CALCULUS: THE EXERCISES

MATH 150: CALCULUS WITH ANALYTIC GEOMETRY I

VERSION 1.3

KEN KUNIYUKI and LALEH HOWARD
SAN DIEGO MESA COLLEGE

WEBSITE FOR FREE E-COPIES OF THESE MATERIALS

http://www.kkuniyuk.com

CONTACT INFORMATION

Ken Kuniyuki:

Email: kkuniyuk@yahoo.com or kkuniyuk@sdccd.edu (San Diego Mesa College)

Website: http://www.kkuniyuk.com

- You may download these and other course notes, exercises, and exams.

Feel free to send emails with suggestions, improvements, tricks, etc.

LICENSING

PARTIAL BIBLIOGRAPHY / SOURCES

Calculus: Anton, Edwards/Penney, Larson, Stewart, Swokowski, Thomas

People: Ken Kuniyuki, Laleh Howard, Tom Teegarden, and many more.

"END OF CHAPTER" MARK

Page E.1.3 is the last page of the Exercises for Chapter 1. Therefore, in the upper right hand corner, there is an additional period: "E.1.3."

This is to help people figure out if they have all the pages for a particular chapter.

CHAPTER 1: REVIEW

FUNCTIONS

1) Assuming $f(x) = x^2$, evaluate $f(-4)$, and evaluate (and expand) $f(a+h)$.

2) If $f(t) = 4t^5 - 13t + 4$, is f a polynomial function? Rational? Algebraic?

3) If $g(w) = \dfrac{w-1}{w^3 + 2w - 3}$, is g a polynomial function? Rational? Algebraic?

4) If $h(x) = \sqrt[3]{x} - x^{2/5}$, is h a polynomial function? Rational? Algebraic?

5) If $f(x) = \sin x$, is f a polynomial function? Rational? Algebraic?

DOMAIN AND RANGE

6) For each function rule below, write the domain of the corresponding function in interval form.

   a) $f(x) = 7x^9 + 4x^6 - 12$

   b) $g(t) = t^{2/3} + 4$

   c) $h(w) = \dfrac{\sqrt[3]{w-4}}{2w^2 - 3w - 2}$

   d) $f(t) = \dfrac{\sqrt{t+2}}{t-5}$

   e) $g(t) = \dfrac{t-5}{\sqrt{t+2}}$

   f) $h(x) = \sqrt{7-x}$

   g) $f(r) = \sqrt{r^2 - 2r - 3}$

   Review Section 2.7 on Nonlinear Inequalities in the Precalculus notes. See the domain discussion coming up in Section 2.2, Example 5.

7) Consider $f(x) = \sqrt{x} - 2$. Graph $y = f(x)$. What is the domain of f in interval form? What is the range of f in interval form?

SYMMETRY

Topic 3, Part A can help with the trigonometry.

8) If $f(x) = x^4 - 3x^2 + 4\cos x$, then the graph of $y = f(x)$ in the usual *xy*-plane is symmetric about what? Why? (What kind of function is f?)

9) If $f(x) = 2x^7 - x - 4\sin x$, then the graph of $y = f(x)$ in the usual *xy*-plane is symmetric about what? Why? (What kind of function is f?)

10) If $g(t) = t^{2/3} + 4$, then is the function g even, odd, or neither?

11) If $h(r) = r\tan r$, then is the function h even, odd, or neither?

12) If $f(x) = x^3 - 2x + 1$, then is the function f even, odd, or neither?

COMPOSITIONS OF FUNCTIONS

13) Let $f(u) = u^2 + \dfrac{1}{u+1}$ and $g(x) = x^4 + x^2$. Find $(f \circ g)(x)$ and $\mathrm{Dom}(f \circ g)$. You do not have to express $(f \circ g)(x)$ as a single fraction.

14) Find rules for functions f and g so that $(f \circ g)(x) = f(g(x)) = (x^4 + x)^8$. (Do not let f or g be the identity function.)

15) Find rules for functions f and g so that $(f \circ g)(t) = f(g(t)) = \sqrt[4]{\dfrac{1}{t}}$. (Do not let f or g be the identity function.)

16) Find rules for functions f and g so that $(f \circ g)(r) = f(g(r)) = \sin(r^2)$. (Do not let f or g be the identity function.)

TRIGONOMETRY

17) Evaluate the following; write "undefined" when appropriate.

a) $\cot \pi$

b) $\sec \pi$

c) $\csc\left(\dfrac{3\pi}{4}\right)$

d) $\sec\left(\dfrac{7\pi}{6}\right)$

e) $\sin\left(\dfrac{5\pi}{3}\right)$

f) $\tan\left(\dfrac{5\pi}{3}\right)$

18) Verify the following trigonometric identities.

a) $\dfrac{\sin(2x)}{\cos^2 x} = 2(\sin x)(\sec x)$

b) $\dfrac{1}{\sec^2 x - 1} = \cot^2 x$

19) Solve the following trigonometric equations; find all real solutions, and write the solution set in set-builder form.

a) $2\sin^2 x + 3\sin x = -1$

b) $2\cos(3x) - 1 = 0$

KNOW THE FOLLOWING

• Domains, ranges, and graphs of the six basic trigonometric functions.

• Fundamental and Advanced Trigonometric Identities in Ch.1, except you do not have to memorize the Product-To-Sum Identities, nor the Sum-To-Product Identities.

CHAPTER 2: LIMITS AND CONTINUITY

When asked to give a limit, give a real number or ∞ or $-\infty$ when appropriate.
If a limit does not exist, and ∞ and $-\infty$ are inappropriate, write "DNE."

SECTION 2.1: AN INTRODUCTION TO LIMITS

Assume that a is a real constant.

BASIC LIMIT THEOREM FOR RATIONAL FUNCTIONS

1) Evaluate $\lim_{x \to -2} \left(3x^4 - x^3 + 1\right)$.

2) Evaluate $\lim_{r \to 3} \dfrac{2r+5}{r^2-2}$.

3) Evaluate $\lim_{x \to \frac{1}{3}} 10$.

4) Evaluate $\lim_{t \to 11} \pi^2$.

ONE- AND TWO-SIDED LIMITS; EXISTENCE OF LIMITS; IGNORING THE FUNCTION AT *a*

5) We have discussed how the numerical / tabular method can help us guess at limits. We will see here how this method can be misleading at times!
Let $f(x) = x - 0.0001$.

   a) Evaluate $f(1)$, $f(0.1)$, and $f(0.01)$.

   b) Evaluate $\lim_{x \to 0} f(x)$. Is the result obvious from the function values in a)?

6) Evaluate: a) $\lim_{r \to 3^+} \dfrac{2r+5}{r^2-2}$, and b) $\lim_{r \to 3^-} \dfrac{2r+5}{r^2-2}$. Compare with Exercise 2.

7) Yes or No: If the one-sided limit $\lim_{x \to a^+} f(x)$ exists, then must the two-sided limit $\lim_{x \to a} f(x)$ exist?

• If your answer is "Yes," then explain why.

• If your answer is "No," then give a counterexample. A counterexample is a situation where the hypothesis (assumption) holds true, but the conclusion does not hold true. Here, the hypothesis is "the one-sided limit $\lim_{x \to a^+} f(x)$ exists," and the conclusion is "the two-sided limit $\lim_{x \to a} f(x)$ must exist."

• An if-then statement is true if and only if no such counterexamples exist.

8) Yes or No: If the two-sided limit $\lim_{x \to a} f(x)$ exists, then must the one-sided limit $\lim_{x \to a^+} f(x)$ exist?

9) Yes or No: If $f(a)$ exists, then must $\lim_{x \to a} f(x)$ equal $f(a)$?

• If your answer is "Yes," then explain why.
• If your answer is "No," then give a counterexample.

10) Assume $f(a)$ does not exist.

• If $\lim_{x \to a} f(x)$ cannot exist, write "cannot exist" and explain why it cannot exist.

• If $\lim_{x \to a} f(x)$ might exist, write "might exist" and give an example.

11) Let the function g be defined piecewise as follows: $g(x) = \begin{cases} x+2, & \text{if } x < -1 \\ x^2 - 1, & \text{if } -1 \le x < 2 \\ \sqrt{x+1}, & \text{if } x \ge 2 \end{cases}$

a) Draw the graph of $y = g(x)$. Remember transformations from Section 1.4 of the Precalculus notes!

b) Evaluate: $\lim_{x \to -1^-} g(x)$, $\lim_{x \to -1^+} g(x)$, and $\lim_{x \to -1} g(x)$.

c) Evaluate: $\lim_{x \to 2^-} g(x)$, $\lim_{x \to 2^+} g(x)$, $\lim_{x \to 2} g(x)$, and $\lim_{x \to 3} g(x)$.

12) Let $f(x)=\dfrac{|x-3|}{x-3}$.

a) Draw the graph of $y=f(x)$. Remember transformations from Section 1.4 of the Precalculus notes!

b) Evaluate: $\lim\limits_{x\to 3^-} f(x)$, $\lim\limits_{x\to 3^+} f(x)$, and $\lim\limits_{x\to 3} f(x)$.

13) (Charles's Law for Ideal Gases). Assuming that we have an ideal gas occupying volume V_0 (measured in liters, let's say) when the temperature of the gas is 0° Celsius, and assuming that the gas is under constant pressure, the volume of the gas when its temperature is T degrees Celsius is given by:

V, or $V(T)=V_0\left(1+\dfrac{T}{273.15}\right)$. Absolute zero is -273.15° Celsius.

a) Evaluate $\lim\limits_{T\to -273.15^+} V(T)$, and interpret the result.

b) Discuss $\lim\limits_{T\to -273.15^-} V(T)$.

SECTION 2.2: PROPERTIES OF LIMITS and ALGEBRAIC FUNCTIONS

1) Evaluate $\lim_{x \to 3} \left(\frac{2x^2\sqrt{x} - 3x}{(x+1)^2} + x \right)$.

2) Evaluate: a) $\lim_{x \to 0^+} \sqrt{-x}$, b) $\lim_{x \to 0^-} \sqrt{-x}$, and c) $\lim_{x \to 0} \sqrt{-x}$.

3) Evaluate: a) $\lim_{x \to 0^+} x^{2/3}$, b) $\lim_{x \to 0^-} x^{2/3}$, and c) $\lim_{x \to 0} x^{2/3}$.

4) Evaluate:

a) $\lim_{x \to 4^+} \sqrt{x-4}$

b) $\lim_{x \to 4^-} \sqrt{x-4}$

c) $\lim_{x \to 4} \sqrt{x-4}$

d) $\lim_{x \to 4^-} \sqrt{4-x}$

e) $\lim_{x \to 9} \sqrt{x-4}$

f) $\lim_{x \to 3} \sqrt{x-4}$

g) $\lim_{x \to -2} \sqrt[3]{x+2}$

5) Evaluate:

a) $\lim_{t \to -4^+} \sqrt{t^2 + 3t - 4}$

b) $\lim_{t \to -4^-} \sqrt{t^2 + 3t - 4}$

c) $\lim_{t \to -4} \sqrt{t^2 + 3t - 4}$

d) $\lim_{t \to 1^+} \sqrt{t^2 + 3t - 4}$

e) $\lim_{t \to 1^-} \sqrt{t^2 + 3t - 4}$

f) $\lim_{t \to 1} \sqrt{t^2 + 3t - 4}$

g) $\lim_{t \to 2^+} \sqrt{t^2 + 3t - 4}$

6) Yes or No: If $\lim_{x \to 2} f(x) = 10$, then must $\lim_{x \to 2} \left[5f(x) - 4 \right] = 46$?

- If your answer is "Yes," then explain why.
- If your answer is "No," then give a counterexample.

7) Assume $\lim_{x \to a^+} \sqrt{f(x)} = 0$.

- If $\lim_{x \to a^-} \sqrt{f(x)}$ cannot exist, write "cannot exist" and explain why it cannot.
- If $\lim_{x \to a^-} \sqrt{f(x)}$ might exist, write "might exist" and give an example.

SECTION 2.3: LIMITS AND INFINITY I

HORIZONTAL ASYMPTOTES (HAs) and "LONG-RUN" LIMITS

1) Let $f(x) = \frac{1}{x} + 2$.

   a) Draw the graph of $y = \frac{1}{x} + 2$.

   b) Evaluate $\lim_{x \to \infty} \left(\frac{1}{x} + 2 \right)$.

   c) What is the equation of the horizontal asymptote (HA) for the graph in a)?

2) For the following, assume that the graph of a function f is given by $y = f(x)$. (See Parts A and C.)

   a) What are the possible numbers of horizontal asymptotes (HAs) that the graph of a nonconstant polynomial function can have?

   b) What are the possible numbers of HAs that the graph of a rational function can have?

   c) What are the possible numbers of HAs that the graph of a function can have?

LIMIT FORMS

3) In Section 2.4, we will discuss the Limit Form $\frac{1}{0^+}$. Fill out the table below and make a conjecture (guess) as to what this Limit Form yields.

$\frac{1}{1/10}$	$\frac{1}{1/100}$	$\frac{1}{1/1000}$	$\frac{1}{1/10{,}000}$

4) For each of the Limit Forms below, find the limit that it yields.
If 0^+ is appropriate, then write 0^+. If 0^- is appropriate, then write 0^-.
It may help to experiment with sequences of numbers and with extreme numbers.
As a last resort, refer to Section 2.5, Part D.

a) $\frac{3}{\infty}$

b) $\frac{2}{-\infty}$

c) $\frac{-4}{-\infty}$

d) $\frac{\infty}{-2}$

e) $\frac{\infty}{0^+}$

f) $4 \cdot \infty$

g) $\infty - 4$

h) $\infty^{-\infty}$

i) 0^∞

j) 3^∞

k) $\left(\frac{1}{3}\right)^\infty$

5) Yes or No: If $\left[\lim_{x\to\infty} f(x) = 0, \text{ and } \lim_{x\to\infty} g(x) = \infty\right]$, then must it be true that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$?

6) Assume $\lim_{x\to\infty} g(x)$ does not exist (DNE) for a function g.

• If $\lim_{x\to\infty} \frac{g(x)}{h(x)}$ cannot exist for a function h, write "cannot exist" and explain why it cannot exist.

• If $\lim_{x\to\infty} \frac{g(x)}{h(x)}$ might exist, write "might exist" and give an example.

"LONG-RUN" LIMITS

7) Evaluate the following "long-run" limits.

a) $\lim_{x \to -\infty} \tan x$

b) $\lim_{x \to \infty} \frac{4}{x^5}$

c) $\lim_{x \to -\infty} \frac{-3}{x^6}$

d) $\lim_{x \to \infty} \frac{4}{x^{2/3}}$

e) $\lim_{x \to -\infty} \frac{\sqrt{3}}{x^{3/2}}$

f) $\lim_{x \to \infty} \frac{1}{x^{-2}}$

8) Evaluate the following "long-run" limits for polynomial functions.

a) $\lim_{x \to \infty} \sqrt{\pi}$

b) $\lim_{x \to \infty} \left(x^5 + 3x^4 - 2\right)$.

i. Apply a "short cut" using Dominant Term Substitution ("DTS").

ii. Also give a more rigorous solution using factoring.

c) $\lim_{x \to -\infty} \left(2x^3 - 6x^2 + x\right)$.

i. Apply a "short cut" using Dominant Term Substitution ("DTS").

ii. Also give a more rigorous solution using factoring.

d) $\lim_{w \to \infty} \left(5w - 4w^4\right)$.

Apply a "short cut" using Dominant Term Substitution ("DTS").

9) Evaluate the following "long-run" limits for rational functions.

a) Assuming $g(r) = \frac{3r^3 + r - 4}{2r^5 - 7r^2}$, evaluate $\lim_{r \to \infty} g(r)$.

i. Use a "short cut" to figure out the answer quickly.
Explain your answer.

ii. Apply a "short cut" using Dominant Term Substitution ("DTS").

iii. Also give a more rigorous solution based on one of the methods seen in Examples 13-15. Show work.

iv. Since g is a rational function, what must $\lim_{r \to -\infty} g(r)$ then be?

v. What is the equation of the horizontal asymptote (HA) for the graph of $s = g(r)$ in the *rs*-plane?

b) $\displaystyle\lim_{x\to-\infty} \frac{7x^4 - 5x}{3x^4 + 2}$.

i. Use a "short cut" to figure out the answer quickly. Explain your answer.

ii. Apply a "short cut" using Dominant Term Substitution ("DTS").

iii. Also give a more rigorous solution based on one of the methods seen in Examples 13-15. Show work.

iv. What is the equation of the horizontal asymptote (HA) for the graph of $y = \dfrac{7x^4 - 5x}{3x^4 + 2}$ in the *xy*-plane?

c) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x}$.

i. Apply a "short cut" using Dominant Term Substitution ("DTS").

ii. Does the graph of $y = \dfrac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x}$ in the *xy*-plane have a horizontal asymptote (HA)?

iii. What is $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2}x^5 + 11x^8 - \pi}{6x^5 + x}$?

d) $\displaystyle\lim_{t\to\infty} \frac{\left(t^3 + 1\right)^2}{4t^6}$.

10) Let $f(x) = \dfrac{-3x^4 + 2x^3 + x^2 - 3x + 2}{x^3 + 1}$.

a) Use Long Division to rewrite $f(x)$ in the form: (*polynomial*) + (*proper rational expression*). Show work.

b) Evaluate $\displaystyle\lim_{x\to\infty} f(x)$.

c) Evaluate $\displaystyle\lim_{x\to-\infty} f(x)$.

d) What is the equation of the slant asymptote (SA) for the graph of $y = f(x)$ in the *xy*-plane?

11) Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{3x^4+1}}{7x^2}$ by first rewriting $\dfrac{\sqrt{3x^4+1}}{7x^2}$ as $\sqrt{(\text{expression in } x)}$.

12) **ADDITIONAL PROBLEM:** Evaluate the following "long-run" limits for algebraic functions.

a) $\lim_{x\to-\infty} \left(3x^{5/3} - 4x + 2 - x^{-4}\right)$.

b) $\lim_{x\to-\infty} \left(3x^{5/4} - 4x + 2 - x^{-4}\right)$.

c) $\lim_{x\to\infty} \dfrac{\sqrt{4x^6+x^2}+2x^2}{5x^3-\sqrt[3]{x}}$. Use Dominant Term Substitution ("DTS").

d) $\lim_{x\to-\infty} \dfrac{\sqrt{4x^6+x^2}+2x^2}{5x^3-\sqrt[3]{x}}$. Use Dominant Term Substitution ("DTS").

e) Judging from your results in c) and d), what are the horizontal asymptotes (HAs) for the graph of $y = \dfrac{\sqrt{4x^6+x^2}+2x^2}{5x^3-\sqrt[3]{x}}$ in the *xy*-plane?

f) $\lim_{z\to\infty} \dfrac{\sqrt[3]{5z^{12}+7z^7}}{z^5+2}$. Use Dominant Term Substitution ("DTS").

g) $\lim_{x\to\infty} \left(\sqrt{x^4+5x^2} - x^2\right)$.

<u>ADDITIONAL PROBLEM: A WORD PROBLEM</u>

13) (Bacterial populations). At midnight, a large petri dish contains 2500 bacteria of the species *E. calculi*. Starting at midnight, a stream is poured into the petri dish that adds 100 *E. calculi* bacteria and 150 *E. coli* bacteria (and no other bacteria) to the dish every second.

a) Find an expression for $p(t)$, the proportion of the bacteria in the petri dish that are *E. calculi* t seconds after midnight, where $t \geq 0$.

b) Find $\lim_{t\to\infty} p(t)$, and interpret the result. Discuss the realism of this problem.

SECTION 2.4: LIMITS AND INFINITY II

VERTICAL ASYMPTOTES (VAs) and INFINITE LIMITS AT A POINT

1) Let $f(x) = \frac{1}{x-3}$.

   a) Draw the graph of $y = \frac{1}{x-3}$.

   b) Evaluate $\lim_{x \to 3^+} \frac{1}{x-3}$.

   c) Evaluate $\lim_{x \to 3^-} \frac{1}{x-3}$.

   d) What is the equation of the vertical asymptote (VA) for the graph in a)?

2) For the following, assume that the graph of a function f is given by $y = f(x)$.

   a) What are the possible numbers of vertical asymptotes (VAs) that the graph of a polynomial function can have?

   b) What are the possible numbers of VAs that the graph of a rational function can have?

   c) What are the possible numbers of VAs that the graph of a function can have?

LIMIT FORMS

3) For each of the Limit Forms below, find the limit that it yields.

   a) $\frac{5}{0^+}$

   b) $\frac{-3}{0^+}$

   c) $\frac{\pi}{0^-}$

   d) $\frac{-\sqrt{2}}{0^-}$

RATIONAL FUNCTIONS

4) Consider $f(x) = \dfrac{3x-2}{x^3 - 3x^2 + 4}$.

a) Factor the denominator. You may want to review Section 2.3 on the Rational Zero Test (Rational Roots Theorem) and Synthetic Division in the Precalculus notes. Show work.

b) Evaluate the following limits at a point. Show work.

i. $\lim\limits_{x \to -1^+} f(x)$

ii. $\lim\limits_{x \to -1^-} f(x)$

iii. $\lim\limits_{x \to -1} f(x)$

iv. $\lim\limits_{x \to 2^+} f(x)$

v. $\lim\limits_{x \to 2^-} f(x)$

vi. $\lim\limits_{x \to 2} f(x)$

vii. $\lim\limits_{x \to 0} f(x)$

c) Evaluate the following "long-run" limits. You may use "short cuts."

i. $\lim\limits_{x \to \infty} f(x)$

ii. $\lim\limits_{x \to -\infty} f(x)$

d) What is the equation of the horizontal asymptote (HA) for the graph of $y = f(x)$ in the *xy*-plane?

e) What are the equations of the vertical asymptotes (VAs)?

f) What is the *x*-intercept of the graph?

g) What is the *y*-intercept of the graph?

h) Based on your results in a) through g), sketch a guess as to what the graph of $y = f(x)$ should look like.

5) Give the rule $f(x)$ for a rational function f such that the graph of $y = f(x)$ in the *xy*-plane has a horizontal asymptote (HA) at $y = 4$ and vertical asymptotes (VAs) at $x = -2$ and $x = 3$.

6) Consider $g(t) = \dfrac{t - 6t^2}{2t^2 - 8t + 6}$.

   a) Find the equations of the vertical asymptotes (VAs) of the graph of $w = g(t)$ in the *tw*-plane. Justify your answer using limits. Show work.

   b) Find the equation of the horizontal asymptote (HA) of the graph of $w = g(t)$. Justify your answer using limits. Show work by using a rigorous method from Section 2.3.

7) Consider $h(z) = \dfrac{z^4 - 3z + 2}{z^2 + 1}$. How many vertical asymptotes (VAs) and horizontal asymptotes (HAs) does the graph of $p = h(z)$ have in the *zp*-plane?

OTHER EXAMPLES

8) Evaluate the following limits at a point.

   a) $\lim_{\theta \to \left(\frac{\pi}{2}\right)^-} \tan\theta$

   b) $\lim_{\theta \to \left(\frac{\pi}{2}\right)^+} \tan\theta$

   c) $\lim_{x \to 0^+} \csc x$

   d) $\lim_{x \to \pi^-} \csc x$

9) (Einstein's Theory of Relativity). A particular object at rest has mass m_0 (measured in kilograms, let's say). The speed of light in a vacuum, denoted by c in physics, is about 186,282 miles per second, or exactly 299,792,458 meters per second; the meter is now defined as a consequence of this. If the object is traveling with speed v, then the mass of the object is given by: m, or $m(v) = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

   a) Evaluate $\lim_{v \to c^-} m(v)$, and interpret the result.

   b) Discuss $\lim_{v \to c^+} m(v)$.

See Michael Fowler's web page on relativistic mass increase:
http://galileo.phys.virginia.edu/classes/109N/lectures/mass_increase.html

SECTION 2.5 : THE INDETERMINATE FORMS $\frac{0}{0}$ AND $\frac{\infty}{\infty}$

1) Give function rules for $f(x)$ and $g(x)$ such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, but $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

2) Same as 1), but give rules such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

3) Evaluate the following limits of the form 0/0 at a point. Show work.

a) $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$

b) $\lim_{r\to 6} \frac{3r^2 - 17r - 6}{36 - r^2}$

c) $\lim_{x\to 25} \frac{\sqrt{x} - 5}{x - 25}$

d) $\lim_{t\to 5} \frac{\sqrt{11+t} - 4}{t - 5}$

e) $\lim_{\theta\to -2} \frac{\frac{1}{\theta} + \frac{1}{2}}{\theta + 2}$

f) $\lim_{x\to \frac{1}{2}} \frac{2x^3 - x^2 + 8x - 4}{2x - 1}$.
(Various methods can be used.)

g) $\lim_{x\to 2} \frac{x^3 - 8}{x - 2}$

4) Sketch a guess as to what the graph of $f(x) = \frac{x^2 - 2x - 3}{x^2 - 3x - 4}$ should look like.

a) Find the domain of f.

b) Find the x-intercept(s), if any.

c) Find the y-intercept, if any.

d) Identify whether f is even, odd, or neither.

e) Write the equations of any asymptotes for the graph. Justify using limits.

f) Find any holes "on" the graph. Justify using limits.

g) Sketch your graph. Incorporate all of the above in your sketch.

SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

1) Find $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right)$, and prove it.

2) Find $\lim_{t \to 0} \left(t^4 + \sin^2 t\right)\cos\left(\frac{t+3}{\sqrt[3]{t^2 - t}}\right)$, and prove it.

3) Assume that there exist real constants c and d such that $c \le f(x) \le d$ for all real values of x (except possibly at 0). Find $\lim_{x \to 0} x^8 f(x)$, and prove it.

4) Find $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$, and prove it. **(← ADDITIONAL PROBLEM)**

5) Find $\lim_{x \to \infty} \frac{\cos x}{x^5}$, and prove it.

6) Find $\lim_{\theta \to -\infty} \frac{5\sin(3\theta)}{4\theta^3}$, and prove it.

7) Evaluate $\lim_{x \to \infty} \frac{5x + \sin x}{x}$.

8) Refer back to Exercise 1. Is it true that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) = \left[\lim_{x \to 0} x^2\right]\left[\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)\right]?$$

Why does this not contradict the list of properties at the beginning of Section 2.2?

SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

ADDITIONAL PROBLEMS (#1-7)

1) Use the ε-δ definition of $\lim_{x \to a} f(x) = L$ to prove that $\lim_{x \to 2} (3x - 7) = -1$.

2) In Exercise 1, given that $\varepsilon = 0.6$, find the largest value of δ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Answer the same question for $\varepsilon = 0.06$ and $\varepsilon = 0.006$.

3) Use the ε-δ definition of $\lim_{x \to a} f(x) = L$ to prove that $\lim_{x \to -8} \left(5 + \frac{1}{4} x \right) = 3$.

4) Consider the statement $\lim_{x \to 4} 6 = 6$, which is of the form $\lim_{x \to a} f(x) = L$.
For any positive real value of ε, what are the positive real values of δ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$?

5) The following statements are of the form $\lim_{x \to a} f(x) = L$. For the given value of ε, find the largest value of δ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Graphs may help.

   a) $\lim_{x \to 9} \sqrt{x} = 3$, given $\varepsilon = 0.1$; give an exact answer.

   b) $\lim_{x \to 2} x^3 = 8$, given $\varepsilon = 0.01$; round off your answer to five significant figures.

6) Give a precise "M-δ" definition of $\lim_{x \to a} f(x) = \infty$, where *a* is a real constant, and the function *f* is defined on an open interval containing *a*, possibly excluding *a* itself.

7) Give a precise "N-δ" definition of $\lim_{x \to a} f(x) = -\infty$, where *a* is a real constant, and the function *f* is defined on an open interval containing *a*, possibly excluding *a* itself.

<u>KNOW THE FOLLOWING</u>

- Precise ε-δ definition of $\lim\limits_{x \to a} f(x) = L$.

- We will not have time to cover the precise definitions of:

$\lim\limits_{x \to a^+} f(x) = L$ $\qquad$ $\lim\limits_{x \to \infty} f(x) = L$ $\qquad$ $\lim\limits_{x \to a} f(x) = \infty$

$\lim\limits_{x \to a^-} f(x) = L$ $\qquad$ $\lim\limits_{x \to -\infty} f(x) = L$ $\qquad$ $\lim\limits_{x \to a} f(x) = -\infty$

SECTION 2.8: CONTINUITY

CLASSIFYING DISCONTINUITIES

1) For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function, and graph the function.

   a) $g(t) = t^2 - 4t + 3$

   b) $f(x) = \dfrac{x^2 - 5x + 6}{x - 2}$

   c) $h(r) = \dfrac{4r + 12}{r^2 + 6r + 9}$

   d) $f(x) = \begin{cases} x^2 + x, & \text{if } x \le 2 \\ 6 - x, & \text{if } x > 2 \end{cases}$

   e) $g(x) = \begin{cases} x^2 + x, & \text{if } x < 2 \\ 8 - x, & \text{if } x > 2 \end{cases}$ (Variation on d))

   f) $h(x) = \begin{cases} x^2 + x, & \text{if } x < 2 \\ 8 - x, & \text{if } x \ge 2 \end{cases}$ (Variation on d) and e))

   On f), discuss the continuity of h at 2, and justify your conclusion.

   g) $f(x) = \dfrac{\left|x^2 - 4\right|}{x^2 - 4}$

2) Your bank account is accruing continuously compounded interest. At noon today, you withdraw \$200 from the account. If the amount of money in your account is plotted against time, what type of discontinuity appears at noon?

CONTINUITY

3) Draw a graph where f is defined on $[a, b]$, and f is continuous on (a, b), but f is not continuous on the closed interval $[a, b]$.

4) Determine A such that the function f defined below is continuous on $\mathbb{R}$:

$$f(x)=\begin{cases}\dfrac{x^2-25}{x-5}, & \text{if } x\neq 5\\ A, & \text{if } x=5\end{cases}$$

5) Let $f(x)=\dfrac{\sqrt{x-1}+\sqrt[3]{x+5}}{x^2-7x+12}$. What are the continuity intervals of f?

6) Let $h(x)=\csc\left(\dfrac{1}{\sqrt{x}}\right)$. Where is h continuous? Show work!

THE INTERMEDIATE VALUE THEOREM (IVT)

7) a) Use the IVT to prove that the following equation has a solution between 1 and 2:
$3x^3-2x^2-2x-5=0$.

b) Use Synthetic Division (see Section 2.3 in the Precalculus notes) to show that $\dfrac{5}{3}$ is such a solution.

8) The height of a projectile t seconds after it is fired is given by
$s(t)=-16t^2+30t+4$ in feet, where $0\le t\le 2$.

a) Use the IVT to prove that the projectile achieves a height of 15 feet sometime within one second after being fired.

b) Find the value for t at which this happens. Show work!

ADDITIONAL PROBLEMS (#9-10)

9) Verify the IVT for $f(x)=x^2+5$ on the x-interval $[1,3]$.

10) Verify the IVT for $f(x)=x^2+4x-1$ on the x-interval $[-1,2]$.

KNOW THE FOLLOWING

- The definitions of continuity of a function at a point and on an open interval.
- Recognize continuity on a closed interval.