(1 - u^2)^2 = (1 + u^2)^2 - (2u)^2$$

or

$$\left(\frac{1-u^2}{1+u^2}\right)^2 = 1 - \left(\frac{2u}{1+u^2}\right)^2,$$

and so the substitution  $t = 2u/(1+u^2)$  will result in

$$\sqrt{1-t^2} = \frac{1-u^2}{1+u^2}$$

and

$$dt = \frac{(1+u^2)(2) - (2u)(2u)}{(1+u^2)^2} du = 2 \frac{1-u^2}{(1+u^2)^2} du.$$

Then the indefinite integral  $\int 1/\sqrt{1-t^2} dt$  becomes

$$\int \frac{1+u^2}{1-u^2} \cdot 2 \frac{1-u^2}{(1+u^2)^2} du = 2 \cdot \int \frac{1}{1+u^2} du$$

and the integrand has been rationalized. Solving  $t = 2u/(1+u^2)$  for  $u$ , we find that

$u = (1 \pm \sqrt{1-t^2})/t$ . Equating  $t$  with  $\sin \theta$ , we see that  $\sqrt{1-t^2} = \cos \theta$  and the expression for  $u$  with the minus sign becomes

$$u = (1 - \cos \theta)/\sin \theta = \tan(\theta/2)$$

and we have derived the " $u = \tan(\theta/2)$ " substitution for the rationalization of trigonometric integrals. (Compare this development with the typical "it has been discovered that ..." treatment in [6], p. 368.)

### Euler's sine sum formula

In the introductory section of [5], Siegel describes Fagnano's study of arc length on the lemniscate (which follows our development of the arcsine of the circle) and speculates that Fagnano's 1718 discovery of a geometric construction to double arc length on the lemniscate resulted from his attempt to rationalize the integrand of the lemniscate sine. Thirty-five years later, Euler extended Fagnano's doubling theorem to an algebraic addition theorem for the lemniscate sine and he shortly thereafter generalized his discovery to elliptic integrals. Siegel ([5], p. 10) describes the aim of his first chapter to be the fuller understanding of Euler's result from the viewpoint of analytic functions on their full domain of definition. We conclude our discussion of the arcsine with Euler's algebraic addition theorem for the arcsine adapted from §585 and 586 of [2] (see also [2], Caput VI for Euler's study of elliptic integrals; [4], Chapter 4; and [5], §2).

Let  $-\pi/2 < \tau < \pi/2$  be a fixed angle and let  $-\pi/2 < \theta, \phi < \pi/2$  be any two angles with  $\theta + \phi = \tau$ . Since the sum  $\theta + \phi$  is constant,  $d(\theta + \phi) = 0$ , and if we set  $u = \sin \theta$  and  $v = \sin \phi$ , we can rewrite  $d(\theta + \phi) = 0$  as

$$d\left(\int_0^u \frac{dt}{\sqrt{1-t^2}} + \int_0^v \frac{dt}{\sqrt{1-t^2}}\right) = 0$$

and so we have the differential equation

$$\frac{du}{\sqrt{1-u^2}} + \frac{dv}{\sqrt{1-v^2}} = 0. \quad (4)$$

We seek an algebraic solution of (4) subject to the condition that  $\theta + \phi = \tau$ .

Euler's main observation was that if we begin with the symmetric second-order equation

$$u^2 + 2auv + v^2 = K^2, \quad (5)$$

where  $a$  and  $K$  are constants, we can complete the square on the left side in either  $u$  or  $v$ . In the first case, we have

$$u^2 + 2auv + a^2v^2 = K^2 + (a^2 - 1)v^2,$$

so that

$$u + av = \sqrt{K^2 + (a^2 - 1)v^2}, \quad (6)$$

while in the second, we have

$$v^2 + 2auv + a^2u^2 = K^2 + (a^2 - 1)u^2,$$

so that

$$v + au = \sqrt{K^2 + (a^2 - 1)u^2}. \quad (7)$$

If we differentiate equation (5), we find

$$2udu + 2avdu + 2audv + 2v dv = 0,$$

which becomes, after collecting terms and using (6) and (7),

$$\frac{du}{\sqrt{K^2 + (a^2 - 1)u^2}} + \frac{dv}{\sqrt{K^2 + (a^2 - 1)v^2}} = 0. \quad (8)$$

If we set  $a^2 - 1 = -K^2$  then equation (8) is the same as (4) and so  $a = \sqrt{1 - K^2}$ . If we can solve (5) for  $K$  using this value of  $a$ , we will find an algebraic solution of (4) that expresses the constant value  $\sin(\theta + \phi)$  in terms of  $\sin \theta$  and  $\sin \phi$ .

Substituting our value for  $a$  into (5) and rearranging, we have

$$(u^2 + v^2) - K^2 = -2\sqrt{1 - K^2}uv$$

and squaring both sides gives

$$(u^2 + v^2)^2 - 2K^2(u^2 + v^2) + K^4 = 4(1 - K^2)u^2v^2. \quad (9)$$

By setting  $a = u$  and  $b = v$  in equation (3) and solving for  $(u^2 + v^2)^2$ , we can rewrite (9) as

$$(u^2 - v^2)^2 - 2K^2((u^2 - v^2) - 2u^2v^2) + K^4 = 0. \quad (10)$$

Since (10) is a quadratic in  $K^2$ , we can complete the square with respect to  $K^2$  to obtain

$$(K^2 + (2u^2v^2 - (u^2 + v^2)))^2 = ((u^2 + v^2) - 2u^2v^2)^2 - (u^2 - v^2)^2. \quad (11)$$

By expanding, collecting terms, and removing common factors, the right side of (11) may be rewritten as  $4u^2v^2(u^2v^2 - (u^2 + v^2) + 1)$ . Taking square roots, we find

$$K^2 = (u^2 + v^2) - 2u^2v^2 + 2uv\sqrt{(1 - u^2)(1 - v^2)}. \quad (12)$$

Regrouping the expression  $(u^2 + v^2) - 2u^2v^2$  as  $(u^2 - u^2v^2) + (v^2 - v^2u^2)$ , we see that the right side of (12) is a perfect square and so

$$K = u\sqrt{1 - v^2} + v\sqrt{1 - u^2}. \quad (13)$$

Changing back to angles,  $K$  is the constant  $\sin(\theta + \phi)$  while  $\sqrt{1 - v^2} = \sqrt{1 - \sin^2\phi} = \cos \phi$  and  $\sqrt{1 - u^2} = \cos \theta$  so we have the sine sum formula

$$\sin(\theta + \phi) = \sin\theta \cos \phi + \sin \phi \cos \theta. \quad (14)$$

Since the fixed value of  $\tau$  played no part in our calculations, we have established (14) for any  $\tau$ .

### **References**

- [1] Arthur Cayley, *An Elementary Treatise on Elliptic Functions*, 2nd ed., 1895. Reprinted by Dover Pub., New York, 1961.
- [2] Leonhard Euler, *Institutiones Calculi Integralis, Volumen Primum*, 1768. Reprint as *Leonhardi Euleri Opera Omnia, Series I, Opera Mathematica, Volumen XI*, Engel et Schlesinger, eds., B.G. Teubner, Leipzig, 1913.
- [3] Louis Leithold, *The Calculus with Analytic Geometry*, 4th ed., Harper & Row, New York, 1981.
- [4] A.I. Markushevich, *The Remarkable Sine Functions* (translated by Scripta Technica, Inc.), Amer. Elsevier, New York, 1966.
- [5] C.L. Siegel, *Topics in Complex Function Theory, vol. I: Elliptic Functions and Uniformization Theory* (translated by Shenitzer and Solitar), Wiley-Interscience, New York, 1969.
- [6] George B. Thomas and Ross L. Finney, *Calculus and Analytic Geometry*, 5th ed., Addison-Wesley, Reading, 1979.