CHAPTER 4: APPLICATIONS OF DERIVATIVES

SECTION 4.1: EXTREMA

1) a) A.Max Value: 23, A.Max Point: $(2, 23)$; A.Min Value: 5; A.Min Point: $(-1, 5)$.

b) A.Max Value: 10, A.Max Point: $(0, 10)$;

A.Min Value: $-\frac{34}{3}$; A.Min Point: $\left(2, -\frac{34}{3}\right)$.

c) A.Max Value: 20, A.Max Point: $(-4, 20)$;

A.Min Value: 12; A.Min Point: $(-2, 12)$.

2) a) $\text{Dom}(f) = (-\infty, \infty)$; CNs: -2, $\frac{3}{16}$, and 2. Hint: Try Factoring by Grouping.

b) $\text{Dom}(g) = (-\infty, -6] \cup [6, \infty)$; CNs: -6 and 6.

c) $\text{Dom}(h) = \left[\frac{1}{4}, \infty\right)$; CN: $\frac{1}{4}$.

d) $\text{Dom}(p) = (-\infty, \infty)$;

CNs: $\left\{ \theta \in \mathbb{R} \,\middle|\, \theta = \frac{\pi}{2} + \pi n, \text{ or } \theta = \frac{\pi}{6} + 2\pi n, \text{ or } \theta = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z}) \right\}$;

Alternatively: $\left\{ \theta \in \mathbb{R} \,\middle|\, \theta = \frac{\pi}{2} + 2\pi n, \text{ or } \theta = \frac{\pi}{6} + \frac{2\pi n}{3} \quad (n \in \mathbb{Z}) \right\}$.

Hint: After differentiating, use a Double-Angle ID.

e) $\text{Dom}(q) = \left\{ x \in \mathbb{R} \,\middle|\, x \neq \pi n \quad (n \in \mathbb{Z}) \right\}$; CNs: NONE.

SECTION 4.2: MEAN VALUE THEOREM (MVT) FOR DERIVATIVES

1) a) f satisfies the hypotheses on $[1, 5]$; $c = 3$.

b) f does not satisfy the hypotheses on $[3, 7]$, because $f(3) \neq f(7)$.

c) f satisfies the hypotheses on $[-6, -1]$; $c = -5$, or $c = -\frac{7}{2} = -3.5$, or $c = -2$.

d) f does not satisfy the hypotheses on $[-4, 4]$, because f is not differentiable at 0, and $0 \in (-4, 4)$; therefore, f is not differentiable on $(-4, 4)$.

2) a) f satisfies the hypotheses on $[1, 4]$; $c = 2$.

Note 1: $-2 \notin (1, 4)$. Note 2: Rolle's Theorem also applies!

b) f satisfies the hypotheses on $[-2, 3]$; $c = \dfrac{-5+\sqrt{139}}{6} = \dfrac{\sqrt{139}-5}{6} \approx 1.1316$.

Note: $\dfrac{-5-\sqrt{139}}{6} = -\dfrac{5+\sqrt{139}}{6} \approx -2.7983$, so $\dfrac{-5-\sqrt{139}}{6} \notin (-2, 3)$.

c) f does not satisfy the hypotheses on $[-8, 8]$, because f is not differentiable at 0, and $0 \in (-8, 8)$; therefore, f is not differentiable on $(-8, 8)$.

d) f satisfies the hypotheses on $[0, 2]$; all real values in $(0, 2)$ satisfy the theorem. (Can you see graphically why this is true?)

SECTION 4.3: FIRST DERIVATIVE TEST

1) a) $\text{Dom}(f) = (-\infty, \infty)$. f is odd, so its graph is symmetric about the origin.

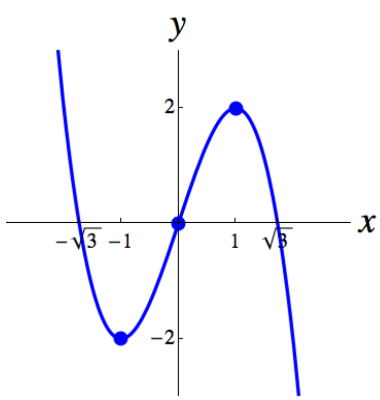
y-intercept: 0, or $(0, 0)$. x-intercepts: $(0, 0)$, $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$.

Holes: None. VAs: None. HAs: None. SAs: None.

Points at critical numbers:

$(-1, -2)$, a local minimum point; $(1, 2)$, a local maximum point;

f is increasing on $[-1, 1]$. f is decreasing on $(-\infty, -1]$, $[1, \infty)$.



b) $\text{Dom}(f) = (-\infty, \infty)$. f is neither even nor odd.

y-intercept: -5, or $(0, -5)$. Holes: None. VAs: None. HAs: None. SAs: None.

Points at critical numbers: $(-5, -105)$, a local minimum point;

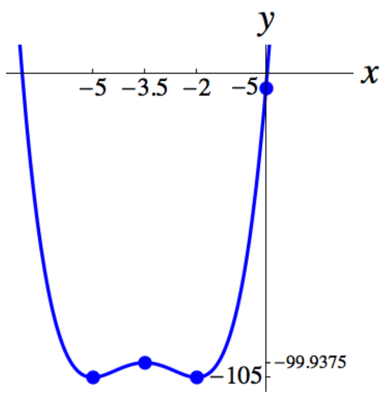
$\left(-\dfrac{7}{2}, -\dfrac{1599}{16}\right)$, or $(-3.5, -99.9375)$, a local maximum point;

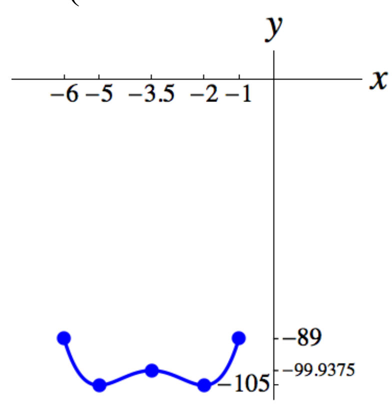
$(-2, -105)$, a local minimum point.

f is increasing on $\left[-5,-\dfrac{7}{2}\right],\left[-2,\infty\right)$; or $\left[-5,-3.5\right],\left[-2,\infty\right)$.

f is decreasing on $\left(-\infty,-5\right],\left[-\dfrac{7}{2},-2\right]$, or $\left(-\infty,-5\right],\left[-3.5,-2\right]$.

Looking back at Section 4.2: (Axes are scaled differently.)





c) $\text{Dom}(f)=(-\infty,4)\cup(4,\infty)$. f is neither even nor odd.

y-intercept: $-\dfrac{1}{4}$, or $\left(0,-\dfrac{1}{4}\right)$. Holes: None.

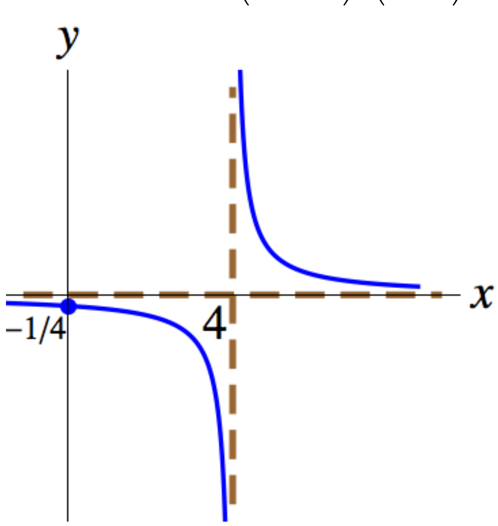
VA: $x=4$, because $\lim\limits_{x\to 4^+} f(x)=\infty$ (or because $\lim\limits_{x\to 4^-} f(x)=-\infty$).

HA: only $y=0$, because $\lim\limits_{x\to\infty} f(x)=0$, and $\lim\limits_{x\to-\infty} f(x)=0$.

SAs: None.

Points at critical numbers: None.

f is decreasing on $(-\infty,4),(4,\infty)$.



d) $\mathrm{Dom}(f) = (-2\pi, 2\pi)$. f is odd, so its graph is symmetric about the origin.

Hints: The derivative of an odd function is even. Try graphing $y = f'(x)$.

y-intercept: 0, or $(0, 0)$.

Holes: None, not counting the excluded endpoints of the graph.
VAs: None. HAs: None. SAs: None.
Points at critical numbers:

$A\left(-\frac{5\pi}{3}, \frac{5\pi + 3\sqrt{3}}{6}\right)$, a local maximum point;

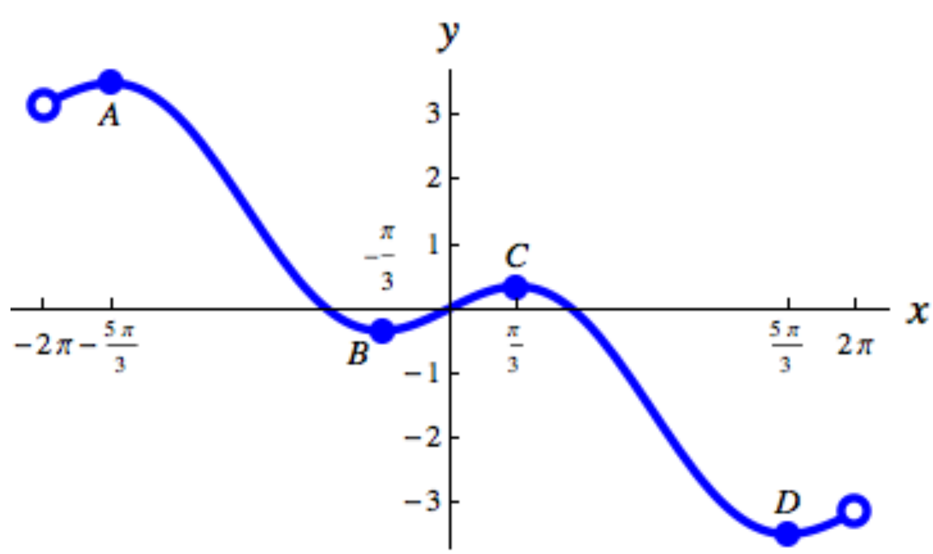
$B\left(-\frac{\pi}{3}, \frac{\pi - 3\sqrt{3}}{6}\right)$, a local minimum point;

$C\left(\frac{\pi}{3}, \frac{3\sqrt{3} - \pi}{6}\right)$, a local maximum point (can use B; f is odd);

$D\left(\frac{5\pi}{3}, -\frac{5\pi + 3\sqrt{3}}{6}\right)$, a local minimum point (can use A; f is odd).

f is increasing on $\left(-2\pi, -\frac{5\pi}{3}\right], \left[-\frac{\pi}{3}, \frac{\pi}{3}\right], \left[\frac{5\pi}{3}, 2\pi\right)$.

f is decreasing on $\left[-\frac{5\pi}{3}, -\frac{\pi}{3}\right], \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$.



2) A local maximum point

SECTION 4.4: SECOND DERIVATIVES

1) a) PIN: 0. Concave up on $(-\infty, 0]$. Concave down on $[0, \infty)$.

PIN corresponds to IP: $(0,0)$.

b) PINs: Both of $\dfrac{-7 \pm \sqrt{3}}{2}$; these are about -2.634 and -4.366.

Concave up on $\left(-\infty, \dfrac{-7-\sqrt{3}}{2}\right], \left[\dfrac{-7+\sqrt{3}}{2}, \infty\right)$, about

$(-\infty, -4.366], [-2.364, \infty)$.

Concave down on $\left[\dfrac{-7-\sqrt{3}}{2}, \dfrac{-7+\sqrt{3}}{2}\right]$, about $[-4.366, -2.364]$.

Both PINs correspond to IPs.

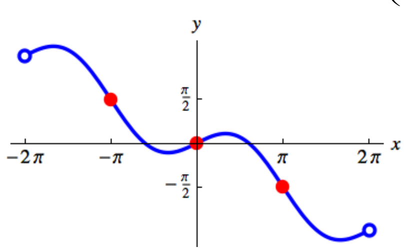
c) PINs: None; observe that 4 is not in $\text{Dom}(f)$.

Concave up on $(4, \infty)$. Concave down on $(-\infty, 4)$. IPs: None.

d) PINs: $-\pi$, 0, and π.

Concave up on $[-\pi, 0], [\pi, 2\pi)$. Concave down on $(-2\pi, -\pi], [0, \pi]$.

All PINs correspond to IPs: $\left(-\pi, \dfrac{\pi}{2}\right)$, $(0,0)$, and $\left(\pi, -\dfrac{\pi}{2}\right)$; see red points.



2) Hints: Verify that $f'(-5) = 0$, and show that $f''(-5) > 0$.

3) Hints: A Power-Reducing trig ID will prove very helpful here.
$g(\theta) = 2 + 2\cos(6\theta)$. $g'(\theta) = -12\sin(6\theta)$. $g''(\theta) = -72\cos(6\theta)$.

a) It is a local maximum point, because $g'\left(\dfrac{\pi}{3}\right) = 0$, and $g''\left(\dfrac{\pi}{3}\right) < 0$.

b) Nothing, because $g'\left(\dfrac{\pi}{4}\right) \neq 0$.

4) Nothing, because $h''(0) = 0$.

5) Employment was decreasing but at a slower and slower rate.

SECTION 4.5: GRAPHING

1) a) $\mathrm{Dom}(f) = (-\infty, \infty)$.

f is neither even nor odd.

y-intercept: -500, or $(0, -500)$. x-intercepts: We will discuss in Section 4.8.

Holes: None. VAs: None. HAs: None. SAs: None.

$f'(x) = -4x^3 + 12x^2 + 96x + 112$.

CNs: -2 and 7. Points at critical numbers:

$(-2, -580)$, neither a local maximum nor a local minimum point;

$(7, 1607)$, a local maximum point.

f is increasing on $(-\infty, 7]$.

f is decreasing on $[7, \infty)$.

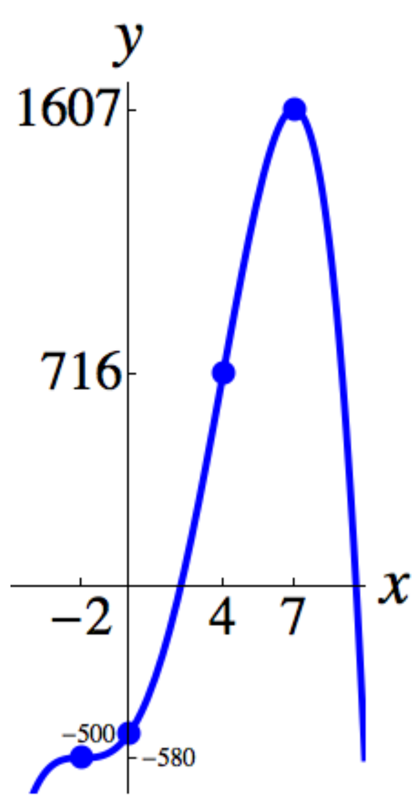
$f''(x) = -12x^2 + 24x + 96$.

PINs: -2 and 4.

Concave up on $[-2, 4]$.

Concave down on $(-\infty, -2], [4, \infty)$.

Both PINs correspond to IPs: $(-2, -580)$ and $(4, 716)$.



(Axes are scaled differently.)

b) $\mathrm{Dom}(f) = (-\infty, -4) \cup (-4, \infty)$.

f is neither even nor odd.

y-intercept: 0, or $(0, 0)$. x-intercept: 0, or $(0, 0)$.

Holes: None.

VA: $x = -4$, because $\lim_{x \to -4^+} f(x) = \infty$ (or because $\lim_{x \to -4^-} f(x) = \infty$).

HA: only $y = \frac{1}{3}$, because $\lim_{x \to \infty} f(x) = \frac{1}{3}$, and $\lim_{x \to -\infty} f(x) = \frac{1}{3}$.

SAs: None.

$$f'(x) = \frac{8x}{3(x+4)^3}.$$

CN: 0. Points at critical numbers:

$(0, 0)$, a local minimum point.

f is increasing on $(-\infty, -4), [0, \infty)$.

f is decreasing on $(-4, 0]$.

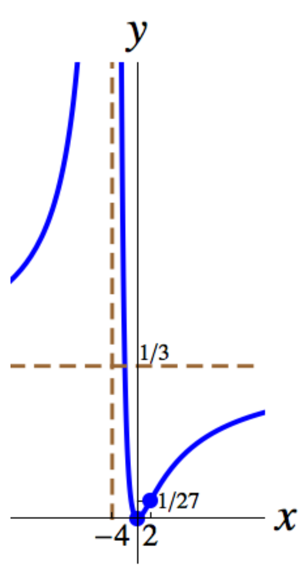
$$f''(x) = \frac{16(2-x)}{3(x+4)^4}.$$

PIN: 2.

Concave up on $(-\infty, -4), (-4, 2]$.

Concave down on $[2, \infty)$.

The PIN does correspond to an IP: $\left(2, \frac{1}{27}\right)$.



(Axes are scaled differently.)

c) $\text{Dom}(f) = (-\infty, \infty)$.

f is neither even nor odd.

y-intercept: 1, or $(0, 1)$. x-intercept: 1, or $(1, 0)$.

Holes: None. VAs: None.

HA: only $y = 1$, because $\lim_{x\to\infty} f(x) = 1$, and $\lim_{x\to-\infty} f(x) = 1$.

SAs: None.

$$f'(x) = \frac{2\left(x^2 - 1\right)}{\left(x^2 + 1\right)^2}.$$

CNs: -1 and 1. Points at critical numbers:

$(-1, 2)$, a local maximum point.

$(1, 0)$, a local minimum point.

f is increasing on $(-\infty, -1], [1, \infty)$.

f is decreasing on $[-1, 1]$.

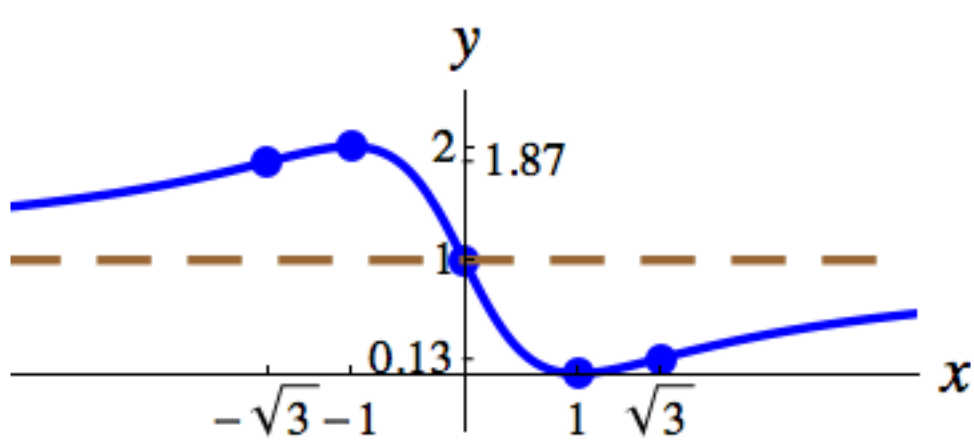
$$f''(x) = \frac{4x\left(3 - x^2\right)}{\left(x^2 + 1\right)^3}.$$

PINs: $-\sqrt{3}$, 0, and $\sqrt{3}$.

Concave up on $\left(-\infty, -\sqrt{3}\right], \left[0, \sqrt{3}\right]$.

Concave down on $\left[-\sqrt{3}, 0\right], \left[\sqrt{3}, \infty\right)$.

The PINs correspond to IPs: $\left(-\sqrt{3}, \text{about } 1.87\right)$, $(0, 1)$, and $\left(\sqrt{3}, \text{about } 0.13\right)$.



d) $\text{Dom}(f) = (-\infty, \infty)$.

f is neither even nor odd.

y-intercept: 0, or $(0, 0)$. x-intercepts: 0 and $\dfrac{27}{8} = 3.375$, or $(0, 0)$ and $(3.375, 0)$.

Holes: None. VAs: None. HAs: None. SAs: None.

$f'(x) = 2x^{-1/3} - 2$, or $\dfrac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$.

CNs: 0 and 1. Points at critical numbers:

$(0, 0)$, a local minimum point.

$(1, 1)$, a local maximum point.

f is increasing on $[0, 1]$.

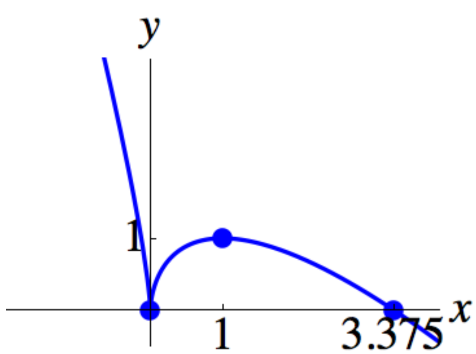
f is decreasing on $(-\infty, 0], [1, \infty)$.

$f''(x) = -\dfrac{2}{3x^{4/3}}$.

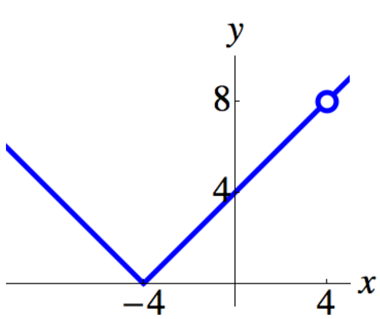
PIN: 0.

Concave down on $(-\infty, 0], [0, \infty)$.

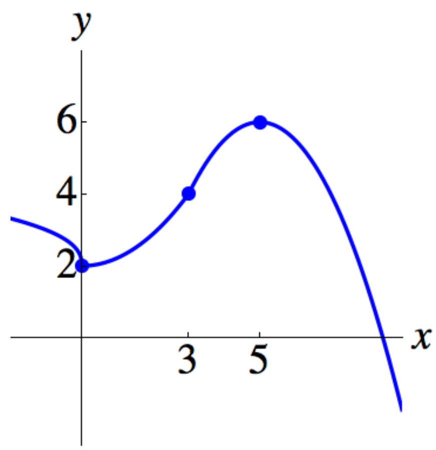
The PIN does not correspond to an IP: no IPs.



2) Make sure to indicate the hole at $(4, 8)$ and the local minimum (and corner) point at $(-4, 0)$.



3) The following is one of infinitely many possible graphs:



SECTION 4.6: OPTIMIZATION

1) We want a cube of side length $2\left(\sqrt[3]{4}\right)\text{ m} = 2\left(2^{2/3}\right)\text{ m} = 2^{5/3}\text{ m} \approx 3.175\text{ m}$.

   It requires $48\left(\sqrt[3]{2}\right)\text{ m}^2 = 48\left(2^{1/3}\right)\text{ m}^2 \approx 60.48\text{ m}^2$ of cardboard.

   Hints: If x is the side length of the square top (or bottom) and y is the height of the box, then surface area $S = 2x^2 + 4xy = 2x^2 + \dfrac{128}{x}$, which is continuous on $(0, \infty)$. $S' < 0$ on $\left(0,\ 2\left(\sqrt[3]{4}\right)\right)$, and $S' > 0$ on $\left(2\left(\sqrt[3]{4}\right),\ \infty\right)$; this verifies that S has an absolute minimum at $x = 2\left(\sqrt[3]{4}\right)\text{ m}$.

2) Optimal dimensions: 4 m × 4 m × 2 m. The box requires 48 m^2 of cardboard. The absence of a top side favors a larger bottom side and allows for a smaller total surface area. (Compare to the pigpen problems in the notes.)

   Hint: Using the notation from Exercise 1, $S = x^2 + 4xy = x^2 + \dfrac{128}{x}$.

3) Base radius $r = \sqrt[3]{\dfrac{2}{\pi}}\text{ m} \approx 0.8603\text{ m}$, and height $h = \sqrt[3]{\dfrac{2}{\pi}}\text{ m} \approx 0.8603\text{ m}$.

   Hint: Surface area $S = \pi r^2 + 2\pi rh = \pi r^2 + \dfrac{4}{r}$.

   The aquarium requires $3\left(\sqrt[3]{4\pi}\right)\text{ m}^2 = 3\left(2^{2/3}\right)\left(\pi^{1/3}\right)\text{ m}^2 \approx 6.975\text{ m}^2$ of glass.

   (It’s easier to use $S = \pi r^2 + 2\pi rh$ instead of $S = \pi r^2 + \dfrac{4}{r}$ to find this.)

   The diameter would be twice the height, so the aquarium would be “squat.”

4) $x = \frac{50}{9}$ ft (by) $y = \frac{25}{4}$ ft, or $5\frac{5}{9}$ ft by $6\frac{1}{4}$ ft, where R has dimensions $3x$ by $2y$.

The total area (enclosed by R) is $\frac{625}{3}\text{ ft}^2 = 208\frac{1}{3}\text{ ft}^2$.

Hint: If R has dimensions $3x$ by $2y$, then total area $A = 6xy = 75x - \frac{27}{4}x^2$.

5) $\frac{15}{13}\text{ sec} = 1\frac{2}{13}\text{ sec} \approx 1.154\text{ sec}$. The corresponding minimum distance is $\sqrt{\frac{45{,}000}{13}}\text{ ft} = \frac{150\sqrt{26}}{13}\text{ ft} \approx 58.83\text{ ft}$, which is just a bit less than the initial 60 ft.

Hint: Squared distance of interest = $104t^2 - 240t + 3600$.

6) About 57.24 feet (floor width w) by 114.47 feet (length l) by 76.31 feet (height). The corresponding cost is about \$157,244.

Hint: Cost $C = 3lw + 4(2lh) + 4(2wh) + 5lw = 16w^2 + \frac{6{,}000{,}000}{w}$.

7) Point: $(2, 5)$. The corresponding minimum distance is $\sqrt{17}\text{ m} \approx 4.123\text{ m}$.

Hint 1: Minimize d^2, the squared distance between points of the form $(x, x^2 + 1)$ and the point $(6, 4)$. $d^2 = x^4 - 5x^2 - 12x + 45$.

Hint 2: Remember the Rational Zero Test and Synthetic Division. See Sections 2.3 and 2.5 in the Precalculus notes.

Note: We also get integers for the coordinates of the closest point on the parabola if the UFO is at $(3,1)$, $(-3,1)$, or $(10,3)$, among others.

8) Hint: Set up a generic rectangle with dimensions l and w. Show that $l = w$ for the largest rectangle.

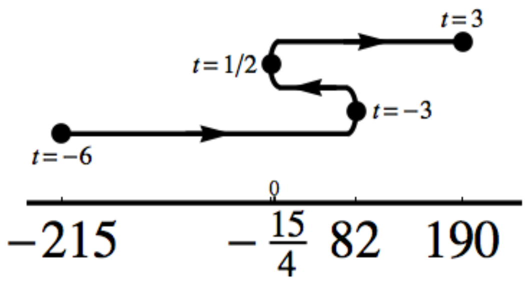
SECTION 4.7: MORE APPLICATIONS OF DERIVATIVES

1) a) $v(t) = 12t^2 + 30t - 18$

b) $(-\infty, -3), \left(\frac{1}{2}, \infty\right)$

c) $\left(-3, \frac{1}{2}\right)$

d)



e) $a(t)=24t+30$

f) $v(-4)=54$, $a(-4)=-66$, moving to the right, slowing down

g) $v(-2)=-30$, $a(-2)=-18$, moving to the left, speeding up

h) $v(0)=-18$, $a(0)=30$, moving to the left, slowing down

i) $v(1)=24$, $a(1)=54$, moving to the right, speeding up

2) a) $P(x)=-3x^2+200x-500$

b) $P'(30)=20\,\dfrac{\$}{\text{device unit}}$, increase production.

c) 33 devices. The CN is $\dfrac{100}{3}=33\dfrac{1}{3}$ devices, and the absolute maximum of P is there if the domain is taken to be $[0,\infty)$. However, an integer number of devices such as 33 or 34 devices would be a more appropriate answer to this problem. $P(33)=\$2833$, and $P(34)=\$2832$, so $P(33)>P(34)$, and 33 devices is a better production level than 34 devices.

SECTION 4.8: NEWTON'S METHOD

1) $x_2\approx 1.91667$, $x_3\approx 1.91294$, $x_4\approx 1.91293$. $\sqrt[3]{7}\approx 1.9129$.

2) $x_2\approx 9.664$, $x_3\approx 9.632$, $x_4\approx 9.631$. Answer: about 9.63.

3) $x_2\approx 0.73911$, $x_3\approx 0.73909$. Answer: about 0.7391. Hint 1: Make sure your calculator is in radian mode! Hint 2: Isolate 0 on one side of the given equation.

4) $x_2=-2$, $x_3=4$. (Note: In fact, the iterates will move further away from 0.) The tangent lines are getting flatter and flatter; that is, the derivatives at our iterates are getting closer to 0. (Note: In the computational field of numerical analysis, derivatives that are close to zero can lead to unstable results.)