CHAPTER 6: APPLICATIONS OF INTEGRALS

SECTION 6.1: AREA

Assume that distances and lengths are measured in meters.

1) For parts a) and b) below, in the usual *xy*-plane …
   i) Sketch the region R bounded by the graphs of the given equations.
      Locate any intersection points of the graphs.
   ii) **Set up the integral(s)** for the area of R by integrating with respect to x.
   iii) **Set up the integral(s)** for the area of R by integrating with respect to y.
   iv) Find the area of the region by using either method from ii) or iii).
      ADDITIONAL PROBLEM: Find the area using the other method.

   a) $y = -x^2$ and $y = x^2 - 8$

   b) $y^2 = 4 - x$ and $x + 2y = 1$

2) In the *tw*-plane, sketch the regions bounded by the graphs of $w = \sin t$ and $w = \cos t$, where t is restricted to the interval $\left[0, 2\pi\right]$, and find the total area of the regions.

3) Find the area of the region bounded by the graphs of the given equations in the usual *xy*-plane. **You do not have to sketch the region.**

   a) $y = 3x^2 - 5$ and $y = x^2 + 5x - 2$

   b) $y = x\sqrt{x^2 + 16}$, $x = 0$, $x = 3$, and $y = 0$

   c) $x - y^3 = 0$ and $x + y + 2y^2 = 0$

SECTION 6.2: VOLUMES OF SOLIDS OF REVOLUTION – DISKS AND WASHERS

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual *xy*-plane, unless otherwise indicated.

1) The region R is bounded by the graphs of $x+2y=2$, $x=0$, and $y=0$.
   a) Sketch the region.
   b) Find the volume of the solid generated if R is revolved about the x-axis. First, use an integral. Then, use the formula for the volume of a cone.
   c) Find the volume of the solid generated if R is revolved about the y-axis. First, use an integral. Then, use the formula for the volume of a cone.

2) The region R is bounded by the graphs of $y=-x^2$ and $y=x^2-8$. You should have sketched and found the area of this region in Section 6.1, Exercise 1a.
   a) Find the volume of the solid generated if R is revolved about the x-axis.
   b) Find the volume of the solid generated if R is revolved about the y-axis.

3) The region R in the *tw*-plane is bounded by the graphs of $w=\sin t$, $w=\cos t$, $t=0$, and $t=\frac{\pi}{4}$. (This region was related to Section 6.1, Exercise 2.) Find the volume of the solid generated if R is revolved about the t-axis. Hint: You will need to use a trig identity.

4) The region R in the *xy*-plane is bounded by the graphs of $y=\cos(2x)$, $y=0$, $x=0$, and $x=\frac{\pi}{4}$. Sketch R. Find the volume of the solid generated if R is revolved about the x-axis. Hint: You will need to use a trig identity.

5) The region R is bounded by the graphs of $y=4x^2$ and $y=2x$. Sketch R and locate any intersection points of the graphs. Find the volume of the solid generated if R is revolved about the y-axis.

6) Find the volume of a right circular cylinder of base radius r and height (or altitude) h using the Disk Method.

7) Find the volume of a right circular cone of base radius r and height (or altitude) h using the Disk Method.

8) Find the volume of a sphere of radius r using the Disk Method.

For the following problems, set up the appropriate integral(s), but do not evaluate. Your final integrals must have no general notation such as *f*, *g*, f', or *r* that can be re-expressed more precisely. You do not have to simplify your final answers.

9) The region R is bounded by the graphs of $x = 4y - y^2$ and $x = 0$. Sketch R and locate any intersection points of the graphs. **Set up the integral(s)** for the volume of the solid generated if R is revolved about the y-axis.

   ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.

10) The region R is bounded by the graphs of $y = x^2$ and $y = 4$. Sketch R and locate any intersection points of the graphs. **Set up the integral(s)** for the volume of the solid generated if R is revolved about …

    a) the line $y = 4$;
       the volume turns out to be the same as for #9 – analyze sketches to see why!
       ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.
    b) the line $y = 5$.
       ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.
    c) the line $x = 3$.
       ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.

11) The region R is bounded by the graph of $x^2 + y^2 = 1$. Sketch R. **Set up the integral(s)** for the volume of the solid generated if R is revolved about the line $x = 5$.

    ADDITIONAL PROBLEM: Evaluate, and give the volume of the solid.

    ADDITIONAL PROBLEM: Find the volume of the solid using Pappus's second centroid theorem, which states that this volume can be found by multiplying the area of the region by the distance traveled by the centroid of the region in one full revolution about the axis of revolution. (Pappus's first centroid theorem deals with surface area. See Pappus's Centroid Theorem at mathworld.wolfram.com.)

    • Note 1: In general, the centroid could lie outside of the region.
    • Note 2: We assume that the axis of revolution does not pass through the region.

SECTION 6.3: VOLUMES OF SOLIDS OF REVOLUTION – CYLINDRICAL SHELLS

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual *xy*-plane, unless otherwise indicated.

1) The region R is bounded by the graphs of $y = \tan\left(x^2\right)$, $y = 0$, $x = 0$, and $x = \dfrac{\sqrt{\pi}}{2}$.
   Set up the integral(s) for the volume of the solid generated by revolving R about the y-axis. You do not have to evaluate the integral(s). (You will in Section 7.4.)

2) The region R is bounded by the graphs of $2x - y = 12$, $x - 2y = 3$, and $x = 4$.
   Sketch R and locate any intersection points of the graphs.
   Find the volume of the solid generated by revolving R about the y-axis.

3) The region R is bounded by the graphs of $x^2 = 4y$ and $y = 4$.
   Sketch R and locate any intersection points of the graphs.
   Find the volume of the solid generated by revolving R about the x-axis.
   Try to visualize this solid.

4) The region R is bounded by the graphs of $x = y^2 + 3$ and $x - 4y = 0$.
   Sketch R and locate any intersection points of the graphs.
   Find the volume of the solid generated by revolving R about the x-axis.

MIXING METHODS

5) The region R is bounded by the graphs of $y = x^3$, $y = 8$, and $x = 0$. Sketch R.
   Locate any intersection points of the graphs. For each part below, **set up the integral(s)** for the volume of the solid generated by revolving R about the indicated axis. Use the indicated variable of integration; you need to figure out which method (Section 6.2 vs. Section 6.3) to use.
   a) axis is the line $x = 0$; variable of integration is x
   b) axis is the line $x = 0$; variable of integration is y
   c) axis is the line $x = 3$; variable of integration is x
   d) axis is the line $x = 3$; variable of integration is y

   ADDITIONAL PROBLEM(S): Evaluate the integrals you have set up in a), b), c), and d), and give the volumes using appropriate units. Your answers to a) and b) will be the same; your answers to c) and d) will be the same.

6) The region R is bounded by the graphs of $y = 4x^2$ and $4x + y = 8$. Sketch R. Locate any intersection points of the graphs. For each part below, **set up the integral(s)** for the volume of the solid generated by revolving R about the indicated axis. Use the indicated variable of integration; you need to figure out which method (Section 6.2 vs. Section 6.3) to use.

   a) axis is the x-axis; variable of integration is x
   b) axis is the line $x = 1$; variable of integration is x
   c) axis is the line $y = 16$; variable of integration is x

   ADDITIONAL PROBLEM(S): Evaluate the integrals you have set up in a), b), and c), and give the volumes using appropriate units.

ADDITIONAL PROBLEM

7) Use the cylindrical shells method to prove that the volume of a sphere of radius r $(r > 0)$ is $\frac{4}{3}\pi r^3$.

SECTION 6.4: VOLUMES BY CROSS SECTIONS

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual xy-plane, unless otherwise indicated.

1) Let B be the region bounded by the graphs of $x = y^2$ and $x = 9$. Sketch B. For each part below, find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x-axis is …

   a) … a square
   b) … a semicircle with diameter lying on B
   c) … an equilateral triangle

2) Let B be the region bounded by the graph of $x^2 + y^2 = a^2$, where a is a positive constant. Sketch B. Find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x-axis is a square.

3) Let B be the region bounded by the graphs of $y = x^2$ and $y = 4$. Sketch B. Find the volume of the solid that has B as its base if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with hypotenuse on B.

SECTION 6.5: ARC LENGTH and SURFACES OF REVOLUTION

Assume that distances and lengths are measured in meters.
Assume that graphs are in the usual *xy*-plane, unless otherwise indicated.

1) Consider the portion of the graph of $y = x^3 + 1$ from the point $A(1, 2)$ to the point $B(3, 28)$.
   a) **Set up the integral(s)** for the arc length of the portion by integrating with respect to x.
   b) **Set up the integral(s)** for the arc length of the portion by integrating with respect to y.
   c) **Set up the integral(s)** for the area of the surface of revolution that is obtained when the portion is revolved about the x-axis.
   d) **Set up the integral(s)** for the area of the surface of revolution that is obtained when the portion is revolved about the y-axis.

2) **Set up the integral(s)** for the area of the surface of revolution that is obtained when the portion of the graph of $y^2 = x$ from $A(1, 1)$ to $B(4, 2)$ is revolved about the …
   a) x-axis. Also, evaluate the integral(s). Give the exact surface area and also an approximation rounded off to four significant digits.
   b) y-axis. Do not evaluate the integral(s).

3) Find the surface area of a sphere of radius r.
   • Technical Note: Simplify the integrand before writing your first integral. There is technically a division-by-zero issue that arises when, say, $x = r$ and/or $x = -r$, which would lead to an "improper integral"; ignore this for now and proceed.

ADDITIONAL PROBLEM

4) Find the lateral surface area (which excludes the base area) of a right circular cone of base radius r and height (or altitude) h using …
   a) the methods of this section.
   b) Pappus's first centroid theorem, which states that this surface area can be found by multiplying the length of the arc being revolved by the distance traveled by the centroid of the arc in one full revolution about the axis of revolution. (See Pappus's Centroid Theorem at mathworld.wolfram.com.)