CHAPTER 3: DERIVATIVES

SECTION 3.1: DERIVATIVES, TANGENT LINES, and RATES OF CHANGE

In these Exercises, use a version of the Limit Definition of the Derivative.
Do **not** use the short cuts that will be introduced in later sections.

1) Let $f(x) = 5x^2 + 1$.

a) Evaluate the difference quotients in the tables below.

$\dfrac{f(3.1) - f(3)}{3.1 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = 0.1$	
$\dfrac{f(3.01) - f(3)}{3.01 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = 0.01$	
$\dfrac{f(3.001) - f(3)}{3.001 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = 0.001$	

$\dfrac{f(2.9) - f(3)}{2.9 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = -0.1$	
$\dfrac{f(2.99) - f(3)}{2.99 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = -0.01$	
$\dfrac{f(2.999) - f(3)}{2.999 - 3}$; this is $\dfrac{f(3+h) - f(3)}{h}$ with $h = -0.001$	

b) Does the table in a) rigorously prove what $f'(3)$ is?

c) Find $f'(3)$ rigorously using a version of the Limit Definition of the Derivative.

2) Let $f(x) = \sqrt{3x-2}$. Consider the graph of $y = f(x)$ in the usual *xy*-plane.

   a) Find the slope of the tangent line to the graph of f at the point $(a, f(a))$, where a is an arbitrary real number in $\left(\frac{2}{3}, \infty\right)$.

   b) Find an equation of the **tangent line** to the graph of f at the point $(9, f(9))$.

   c) Find an equation of the **normal line** to the graph of f at the point $(9, f(9))$.

3) The position function s of a particle moving along a coordinate line is given by $s(t) = 2t - 3t^2$, where time t is measured in seconds, and $s(t)$ is measured in centimeters.

   a) Find the **average velocity** of the particle in the following time intervals:

      i. $[1, 1.1]$ ii. $[1, 1.01]$

   b) Find the **[instantaneous] velocity** of the particle at time $t = 1$.

SECTION 3.2: DERIVATIVE FUNCTIONS and DIFFERENTIABILITY

1) Let $f(x)=\dfrac{1}{x^2}$.

   a) Use the Limit Definition of the Derivative to find $f'(x)$.

   b) Use the short cuts to find $f'(x)$.

2) Let $r(x)=x^4$.

   a) Use the Limit Definition of the Derivative to find $r'(x)$.
      Hint: Use the Binomial Theorem from Section 9.5 in the Precalculus notes.

   b) Use the short cuts to find $r'(x)$.

3) Let $f(x)=9\left(\sqrt[3]{x^2}\right)$. Find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.
   Do not leave negative exponents in your final answers.
   You do not have to simplify radicals or rationalize denominators.

4) Let $y=x^{10}$. What is $\dfrac{d^{20}y}{dx^{20}}$? You shouldn't have to show any work!

5) The position function s of a particle moving along a coordinate line is given by $s(t)=4t^5$, where time t is measured in hours, and $s(t)$ is measured in miles.

   a) Determine the velocity function [rule] $v(t)$. (Use the short cuts.)

   b) Determine the velocity of the particle at times $t=1$, $t=2$, and $t=-4.7$.

   c) Determine the acceleration function [rule] $a(t)$. (Use the short cuts.)

   d) Determine the acceleration of the particle at times $t=1$, $t=2$, and $t=-4.7$.

6) For each part below, answer "Yes" or "No."

a) If $f(x) = x^4 - 3x + 1$, is f differentiable everywhere on $\mathbb{R}$?

b) If $g(x) = |3x - 8|$, is g differentiable at $\frac{8}{3}$?

c) If $h(t) = \sqrt[3]{t^2}$, is h differentiable at 0?

d) If $p(x) = \begin{cases} 4x+3, & \text{if } x \le -1 \\ x^2 - 1, & \text{if } x > -1 \end{cases}$, is p differentiable at -1?

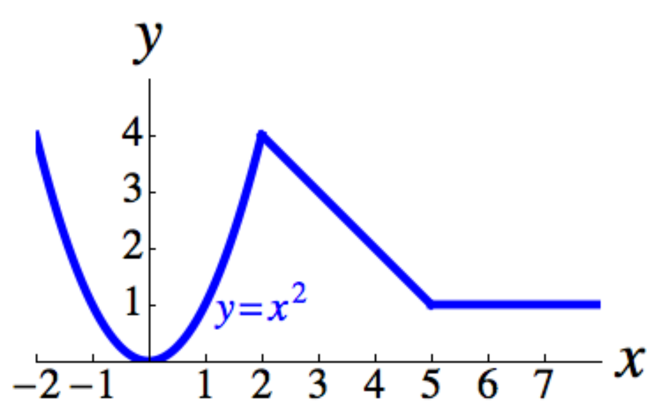
e) If $q(x) = \frac{3}{x+2}$, is q differentiable on the interval $(-10, 0)$?

f) If $q(x) = \frac{3}{x+2}$, is q differentiable on the interval $(0, 10)$?

7) Determine whether or not the graph of f has a **vertical tangent line** at the point $(0, 0)$ and whether f has a **corner**, a **cusp**, or **neither** at $(0, 0)$.

a) $f(x) = x^{4/5}$ b) $f(x) = x^{3/5}$ c) $f(x) = |x|$

8) The graph of f is given below. Sketch the graph of f'.



SECTION 3.3: TECHNIQUES OF DIFFERENTIATION

1) Let $g(w) = 3w^2 - 5w + 4$.

    a) Use the Limit Definition of the Derivative to find $g'(w)$.

    b) Use the short cuts to find $g'(w)$.

2) Find the following derivatives. Simplify where appropriate.
Do not leave negative exponents in your final answer.
You do not have to simplify radicals or rationalize denominators.
From this point on, use the short cuts for differentiation, unless otherwise stated.

    a) Let $f(x) = 5x^3 - \dfrac{3}{x^2} + \dfrac{1}{\sqrt{x}} + \dfrac{\sqrt[3]{x}}{6} - 2$. Find $f'(x)$.

    b) Find $\dfrac{d}{dt}\left(\dfrac{3t-2}{5-t}\right)$.

    c) Find $D_z\left[\left(z^2 - 4\right)^2\right]$. Also, factor your result completely over the integers.

    d) Let $q = \left(w^2 - 3w + 1\right)\left(w^3 - 2\right)$. Find $\dfrac{dq}{dw}$.
    Use the Product Rule of Differentiation.
    After finding all relevant derivatives, you do not have to simplify further.

    e) Let $y = \dfrac{x^2 + 5x - 1}{2x^2}$. Find $\dfrac{dy}{dx}$; that is, find y'.
    i. Do this without using the Quotient Rule of Differentiation.
    ii. Do this using the Quotient Rule of Differentiation, and check that your answer is equivalent to your answer in i.

    f) Let $N(x) = \dfrac{\dfrac{4}{x^2}}{\dfrac{3}{x} + 2}$. Find $N'(x)$. Hint: Simplify first.

    g) Let $W(x) = (3x)^2$. Find $W'(x)$.

    h) Let $S(x) = (3x + 1)^{-2}$. Find $S'(x)$. You will be able to find a shorter solution when you learn the Generalized Power Rule in Section 3.6.

3) The position function s of a particle moving along a coordinate line is given by $s(t) = 4t^3 + 15t^2 - 18t + 1$, where time t is measured in minutes, and $s(t)$ is measured in feet.

   a) Determine the velocity function [rule] $v(t)$. (Use the short cuts.)

   b) Determine the velocity of the particle at times $t = 1$, $t = 2$, and $t = -4.7$.

   c) Determine the acceleration function [rule] $a(t)$. (Use the short cuts.)

   d) Determine the acceleration of the particle at times $t = 1$, $t = 2$, and $t = -4.7$.

4) Consider the graph of $y = \dfrac{8}{x^2 + 4}$.

   a) Find an equation of the **tangent line** to the graph at the point $P(2,1)$.

   b) Find an equation of the **normal line** to the graph at the point $P(2,1)$.

5) (Product Rule for Products of Three Factors).
Assume that f, g, and h are functions that are everywhere differentiable on $\mathbb{R}$.
Use the Product Rule for Products of Two Factors given for this section to prove:

$$D_x\left[f(x)g(x)h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

6) (Preview of the Generalized Power Rule in Section 3.6).
Use Exercise 5 to find an expression for $D_x\left(\left[f(x)\right]^3\right)$.

7) Let $f(x) = x^3 - \dfrac{5}{2}x^2 - 2x + 1$, and let Point P be at $(1, f(1))$.

   a) Find the points on the graph of $y = f(x)$ at which the tangent line is horizontal.

   b) Find an equation of the **tangent line** to the graph of f at Point P.

   c) Find an equation of the **normal line** to the graph of f at Point P.

   d) Find the points on the graph of $y = f(x)$ at which the tangent line has a slope of 10.

8) A new military plane is flying over the ocean. The plane's flight path can be modeled by the graph of $y = x^2 + 100$, where x corresponds to horizontal position in feet, and y corresponds to the height of the plane in feet.

   a) A target is placed at the point $P(0, 0)$. The plane is equipped with a double-tipped missile that can be shot in either direction tangent to the plane's path at the point the missile is shot. Find the two points along its path where the plane can shoot the missile in order to hit the target.

   b) Repeat part a), except assume that the target is placed at the point $Q(24, 0)$.

   Optional / Just for fun: Sketch the graph of $y = x^2 + 100$, and see why your answers to part b) differ the way they do from your answers to part a).

SECTION 3.4: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

LIMITS (ADDITIONAL PROBLEMS)

1) Evaluate the following limits.

a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{4x}$

c) $\displaystyle\lim_{\theta\to 0}\frac{\tan^3(10\theta)}{\theta^3}$

b) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{\sin(3x)}$

d) $\displaystyle\lim_{x\to 0}\frac{1-\cos^3 x}{x}$

DERIVATIVES

2) Find the following derivatives. Simplify where appropriate.

a) Let $f(x)=x^5\cos x$. Find $f'(x)$.

b) Let $g=\dfrac{\sin w}{1+\cos w}$. Find $\dfrac{dg}{dw}$.

c) Find $D_r(\csc r-\cot r)$.

d) Find $D_\alpha\left(7\sec\alpha+4\alpha^2-2\right)$.

e) Find $D_\theta\left(\theta^2\tan\theta\right)$.

f) Let $k(\beta)=\cos\beta\sec\beta$. Find $k'(\beta)$, and find the domain of k'.

TANGENT AND NORMAL LINES

3) Find the x-coordinates of all points on the graph of $y=2\cos x+\left(\sqrt{2}\right)x$ at which the tangent line is horizontal.

4) Consider the graph of $y = 1 + 2\cos x$.

   a) Find the x-coordinates of all points on the graph at which the tangent line is horizontal. (Suggestion: Sketch the graph, and see if your answer makes sense.)

   b) Find the x-coordinates of all points on the graph at which the tangent line is perpendicular to the line $y = -\dfrac{\sqrt{3}}{3}x + 4$.

   c) Find equations of the tangent line and the normal line to the graph at the point where the graph crosses the y-axis.

5) Find equations of the tangent line and the normal line to the graph of $y = \tan x - \sqrt{2}\sin x$ at the point $P\left(-\dfrac{3\pi}{4}, 2\right)$.

6) Find the x-coordinates of all points on the graph of $y = 2\sin x - \cos(2x)$ at which the tangent line is horizontal. Note: This problem will be revisited later when we discuss the Chain Rule.

PROOFS (SEE ALSO SECTION 3.6 EXERCISES)

In 7) and 8) below, do **not** use the Limit Definition of the Derivative.
You may use your knowledge of the derivatives of $\sin x$ and $\cos x$ without proof.

7) Prove: $D_x(\cot x) = -\csc^2 x$.

8) Prove: $D_x(\csc x) = -(\csc x)(\cot x)$.

KNOW THE FOLLOWING

- The derivatives of the six basic trig functions.

- The proofs for the derivatives of $\sin x$ and $\cos x$ using the Limit Definition of the Derivative.

- The proofs for the derivatives of the other four basic trig functions, assuming you have knowledge of the derivatives of $\sin x$ and $\cos x$.

SECTION 3.5: DIFFERENTIALS and LINEARIZATION OF FUNCTIONS

1) Use differentials and $\sqrt[4]{16}$ to approximate $\sqrt[4]{15.92}$.
   (Just for fun, compare your approximation to a calculator's result for $\sqrt[4]{15.92}$.)

2) Let $f(x) = -3x^3 + 8x - 7$. Find a linear approximation for $f(3.96)$ if x changes from 4 to 3.96.

3) Find a linear approximation for $\sec 31.5^\circ$ by using the value of $\sec 30^\circ$.
   Give your result as an exact value, and also give a decimal approximation rounded off to four significant digits.

SECTION 3.6: CHAIN RULE

1) Find the following derivatives. Simplify where appropriate.

a) Let $f(x)=\left(x^2-3x+8\right)^3$. Find $f'(x)$.

b) Let $n=\dfrac{1}{\left(m^2+4\right)^5}$. Find $\dfrac{dn}{dm}$.

c) Let $h(x)=\left(x^2-\dfrac{1}{x^2}\right)^6$. Find $h'(x)$.

d) Find $D_t\left[\tan^3(6t)\right]$.

e) Find $D_x\left[x^3\sin^2(2x)\right]$.

f) Find $D_t\left[\sqrt[3]{8t^3+27}\right]$.

g) Find $D_\theta\left(\left[\pi-6\theta^5+\csc(5\theta)\right]^7\right)$

h) Find $D_w\left[\dfrac{1+\sin(2w)}{\cos(2w)}\right]$. Instead of applying the Quotient Rule of Differentiation, rewrite the expression first for a quicker solution.

i) Redo h), but apply the Quotient Rule of Differentiation and simplify.

j) Find $D_\phi\left[\cos\left(\sqrt{\phi}\right)+\sqrt{\cos\phi}\right]$.

k) Find $D_x\left[(6x-7)^3\left(8x^2+9\right)^4\right]$, factor your result completely over the integers, and simplify.

l) Find $D_x\left[\dfrac{\left(x^2-3\right)^2}{\sqrt{x^2+5}}\right]$. Write your answer as a single simplified fraction.

Do not leave negative exponents in your final answer.
You do not have to simplify radicals or rationalize denominators.

2) Let $S(x) = (3x+1)^{-2}$. Find $S'(x)$. Compare your solution to your longer solution in Section 3.3, Exercise 2h.

3) Find $D_x\left(\left[f(x)\right]^3\right)$. Refer back to Section 3.3, Exercise 6.

4) Find the x-coordinates of all points on the graph of $y = 2\sin x - \cos(2x)$ at which the tangent line is horizontal. (This exercise was introduced in Section 3.4, Exercise 6, but it should be easier now.)

5) Assume $y = f(u)$, $u = g(t)$, and $t = h(r)$, where f, g, and h are functions that are all differentiable everywhere on $\mathbb{R}$. Given that $\frac{dy}{du} = 2$, $\frac{du}{dt} = 7$, and $\frac{dt}{dr} = 3$, find $\frac{dy}{dr}$.

6) We will find $D_x\left[\left(x^3\right)^5\right]$ in three different ways. Observe that all three results are equivalent.

   a) Simplify $\left(x^3\right)^5$, and use the Basic Power Rule of Differentiation.

   b) Do not simplify $\left(x^3\right)^5$, use the Generalized Power Rule of Differentiation, and simplify.

   c) (Seeing the Chain Rule in action). Let $u = x^3$, let $y = u^5$, use the Chain Rule to find $\frac{dy}{dx}$, and simplify. Write your answer in terms of x alone, not x and u.

7) We will find $D_x\left(\frac{1}{x^2+1}\right)$ in three different ways. Observe that all three results are equivalent.

   a) Use the Reciprocal Rule or the Quotient Rule of Differentiation, and simplify.

   b) Rewrite $D_x\left(\frac{1}{x^2+1}\right)$ as $D_x\left[\left(x^2+1\right)^{-1}\right]$, use the Generalized Power Rule of Differentiation, and simplify.

   c) (Seeing the Chain Rule in action). Let $u=x^2+1$, let $y=\frac{1}{u}$, use the Chain Rule to find $\frac{dy}{dx}$, and simplify. Write your answer in terms of x alone, not x and u.

8) The graph of $y=\sqrt{a^2-x^2}$, where $a>0$, is the upper half of a **circle** of radius a centered at O (the origin). Show that the **tangent line** to any point P on the graph is **perpendicular** to the line segment $\overline{OP}$. (We will revisit this in Section 3.7, Exercise 4. This issue was brought up in Section 3.4, Part E.)

9) We can use the Cofunction Identities to prove that $D_x(\csc x)=-(\csc x)(\cot x)$.

   Complete the proof: $D_x(\csc x) = D_x\left[\sec\left(\frac{\pi}{2}-x\right)\right] = \ldots = -(\csc x)(\cot x)$.

   (You may use the Generalized Trigonometric Rule for the secant function.) This helps explain the patterns we find between pairs of derivative rules for cofunctions.

10)

   a) Find $D_x\left[\sin(2x)\right]$.

   b) If we didn't have the Chain Rule, we may have to resort to the Double-Angle Identities. Find $D_x\left[\sin(2x)\right]$ using the Double-Angle Identities.

   c) How does $D_x\left[\sin(2x)\right]$ compare with $D_x(\sin x)$? What does this tell us about how the graph of $y=\sin(2x)$ differs from the graph of $y=\sin x$?

SECTION 3.7: IMPLICIT DIFFERENTIATION

1) Consider the given equation $xy^2 + y = 4$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

   a) Find $\frac{dy}{dx}$, also known as y'.

   b) Verify that the point $P\left(\frac{1}{2}, 2\right)$ lies on the graph of the given equation.

   c) Evaluate $\left[\frac{dy}{dx}\right]_{\left(\frac{1}{2}, 2\right)}$.

   d) Find an equation of the tangent line to the graph of the given equation at the point $P\left(\frac{1}{2}, 2\right)$.

2) Consider the given equation $5x^2 - 2xy + 3x^3y^4 - 4y^2 = 44$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

   a) Find $\frac{dy}{dx}$, also known as y'.

   b) Verify that the point $P(2, -1)$ lies on the graph of the given equation.

   c) Evaluate $\left[\frac{dy}{dx}\right]_{(2, -1)}$.

   d) Find an equation of the tangent line to the graph of the given equation at the point $P(2, -1)$.

3) Consider the given equation $\sin\left(\sqrt{y}\right) + x\cos y = 3$. Assume that it "determines" an implicit differentiable function f such that $y = f(x)$.

   a) Find $\dfrac{dy}{dx}$, also known as y'.

   b) Verify that the point $P(3, 0)$ lies on the graph of the given equation.

   c) Evaluate your result from a) at the point $P(3, 0)$. It turns out that P is an endpoint of the graph of the given equation, and your result here corresponds to a kind of one-sided derivative.

4) Use Implicit Differentiation to show that the **tangent line** to any point P on a **circle** with center O (the origin) is **perpendicular** to the line segment $\overline{OP}$. (This is another approach to Section 3.6, Exercise 8. This issue was brought up in Section 3.4, Part E.)

SECTION 3.8: RELATED RATES

1) Assuming $3x^2y + 2x = -32$, where x and y are differentiable functions of t, and $\dfrac{dy}{dt} = -4$ when $x = 2$ and $y = -3$, find $\dfrac{dx}{dt}$.

2) (Balloon problem). Air is being pumped into a spherical balloon at the rate of 1.754 cubic centimeters per second. The balloon maintains a spherical shape throughout. How fast is the radius of the balloon changing when the diameter is 2.736 centimeters in length? Round off your answer to four significant digits. Keep intermediate results exact.

3) (Ladder problem). A wall stands upright and perpendicular from the flat ground. A 25-foot long ladder leans against the wall. The bottom of the ladder is moved away from the building horizontally (along a line perpendicular to the wall) at a rate of 30 inches per minute (until the ladder lies flat on the ground). How fast is the top of the ladder sliding down the building when the top of the ladder is 10 feet above the ground? Round off your answer to four significant digits.
Keep intermediate results exact.

4) (Two bug problem). A ladybug crawls out of a small hole in a large wall and crawls to the right at a rate of 3 inches per minute. Forty-five seconds later, a tick crawls out of the hole and crawls up at a rate of 2 inches per minute. How fast is the distance between the ladybug and the tick changing four minutes after the ladybug crawls out of the hole? Round off your answer to three significant digits. Keep intermediate results exact.

5) (Boyle's Law for Ideal Gases). If the temperature and the mass of a confined ideal gas are fixed, then $PV = k$, where P is the pressure and V is the volume of the gas, and k is a constant. How fast is the volume of the gas changing at the moment that the pressure is $45\,\dfrac{\text{lb}}{\text{in}^2}$, the volume is 60 in^3, and the pressure is increasing at a rate of $3\,\dfrac{\text{lb}}{\text{in}^2}$ per minute? Give an exact answer.

6) (Cylinder problem). A right circular cylinder's volume is shrinking at the rate of $15\,\dfrac{\text{cm}^3}{\text{hr}}$ in such a way that its base radius is always twice its height. The cylinder retains a right circular cylindrical shape. How fast is the base radius changing when the height is 1 meter? Give an exact answer.

7) (Parallel resistor problem). Two parallel resistors have resistances R_1 and R_2. If the total resistance is R, then $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. R_1 is increasing at $0.04\,\dfrac{\text{ohm}}{\text{sec}}$, and R_2 is decreasing at $0.03\,\dfrac{\text{ohm}}{\text{sec}}$. How fast is the total resistance changing at the moment that R_1 is 3 ohms and R_2 is 4 ohms? Round off your answer to four significant digits.

8) (Airplane problem). A military plane maintains an altitude of 15,000 feet over a vast flat desert. It flies at a constant speed on a line that will take it directly over an observer on the ground. At noon, the angle of elevation from the observer's shoes to the plane is 30 degrees, and the angle of elevation is increasing at a rate of 2.2 degrees per second. Find the speed of the plane at noon. Round off your answer to five significant digits. What is the speed in miles per hour (mph)? Remember that there are 5280 feet in one mile.