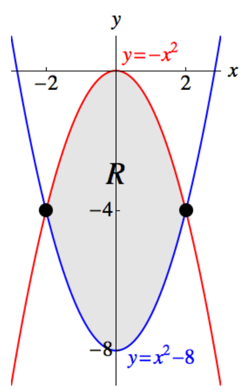
# CHAPTER 6: APPLICATIONS OF INTEGRALS

## SECTION 6.1: AREA

1)

a)

i)




ii) $\int_{-2}^{2}\left[\left(-x^2\right)-\left(x^2-8\right)\right]dx$, or $2\int_{0}^{2}\left[\left(-x^2\right)-\left(x^2-8\right)\right]dx$, or

$4\int_{0}^{2}\left[\left(-x^2\right)-\left(-4\right)\right]dx$ (by symmetry)

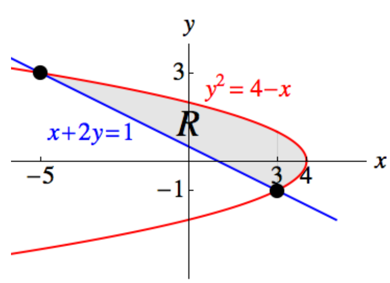
iii) $\int_{-8}^{-4} 2\sqrt{y+8}\,dy \;+\; \int_{-4}^{0} 2\sqrt{-y}\,dy$, or $2\int_{-8}^{-4}\sqrt{y+8}\,dy \;+\; 2\int_{-4}^{0}\sqrt{-y}\,dy$,

or $4\int_{-4}^{0}\sqrt{-y}\,dy$ (by symmetry)

iv) $\dfrac{64}{3}\text{ m}^2$, or $21\dfrac{1}{3}\text{ m}^2$

b)

i)

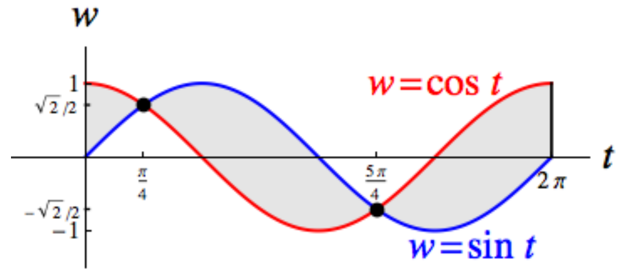



ii) $\int_{-5}^{3}\left(\sqrt{4-x}-\dfrac{1-x}{2}\right)dx \;+\; \int_{3}^{4} 2\sqrt{4-x}\,dx$

iii) $\int_{-1}^{3}\left[\left(4-y^2\right)-\left(1-2y\right)\right]dy$

iv) $\dfrac{32}{3}\text{ m}^2$

2)




$4\sqrt{2}$ $\text{m}^2$. Hint: The setup is given by:

$$\int_0^{\pi/4} (\cos t - \sin t)\, dt + \int_{\pi/4}^{5\pi/4} (\sin t - \cos t)\, dt + \int_{5\pi/4}^{2\pi} (\cos t - \sin t)\, dt .$$

3)

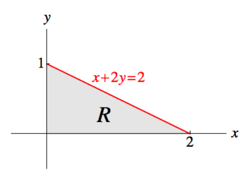
a) $\dfrac{343}{24}$ $\text{m}^2$, or $14\dfrac{7}{24}$ $\text{m}^2$. Hint: Setup is: $\int_{-1/2}^{3} \left[ \left(x^2 + 5x - 2\right) - \left(3x^2 - 5\right) \right] dx$.

b) $\dfrac{61}{3}$ $\text{m}^2$, or $20\dfrac{1}{3}$ $\text{m}^2$. Hint: Setup is: $\int_0^3 x\sqrt{x^2 + 16}\, dx$.

c) $\dfrac{1}{12}$ $\text{m}^2$. Hint: Setup is: $\int_{-1}^{0} \left[ \left(-y - 2y^2\right) - \left(y^3\right) \right] dy$.

## SECTION 6.2: VOLUMES OF SOLIDS OF REVOLUTION – DISKS AND WASHERS

1) a)




b) $\dfrac{2\pi}{3}$ $\text{m}^3$. Hint: Setup is: $\int_0^2 \pi \left(\dfrac{2-x}{2}\right)^2 dx$.

c) $\dfrac{4\pi}{3}$ $\text{m}^3$. Hint: Setup is: $\int_0^1 \pi (2 - 2y)^2\, dy$.

2) a) $\dfrac{512\pi}{3}$ $\text{m}^3$. Hint: Setup is: $\int_{-2}^{2} \left[ \pi\left(0 - \left(x^2 - 8\right)\right)^2 - \pi\left(0 - \left(-x^2\right)\right)^2 \right] dx$, or

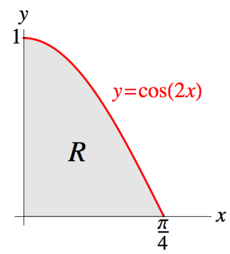
$2\int_0^2 \left[ \pi\left(0 - \left(x^2 - 8\right)\right)^2 - \pi\left(0 - \left(-x^2\right)\right)^2 \right] dx$ by symmetry.

b) $16\pi$ $\text{m}^3$. Hint 1: Setup is: $\int_{-8}^{-4} \pi(y + 8)\, dy + \int_{-4}^{0} \pi(-y)\, dy$, or

$2\int_{-4}^{0} \pi(-y)\, dy$ by symmetry. Hint 2: You only want to revolve half of the region $360^\circ$ around the axis of revolution.

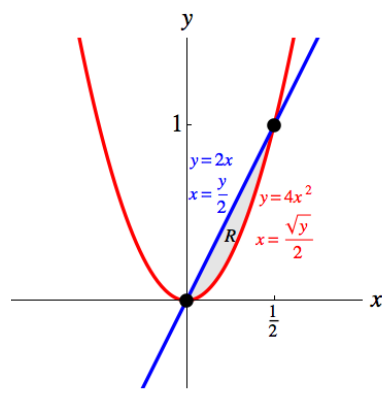
3) $\dfrac{\pi}{2}$ m$^3$. Hint: Setup is: $\int_0^{\pi/4} \left[\pi(\cos t)^2 - \pi(\sin t)^2\right] dt$.

4)




$\dfrac{\pi^2}{8}$ m$^3$. Hint: Setup is: $\int_0^{\pi/4} \pi\left[\cos(2x)\right]^2 dx = \pi \int_0^{\pi/4} \dfrac{1+\cos(4x)}{2}\, dx$.

5)



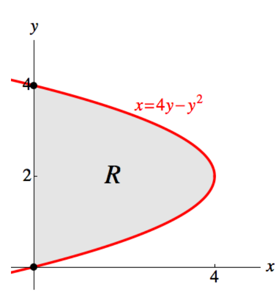

$\dfrac{\pi}{24}$ m$^3$. Hint: Setup is: $\int_0^1 \left[\pi\left(\dfrac{\sqrt{y}}{2}\right)^2 - \pi\left(\dfrac{y}{2}\right)^2\right] dy$.

6) $\pi r^2 h$ (in cubic meters). Hint: Possible setup is: $\int_0^h \pi r^2\, dx$.

7) $\dfrac{1}{3}\pi r^2 h$ (in cubic meters). Hint: Possible setup is: $\int_0^h \pi\left(\dfrac{r}{h}x\right)^2 dx$.

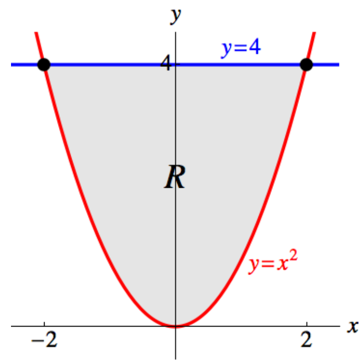
8) $\dfrac{4}{3}\pi r^3$ (in cubic meters). Hint: Possible setup is: $2\int_0^r \pi\left(\sqrt{r^2 - x^2}\right)^2 dx$.

9)




$\int_0^4 \pi\left(4y - y^2\right)^2 dy$ (in cubic meters). Additional Problem: The volume is $\dfrac{512\pi}{15}$ m$^3$.

10)

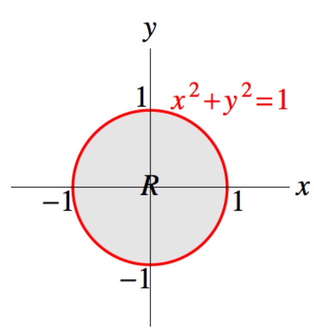



a) $\int_{-2}^{2} \pi\left(4-x^2\right)^2\,dx$ (in cubic meters), or $2\int_{0}^{2} \pi\left(4-x^2\right)^2\,dx$ (in cubic meters) by exploiting symmetry. Additional Problem: The volume is $\dfrac{512\pi}{15}$ m$^3$. (Why do we get the same answer as we did for #9? Draw graphs and see!)

b) $\int_{-2}^{2} \left[\pi\left(5-x^2\right)^2-\pi(1)^2\right]dx$ (in cubic meters), or $2\int_{0}^{2} \left[\pi\left(5-x^2\right)^2-\pi(1)^2\right]dx$ (in cubic meters) by exploiting symmetry. Additional Problem: The volume is $\dfrac{832\pi}{15}$ m$^3$.

c) $\int_{0}^{4} \left[\pi\left(3-\left[-\sqrt{y}\right]\right)^2-\pi\left(3-\sqrt{y}\right)^2\right]dy$ (in cubic meters). Additional Problem: The volume is $64\pi$ m$^3$.

11)

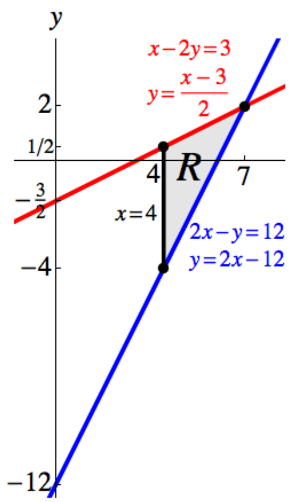



$\int_{-1}^{1} \left[\pi\left(5-\left[-\sqrt{1-y^2}\right]\right)^2-\pi\left(5-\sqrt{1-y^2}\right)^2\right]dy$ (in cubic meters), or $2\int_{0}^{1} \left[\pi\left(5-\left[-\sqrt{1-y^2}\right]\right)^2-\pi\left(5-\sqrt{1-y^2}\right)^2\right]dy$ (in cubic meters) by exploiting symmetry. Additional Problems: The volume is $10\pi^2$ m$^3$. Geometry may help!

## SECTION 6.3: VOLUMES OF SOLIDS OF REVOLUTION – CYLINDRICAL SHELLS

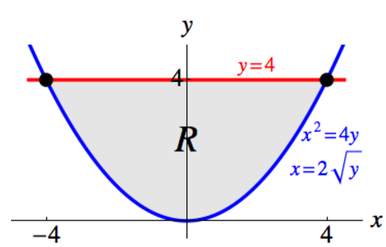
1) $\int_0^{\sqrt{\pi/2}} 2\pi x \tan\left(x^2\right) dx$ (in cubic meters). Observe that, as $x$ varies from 0 to $\frac{\sqrt{\pi}}{2}$, $x^2$ varies from 0 to $\frac{\pi}{4}$, so $\tan\left(x^2\right) \geq 0$ on the interval $\left[0, \frac{\sqrt{\pi}}{2}\right]$.

2)



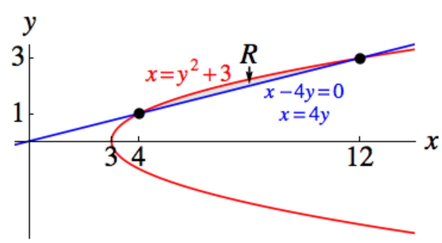

$\frac{135\pi}{2}$ m$^3$. Hint: Setup is: $\int_4^7 2\pi x\left[\frac{x-3}{2} - \left(2x-12\right)\right] dx$.

3)



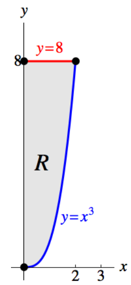

$\frac{512\pi}{5}$ m$^3$. Hint: Setup is: $2\int_0^4 2\pi y\left(2\sqrt{y}\right) dy$ by symmetry. Visualize the solid: Imagine packing foam. The solid corresponds to the space between a "squished convex" hourglass and a cylinder in which it fits snugly.

4)




$\frac{16\pi}{3}$ m$^3$. Hint: Setup is: $\int_1^3 2\pi y\left[4y - \left(y^2+3\right)\right] dy$.

5)




a) $\int_0^2 2\pi x\left(8-x^3\right)dx$ (in cubic meters). Hint: Use cylinders / cylindrical shells.

Additional Problem: The volume is $\frac{96\pi}{5}$ m$^3$.

b) $\int_0^8 \pi\left(\sqrt[3]{y^2}\right)dy$ (in cubic meters). Hint: Use disks.

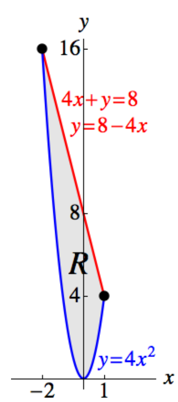
Additional Problem: The volume is $\frac{96\pi}{5}$ m$^3$, same as for a).

c) $\int_0^2 2\pi\left(3-x\right)\left(8-x^3\right)dx$ (in cubic meters). Hint: Use cylinders / cylindrical shells. Additional Problem: The volume is $\frac{264\pi}{5}$ m$^3$.

d) $\int_0^8 \left[\pi\left(3\right)^2-\pi\left(3-\sqrt[3]{y}\right)^2\right]dy$ (in cubic meters). Hint: Use washers.

Additional Problem: The volume is $\frac{264\pi}{5}$ m$^3$, same as for c).

6)




a) $\int_{-2}^1 \left[\pi\left(8-4x\right)^2-\pi\left(4x^2\right)^2\right]dx$ (in cubic meters). Hint: Use washers.

Additional Problem: The volume is $\frac{1152\pi}{5}$ m$^3$.

b) $\int_{-2}^1 2\pi\left(1-x\right)\left(\left[8-4x\right]-\left[4x^2\right]\right)dx$ (in cubic meters). Hint: Use cylinders / cylindrical shells. Additional Problem: The volume is $54\pi$ m$^3$.

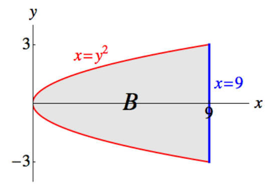
c) $\int_{-2}^{1}\left[\pi\left(16-4x^2\right)^2-\pi\left(16-\left[8-4x\right]\right)^2\right]dx$ (in cubic meters).

Hint: Use washers. Additional Problem: The volume is $\frac{1728\pi}{5}\text{ m}^3$.

7) Hint: Setup is: $2\int_{0}^{r}2\pi x\sqrt{r^2-x^2}\,dx$.

## SECTION 6.4: VOLUMES BY CROSS SECTIONS

1) Sketch of $B$:

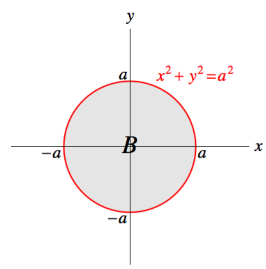



a) $162\text{ m}^3$. Hint: Setup is: $\int_0^9\left(2\sqrt{x}\right)^2dx$.

b) $\frac{81\pi}{4}\text{ m}^3$. Hint: Setup is: $\int_0^9\frac{1}{2}\pi\left(\sqrt{x}\right)^2dx$.

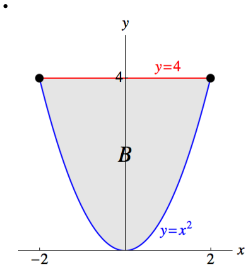
c) $\frac{81\sqrt{3}}{2}\text{ m}^3$. Hint: Setup is: $\int_0^9\sqrt{3}x\,dx$.

2) Sketch of $B$:




$\frac{16a^3}{3}\text{ m}^3$. Hint: Setup is: $2\int_0^a\left(2\sqrt{a^2-x^2}\right)^2dx$.

3) Sketch of $B$:




$\frac{128}{15}\text{ m}^3$, or $8\frac{8}{15}\text{ m}^3$.

Hint: Setup is: $2\int_0^2\frac{1}{2}\left[\frac{1}{\sqrt{2}}\left(4-x^2\right)\right]^2dx$, or $2\int_0^2\frac{1}{2}\cdot\frac{1}{2}\left(4-x^2\right)^2dx$.

# SECTION 6.5: ARC LENGTH and SURFACES OF REVOLUTION

Note: Observe that the integrands are continuous on the closed intervals of interest.

1)

a) $\int_1^3 \sqrt{1+\left(3x^2\right)^2}\, dx$ (in meters)

b) $\int_2^{28} \sqrt{1+\left[\frac{1}{3(y-1)^{2/3}}\right]^2}\, dy$ (in meters)

c) $\int_1^3 2\pi\left(x^3+1\right)\sqrt{1+\left(3x^2\right)^2}\, dx$ (in square meters)

d) $\int_2^{28} 2\pi(y-1)^{1/3}\sqrt{1+\left[\frac{1}{3(y-1)^{2/3}}\right]^2}\, dy$ (in square meters)

2)

a) $\int_1^4 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\, dx = \frac{\pi}{6}\left[17\sqrt{17}-5\sqrt{5}\right]\text{m}^2 \approx 30.85\,\text{m}^2$;

b) $\int_1^2 2\pi y^2\sqrt{1+4y^2}\, dy$ (in square meters)

3) $4\pi r^2$ (in square meters). Hint: Setup is: $2\int_0^r 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2}\, dx$.

Note: The above setup leads to the integral $\int_0^r 4\pi r\, dx$, which has a constant integrand. This implies that, on a fine regular partition (imagine forcing an unhusked coconut through a shredder), the corresponding pieces of the sphere have approximately equal surface areas. Although the "average radii" of these pieces are shrinking as, say, $x \to r^-$, the pieces are also slanting more steeply. (If you eat the shredded coconut pieces, the "end pieces" will be about as filling as the "middle pieces.")

4) a) and b). $\pi r\sqrt{r^2+h^2}$ (in square meters). This can be thought of as $\pi r l$ (in square meters), where $l=\sqrt{r^2+h^2}$, the slant height of the cone.

Hint on a): Evaluate $\int_0^h 2\pi\left(\frac{r}{h}x\right)\sqrt{1+\left(\frac{r}{h}\right)^2}\, dx$.