CHAPTER 4: APPLICATIONS OF DERIVATIVES

SECTION 4.1: EXTREMA

1) For each part below, find the absolute maximum and minimum values of f on the given interval. Also give the absolute maximum and minimum points on the graph of $y = f(x)$. Show work!

   a) $f(x) = 15 + 8x - 2x^2$ on $[-1, 3]$

   b) $f(x) = \frac{1}{4}x^4 - \frac{5}{3}x^3 - 3x^2 + 10$ on $[-1, 2]$

   c) $f(x) = x^2 - \frac{16}{x}$ on $[-4, -1]$

2) For each part below, find the domain and the critical number(s) ("CNs") of the function with the indicated rule. If there are no critical numbers, write "NONE."

   a) $f(x) = 4x^4 - x^3 - 32x^2 + 12x + 13$

   b) $g(t) = \sqrt{t^2 - 36}$

   c) $h(x) = x\sqrt{4x - 1}$

   d) $p(\theta) = \frac{1}{2}\cos(2\theta) + \sin(\theta)$

   e) $q(x) = \cot(x)$

SECTION 4.2: MEAN VALUE THEOREM (MVT) FOR DERIVATIVES

1) For each part below, determine whether or not f satisfies the hypotheses of Rolle's Theorem on the given interval $\left[a, b\right]$. If it does not, explain why not.

   If it does, find all real values c in $\left(a, b\right)$ that satisfy the conclusion of the theorem; i.e., $f'\left(c\right) = 0$.

   a) $f\left(x\right) = x^2 - 6x + 10$ on $\left[1, 5\right]$

   b) $f\left(x\right) = x^2 - 6x + 10$ on $\left[3, 7\right]$

   c) $f\left(x\right) = x^4 + 14x^3 + 69x^2 + 140x - 5$ on $\left[-6, -1\right]$;

   Hint: -2 is a value for c that satisfies $f'\left(c\right) = 0$.

   d) $f\left(x\right) = \left|x\right|$ on $\left[-4, 4\right]$

2) For each part below, determine whether or not f satisfies the hypotheses of the Mean Value Theorem (MVT) for Derivatives on the given interval $\left[a, b\right]$.

   If it does not, explain why not. If it does, find all real values c in $\left(a, b\right)$ that satisfy the conclusion of the theorem; i.e., $f'\left(c\right) = \dfrac{f\left(b\right) - f\left(a\right)}{b - a}$.

   a) $f\left(x\right) = x + \dfrac{4}{x}$ on $\left[1, 4\right]$

   b) $f\left(x\right) = 2x^3 + 5x^2 - 4x + 3$ on $\left[-2, 3\right]$

   c) $f\left(x\right) = x^{2/3}$ on $\left[-8, 8\right]$

   d) $f\left(x\right) = 3x + 1$ on $\left[0, 2\right]$

KNOW THE FOLLOWING

- Rolle's Theorem
- The Mean Value Theorem (MVT) for Derivatives

SECTION 4.3: FIRST DERIVATIVE TEST

1) For each part below, sketch the graph of $y = f(x)$.

   • Find the domain of f.
   • State whether f is even, odd, or neither, and incorporate any corresponding symmetry in your graph.
   • Find the y-intercept, if any. You do not have to find x-intercepts.
   • Find and indicate on your graph any holes, vertical asymptotes (VAs), horizontal asymptotes (HAs), and slant asymptotes (SAs), and justify them using limits.
   • Find all points at critical numbers (if any). Indicate these points on your graph.
   • Use the First Derivative Test to classify each point at a critical number as a local maximum point, a local minimum point, or neither. (The next instruction may help.)
   • Find the intervals on which f is increasing / decreasing, and have your graph show that.

   a) $f(x) = -x^3 + 3x$. You can find the x-intercepts here.

   b) $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$.

   Hint 1: You studied this in Section 4.2, Exercise 1c.
   Hint 2: The Remainder Theorem from Section 2.3 in the Precalculus notes may help you with function evaluations.
   Optional / For fun: See how the graph relates to Section 4.2, Exercise 1c.

   c) $f(x) = \dfrac{1}{x-4}$

   d) $f(x) = \sin(x) - \dfrac{1}{2}x$; restrict the domain to $(-2\pi, 2\pi)$. Symmetry will help!

2) f is a function such that f and f' are continuous everywhere on $\mathbb{R}$, 7 is the only critical number of f, $f'(\pi) = 11$, and $f'(13) = -2$. According to the First Derivative Test, is the point $(7, f(7))$ on the graph of $y = f(x)$ a local maximum point, a local minimum point, or neither?

SECTION 4.4: SECOND DERIVATIVES

1) Let's revisit Section 4.3, Exercise 1. For each part below …
   - Find the Possible Inflection Numbers (PINs), if any.
   - Find the x-intervals on which the graph of $y = f(x)$ is concave up / concave down, and see how this is consistent with your graph in Section 4.3.
   - For each PIN, state whether or not the corresponding point on the graph is an inflection point (IP). Find the y-coordinate(s) of the point(s) in a) and d) but not b) and c).

   a) $f(x) = -x^3 + 3x$

   b) $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$

   c) $f(x) = \dfrac{1}{x-4}$

   d) $f(x) = \sin(x) - \dfrac{1}{2}x$; restrict the domain to $(-2\pi, 2\pi)$.
   Find the inflection points (IPs).

2) Let $f(x) = x^4 + 14x^3 + 69x^2 + 140x - 5$, as in Exercise 1b.
   In Section 4.3, Exercise 1b, you should have used the First Derivative Test to show that f has a local minimum at $x = -5$. Prove this using the Second Derivative Test, instead.

3) Let $g(\theta) = 4\cos^2(3\theta)$. Hint: A trig ID will prove very helpful here.

   a) What does the Second Derivative Test tell us about the point on the graph of $y = g(\theta)$ where $\theta = \dfrac{\pi}{3}$? Justify your answer.

   b) What does the Second Derivative Test tell us about the point on the graph of $y = g(\theta)$ where $\theta = \dfrac{\pi}{4}$? Justify your answer.

4) Let $h(t) = t^6 + 3t^4$. What does the Second Derivative Test tell us about the point $(0, 0)$ on the graph of $y = h(t)$? Justify your answer.

5) In 2009, supporters of President Obama said that, when it came to employment during his first year in office, the first derivative was negative, but the second derivative was positive. Interpret this statement.

SECTION 4.5: GRAPHING

1) For each part below, sketch the graph of $y = f(x)$.

- Find the domain of f.
- State whether f is even, odd, or neither, and incorporate any corresponding symmetry in your graph.
- Find the y-intercept, if any. You do not have to find x-intercepts in a), but find any x-intercepts in the others.
- Find and indicate on your graph any holes, vertical asymptotes (VAs), horizontal asymptotes (HAs), and slant asymptotes (SAs), and justify them using limits.
- Find the critical numbers (CNs), if any.
- Find all points at critical numbers (if any). Indicate these points on your graph.
- Use the First Derivative Test to classify each point at a critical number as a local maximum point, a local minimum point, or neither. (The next instruction may help.)
- Find the intervals on which f is increasing / decreasing, and have your graph show that.
- Find the "Possible Inflection Numbers" (PINs), if any.
- Find the x-intervals on which the graph of $y = f(x)$ is concave up / concave down, and have your graph show that.
- For each PIN, state whether or not the corresponding point on the graph is an inflection point (IP). Find any IPs.

a) $f(x) = -x^4 + 4x^3 + 48x^2 + 112x - 500$.
Hint: -2 is one of the critical numbers.
We will discuss the x-intercepts in Section 4.8.

b) $f(x) = \dfrac{x^2}{3(x+4)^2}$

c) $f(x) = \dfrac{(x-1)^2}{x^2+1}$
In c), give the y-coordinates of any inflection points (IPs) as integers or as decimals rounded off to two decimal places.

d) $f(x) = \sqrt[3]{x^2}\left[3 - 2\left(\sqrt[3]{x}\right)\right]$. Hint: Rewrite by multiplying first!

2) Sketch the graph of $y = \left| \dfrac{x^2 - 16}{x - 4} \right|$. Indicate all features of interest.

   Hint: Simplify first.

3) Draw the graph of a function f with the following properties:

   $f(0) = 2$,
   $f(3) = 4$,
   $f(5) = 6$,

   $f'(0)$ does not exist (DNE) (or is "undefined"),
   $f'(5) = 0$,
   $f'(x) < 0$ wherever $x < 0$ or $x > 5$,
   $f'(x) > 0$ wherever $0 < x < 5$,

   $f''(0)$ does not exist (DNE) (or is "undefined"),
   $f''(3) = 0$,
   $f''(x) < 0$ wherever $x < 0$ or $x > 3$, and
   $f''(x) > 0$ wherever $0 < x < 3$.

   There are infinitely many different possible graphs that will work here.

SECTION 4.6: OPTIMIZATION

Give exact measurements (with the possible exception of #1), and round off irrational results to four significant digits.

1) (Closed box problem). We need a closed rectangular cardboard box with a square top, a square bottom, and a volume of 32 m^3. Find the dimensions of the valid box that requires the least amount of cardboard, and find the amount of cardboard needed. (Ignore the thickness of the cardboard.) Round off measurements to four significant digits. **ADDITIONAL PROBLEM:** Give the exact measurements.

2) (Open box problem). Repeat Exercise 1, except the box must have an open top. Explain why the optimal dimensions here are different compared to Exercise 1.

3) (Aquarium problem). A glass aquarium is to be shaped as a right circular cylinder with an open top and a capacity of two cubic meters. Find the dimensions of the valid cylinder that requires the least amount of glass, and find that amount of glass. (Ignore the thickness of the glass.) How would the diameter compare to the height?

4) (Six pigpen problem). A 2×3 array of six congruent rectangular pigpens (that all look the same from above) will be in the overall shape of a rectangle *R*. We may use 100 feet of fencing to form the boundaries of the pigpens. Find the dimensions for a single pigpen that will maximize the total area of all the pigpens, and find this total area. (The fencing separating the pigpens has constant height, so we may ignore height in our calculations. Also, assume the boundaries between pigpens are not double-fenced; that is, assume that the thickness of the fencing between pigpens is the same as the thickness of the fencing along the outer boundary, *R*.)

5) (Chase problem). At midnight, you are 60 feet due north of Jonas. You run due east at ten feet per second. Jonas walks two feet per second due north. How many seconds after midnight is Jonas closest to you? What is the corresponding minimum distance separating you and Jonas?

6) (Cheap building problem). We need to build a building in the shape of a rectangular box with a capacity of 500,000 cubic feet. We require the length of the floor to be twice the width. The floor will cost \$3 per square foot, the vertical walls will cost \$4 per square foot, and the ceiling will cost \$5 per square foot. To the nearest hundredth of a foot, what are the dimensions of the cheapest building that we can build? What is that building's cost to the nearest dollar?

7) (Closest point problem). A military aircraft flies along the parabola $y = x^2 + 1$ in the usual Cartesian *xy*-plane. (Distance is measured in meters.) A UFO is hovering at the point $(6,4)$. Find the point on the parabola that is closest to the UFO, and find the corresponding minimum distance.

8) (Big squares problem). Prove that, among all rectangles with fixed perimeter *p*, where $p > 0$, the largest in area is a square.

SECTION 4.7: MORE APPLICATIONS OF DERIVATIVES

1) The position function s of a particle moving along a coordinate line is given by $s(t) = 4t^3 + 15t^2 - 18t + 1$, where time t is measured in minutes, and $s(t)$ is measured in feet. (We saw this function in Section 3.2, Exercise 6.) Assume the positive direction extends to the right and the negative direction extends to the left.

   a) Determine the velocity function [rule] $v(t)$.

   b) Determine the time intervals in which the particle moves to the right. Use a sign chart.

   c) Determine the time intervals in which the particle moves to the left. Use a sign chart.

   d) Draw a schematic representing the motion of the particle in the time interval $[-6, 3]$, as we have done in the notes.

   e) Determine the acceleration function [rule] $a(t)$.

   f) Evaluate $v(-4)$ and $a(-4)$. At time $t = -4$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down? (More precisely, we are referring to an open interval containing $t = -4$ minutes. The same idea goes for g), h), and i) below.)

   g) Evaluate $v(-2)$ and $a(-2)$. At time $t = -2$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down?

   h) Evaluate $v(0)$ and $a(0)$. At time $t = 0$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down?

   i) Evaluate $v(1)$ and $a(1)$. At time $t = 1$ (minutes), is the particle moving to the right or to the left? Is it speeding up or slowing down?

2) Assume that, if we produce x units of a device, we sell all x devices. We sell each device for \$200, so the revenue function is modeled by $R(x) = 200x$. The cost function is modeled by $C(x) = 3x^2 + 500$.

   a) Find the profit function [rule], $P(x)$.

   b) Find the marginal profit at $x = 30$ devices. Based on this result alone, would you be inclined to increase or decrease production?

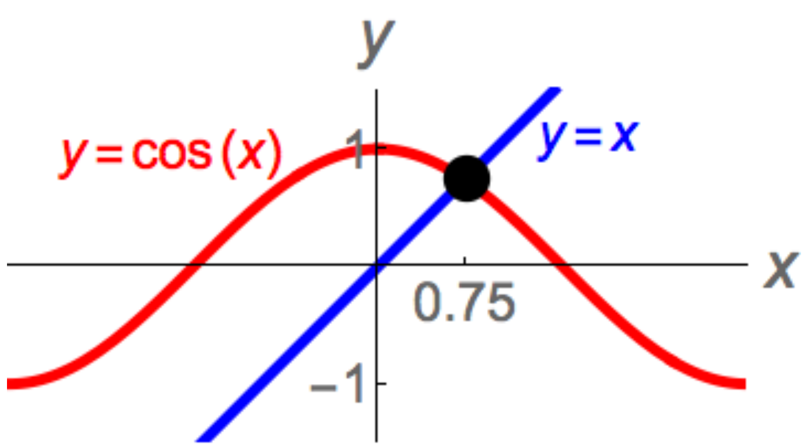
   c) Find the optimal number of devices to produce.

SECTION 4.8: NEWTON'S METHOD

1) Use Newton's Method to approximate $\sqrt[3]{7}$ to four decimal places.
   Use $x_1 = 2$ as your seed. Round off intermediate iterates to five decimal places.

2) Let $f(x) = -x^4 + 4x^3 + 48x^2 + 112x - 500$.
   We saw this in Section 4.5, Exercise 1a. We believe that the graph of $y = f(x)$ has an x-intercept at about $(10, 0)$. (Look at the graph in the Answers, Section 4.5.)
   Use Newton's Method to approximate this x-intercept to two decimal places.
   Use $x_1 = 10$ as your seed. Round off intermediate iterates to three decimal places.
   <u>Note</u>: There do exist quartic formulas for finding the zeros of f exactly, but they are very involved!

3) Use Newton's Method to approximate a real solution to the equation $\cos(x) = x$ to four decimal places. Judging from the graphs of $y = \cos(x)$ and $y = x$ below, use $x_1 = 0.75$ as your seed; it appears that the equation has only one real solution. Round off intermediate iterates to five decimal places.



4) (A failure of Newton's Method). Let $f(x) = \sqrt[3]{x}$. We know that the only real zero of f is 0. Begin with the seed $x_1 = 1$. Use Newton's Method to obtain x_2 and x_3. (Do you believe that further iterates will approach 0? It will help to remember what the graph of the function looks like.) How are the tangent lines changing?