CHAPTER 7: LOGARITHMIC and EXPONENTIAL FUNCTIONS

SECTION 7.1: INVERSE FUNCTIONS

1) Let $f(x) = 3x + 4$.

   a) What is the slope of the line with equation $y = f(x)$?

   b) Find $f^{-1}(x)$, the rule for the inverse function of f.
   You may want to review Section 1.9 on Inverse Functions in the Precalculus notes.

   c) What is the slope of the line with equation $y = f^{-1}(x)$?
   Compare this slope with the slope from part a).

2) Let $f(x) = x^3$. Observe that $f(2) = 8$.

   a) Find $f'(2)$.

   b) Let $g(x) = f^{-1}(x)$, the rule for the inverse function of f. Find $g(x)$.

   c) Find $g'(8)$. Compare this with your answer from part a).

SECTION 7.2: ln x

1) Find the following derivatives. Simplify where appropriate.
Do not leave negative exponents in your final answer.
You do not have to simplify radicals or rationalize denominators.

a) Let $f(x) = \ln(5x^3 - x + 1)$. Find $f'(x)$.

b) Find $\frac{d}{dx}\left[\ln(x^3 + x^2)\right]$.

c) Find $D_x\left[\ln|3x + 7|\right]$.

d) Let $g(t) = \ln\left(|7 - 4t|^{10}\right)$. Find $g'(t)$.

e) Let $y = \ln(x^3) + (\ln x)^3$. Find $\frac{dy}{dx}$.

f) Find $\frac{d}{dw}\left[\frac{w^2 \ln w}{\ln\left(\frac{1}{w}\right)}\right]$. Hint: Simplify first!

g) Find $\frac{d}{dw}\left[\frac{w^2 \ln w}{1 + \ln w}\right]$.

2) Let $y = \ln\left[\frac{(x^4 + 1)^3\left(\sqrt{x}\right)}{(3x - 4)^5}\right]$. Find $\frac{dy}{dx}$.

Before performing any differentiation, apply appropriate laws of logarithms wherever they apply. You do not have to write your final answer as a single fraction.

3) Find $D_x\left(\ln|\sec x|\right)$. Based on your result, write the corresponding indefinite integral statement. We will discuss this further in Section 7.4.

4) We will find $D_\theta\left(\cos^7\theta\right)$ in two different ways.

a) Apply the Generalized Power Rule of Differentiation directly.

b) Use Logarithmic Differentiation. Apply appropriate laws of logarithms wherever they apply. Observe that your answers to a) and b) should be equivalent, at least where $\cos\theta \neq 0$.

5) We will find $D_x\left[\left(3x^2+2\right)^4\left(\sqrt{3x+5}\right)\right]$ in two different ways.

   a) Apply the Product Rule and the Generalized Power Rule of Differentiation directly. Simplify completely, and write your final answer as a single non-compound fraction. Do not leave negative exponents in your final answer.
   b) Use Logarithmic Differentiation. Apply appropriate laws of logarithms wherever they apply. For now, you do not have to write your final answer as a single fraction.
   c) Simplify your answer to part b) completely, and write your final answer as a single non-compound fraction. Compare with your answer to part a).

6) Let $f(x)=\ln(\ln x)$. Consider the graph of f in the usual xy-plane.

   a) What is the domain of f?
   b) Find $f'(x)$.
   c) Find $f''(x)$.
   d) What do f' and f'' tell us about the graph of f?
   e) Find an equation of the tangent line to the graph of f at the point $\left(e^2, \ln 2\right)$.

7) Use derivatives to explain why x increases (or "grows") faster than $\ln x$ does as x increases through the interval $(1, \infty)$.

8) Use Logarithmic Differentiation to prove the Power Rule of Differentiation: $D_x\left(x^n\right)=nx^{n-1}$, where n is an arbitrary real number. Assume $x \neq 0$.

SECTION 7.3: e^x

1) Find the following derivatives. Simplify where appropriate.
   Do not leave negative constant exponents in your final answer.
   You do not have to simplify radicals or rationalize denominators.
   a) Let $f(x) = e^{-8x}$. Find $f'(x)$.
   b) Find $\dfrac{d}{dx}\left(\sqrt{1+2x+3e^{4x}}\right)$.
   c) Find $D_x\left(e^{4x^3+x}\right)$.
   d) Let $g(t) = e^{e^{e^t}}$. Find $g'(t)$.
   e) Let $y = \dfrac{x^2 e^x}{e^x+1}$. Find $\dfrac{dy}{dx}$.
   f) Find $D_x\left(\dfrac{1}{e^x} + e^{\frac{1}{x}}\right)$.
   g) Find $D_x\left(e^{x\ln x}\right)$.
   h) Find $\dfrac{d}{d\theta}\left[e^{\ln(\sin\theta)} + \ln\left(e^{\cos\theta}\right)\right]$, where $\sin\theta > 0$. Hint: Simplify first!
   i) Find $D_x\left[\sec\left(e^x\right)\right]$.
   j) Let $h(r) = \tan^5\left(4e^{6r}\right)$. Find $h'(r)$.
   k) Find $\dfrac{d}{dx}\left[e^{3\csc(2x)+1}\right]$.
   l) Let $f(\theta) = \ln\left[\sin\left(e^{-\theta}\right)\right]$. Find $f'(\theta)$.
   m) Let $g(x) = e^{4x}\cot\left(\sqrt{x}\right)$. Find $g'(x)$.
   n) Find $D_x\left(e^{\pi}\right)$.

2) Consider the given equation $\ln\left(x^2 y\right)+2y^6-x^2=7+e^6$, where $x>0$. Assume that it "determines" an implicit differentiable function f such that $y=f(x)$.

a) Find $\frac{dy}{dx}$, also known as y'.

b) Verify that the point $P\left(e^3, e\right)$ lies on the graph of the given equation.

c) Evaluate $\left[\frac{dy}{dx}\right]_{\left(e^3, e\right)}$.

3) Consider the given equation $e^{xy}=\sec y$. Assume that it "determines" an implicit differentiable function f such that $y=f(x)$. Find $\frac{dy}{dx}$, also known as y'.

4) (Radioactive Decay). The amount remaining (in grams) of a radioactive substance t minutes after noon is given by $f(t)=ae^{-bt}$, where a and b are positive real constants. Show that the rate of decay of the substance is directly proportional to the amount of the substance that remains. (Note: a is the "initial" amount of the substance remaining at $t=0$.) You will come back to these ideas when you study Differential Equations.

5) **ADDITIONAL PROBLEM:** (Statistics). The standard normal probability density function in statistics is given by: $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Sketch the graph of $y=f(x)$ in the usual *xy*-plane.

- Find the domain of *f*.
- Comment on the signs of values of *f*.
- State whether f is even, odd, or neither, and incorporate any corresponding symmetry in your graph.
- Find and indicate on your graph any horizontal asymptotes (HAs), and justify them using limits.
- Locate any local maximum points, local minimum points, and inflection points.
- Find the intervals on which f is increasing / decreasing, and have your graph show that.
- Find the *x*-intervals on which the graph of $y=f(x)$ is concave up / concave down, and have your graph show that.

SECTION 7.4: INTEGRATION and LOG / EXP. FUNCTIONS

1) Evaluate the following indefinite integrals. You may use C, D, etc. as representing arbitrary constants.

   a) $\int \frac{1}{2x-3}\,dx$. Try using Guess-and-Check here.

   b) $\int \left(e^{7x} + \frac{1}{e^{7x}} \right) dx$. Try rewriting and using Guess-and-Check here.

   c) $\int \tan(3x)\,dx$

   d) $\int \cot\left(\frac{x}{5}\right) dx$

   e) $\int \frac{5x}{4x^2+3}\,dx$

   f) $\int x^2 e^{5x^3}\,dx$

   g) $\int 6\theta \sec\left(\theta^2 + e\right) d\theta$

   h) $\int \frac{x-2}{x^2-4x+5}\,dx$

   i) $\int \frac{(t+3)^2}{t}\,dt$

   j) $\int (\csc x + 4)^2\,dx$

   k) $\int \frac{\sin(\ln x)}{x}\,dx$

   l) $\int \frac{\pi e^{\sqrt{x}}}{7\sqrt{x}}\,dx$

   m) $\int \frac{\sec^2 x + 1}{x + \tan x}\,dx$

   n) $\int \frac{(e^x+1)^2}{e^x}\,dx$

   o) $\int \frac{e^x + e^{-x}}{\left(e^x - e^{-x}\right)^2}\,dx$

p) $\int \frac{e^x}{e^x + 1}\, dx$

q) $\int \frac{e^x}{\cos\left(3e^x - e\right)}\, dx$

r) $\int \frac{e^{\sin x}}{\sec x}\, dx$

s) $\int \frac{\cos^2 \theta}{\sin \theta}\, d\theta$

t) **ADDITIONAL PROBLEM:** $\int \frac{\csc\left(e^{-2x}\right)\cot\left(e^{-2x}\right)}{e^{2x}\left[1+\csc\left(e^{-2x}\right)\right]}\, dx$

2) We will consider the definite integral $\int_1^2 \frac{1}{x}\, dx$.

a) Use the Trapezoidal Rule to approximate $\int_1^2 \frac{1}{x}\, dx$. Use a regular partition with $n = 4$ subintervals. Round off your answer to three significant digits.

b) Evaluate $\int_1^2 \frac{1}{x}\, dx$ using the Fundamental Theorem of Calculus. Give an exact answer, and also approximate it to three significant digits. Compare with part a).

3) Show that $\int \ln x\, dx = x \ln x - x + C$ by showing that $D_x\left(x \ln x - x\right) = \ln x$. In Chapter 9, you will use Integration by Parts to perform the integration directly.

4) **ADDITIONAL PROBLEM:** $\int \csc x\, dx = \ln\left|\csc x - \cot x\right| + C$. Alternatively, $\int \csc x\, dx = -\ln\left|\csc x + \cot x\right| + C$. Show that $-\ln\left|\csc x + \cot x\right| = \ln\left|\csc x - \cot x\right|$.

5) (Differential equations and motion). The velocity of a particle traveling along a coordinate line is given by $v(t) = 4e^{2t} + 3e^{-2t}$ in feet per second, where t is measured in seconds. We are given the initial condition $s(0) = 4$ feet, meaning that the position of the particle at time $t = 0$ is 4 feet along the line. Find the position function rule $s(t)$ for the particle.

6) (Volume of a solid of revolution). The region R in the usual xy-plane is bounded by the graphs of $y = e^{-x^2}$, $x = 0$, $x = 1$, and $y = 0$. Find the volume of the solid generated if R is revolved about the y-axis. Give an exact answer and also an approximate answer rounded off to four significant digits. (Assume that distances and lengths are measured in meters.)

7) (Volume of a solid of revolution). In Section 6.3, Exercise 1, you should have found that the volume of the indicated solid was given by: $\int_0^{\sqrt{\pi/2}} 2\pi x \tan\left(x^2\right)\,dx$. Evaluate this integral, and give the volume using appropriate units. Give an exact answer and also an approximate answer rounded off to four significant digits. (Assume that distances and lengths are measured in meters.)

SECTION 7.5: BEYOND *e* – NONNATURAL BASES

1) Find the following derivatives. Simplify where appropriate.
   Do not leave negative constant exponents in your final answer.
   You do not have to simplify radicals or rationalize denominators.
   a) Find $D_x\left(x^e + e^e + e^x + 2^x - 2^{3x^4+x}\right)$.
   b) Find $D_x\left(\ln x + \log_2 x\right)$.
   c) Let $f(x) = \log\left[\left|x^7 - 4x^3 + 2\right|^4\right]$. Find $f'(x)$.
   d) Find $\dfrac{d}{dx}\left(x^\pi \pi^x\right)$.
   e) Let $g(t) = 3^{\sec(5t)}$. Find $g'(t)$.
   f) Let $h(r) = \log_6(\ln r)$. Find $h'(r)$.
   g) Find $D_x\left[(x+3)^{x^2}\right]$. Try to simplify your final answer.
   h) Find $D_x\left(x^{\tan x}\right)$. Try to simplify your final answer.

2) (Revisiting Exercise 1h). An alternative to Logarithmic Differentiation may be performed as follows. Disregarding domain issues, observe that:
   $x^{\tan x} = e^{\ln\left(x^{\tan x}\right)} = e^{(\tan x)(\ln x)}$. Find $D_x\left(x^{\tan x}\right)$ by finding $D_x\left[e^{(\tan x)(\ln x)}\right]$.

3) Evaluate the following indefinite integrals. You may use C, D, etc. as representing arbitrary constants.
   a) $\int \left(x^e + e^e + e^x + 2^x - 9\pi^x\right) dx$; b) $\int 7^{5x+3}\,dx$; c) $\int \dfrac{1}{x \log x}\,dx$

4) Evaluate $\displaystyle\int_0^1 \frac{10^x}{10^x + 1}\,dx$.