CHAPTER 1: REVIEW

TOPIC 1: FUNCTIONS

PART A: AN EXAMPLE OF A FUNCTION

Consider a function f whose rule is given by $f(x) = x^2$.
As a short cut, we often say, "the function $f(x) = x^2$."

Warning: $f(x)$ is referred to as "f **of** x" or "f **at** x." It does **not** mean "f **times** x."

x is the input (or argument) for f, and x^2 is the output or function value.

$$x \rightarrow \boxed{f} \rightarrow x^2$$

This function squares its input, and the result is its output.

Note: The rule for this function could have been given as: $f(u) = u^2$, for example.

Example:

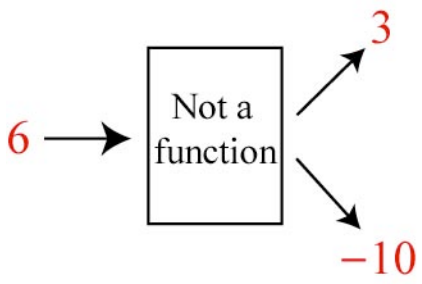
$$\begin{aligned} f(3) &= (3)^2 \\ &= 9 \end{aligned}$$

$$3 \rightarrow \boxed{f} \rightarrow 9$$

We can think of a function as a **calculator button**. In fact, your calculator should have a "squaring" button labeled x^2.

f is a function, because no “legal” input yields more than one output.

There is no function button on a calculator that ever outputs two or more values at the same time. The calculator never outputs, “I don’t know. The answer could be 3 or -10.”



Note: A function is a special type of relation. Relations that are not functions permit multiple outputs for a legal input.

PART B: POLYNOMIAL FUNCTIONS

A polynomial expression in x can be written in the form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 ,$$

where n is a nonnegative integer called the degree of the polynomial,
the a_i coefficients are typically real numbers, and the leading coefficient $a_n \neq 0$.

A polynomial function has a rule that can be written as:

$$f(x) = \text{polynomial in } x .$$

Example

$4x^3 - \frac{5}{2}x^2 + 1$ is a 3rd-degree polynomial in x with leading coefficient 4.

The rule $f(x) = 4x^3 - \frac{5}{2}x^2 + 1$ corresponds to a polynomial function f.

PART C: RATIONAL FUNCTIONS

A rational expression in x can be expressed in the form: $\dfrac{\text{polynomial in } x}{\text{nonzero polynomial in } x}$.

Examples: $\dfrac{1}{x}$, $\dfrac{5x^3-1}{x^2+7x-\sqrt{2}}$, x^7+x $\left(\text{which equals } \dfrac{x^7+x}{1}\right)$

Observe in the second example that irrational numbers such as $\sqrt{2}$ are permissible.

The last example correctly suggests that all polynomials are rational expressions.

A rational function has a rule that can be written as:

$$f(x) = \text{rational expression in } x.$$

PART D: ALGEBRAIC FUNCTIONS

An algebraic expression in x looks like a rational expression, except that radicals and exponents that are noninteger rational numbers $\left(\text{such as } \dfrac{5}{7}\right)$ are allowed even when x appears in a radicand or in a base (but not in an exponent).

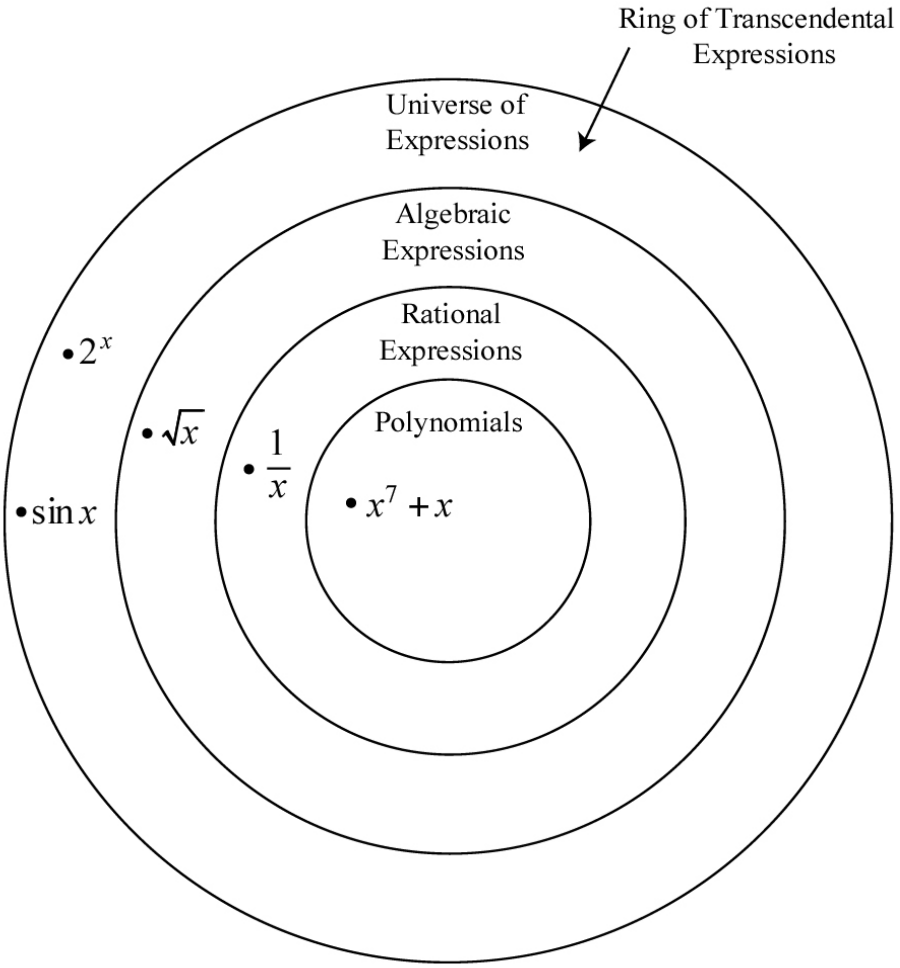
Examples: $\sqrt{x}$, $\dfrac{x^3+7x^{5/7}}{x-\sqrt[3]{x+5}+4}$

All rational expressions are algebraic.

An algebraic function has a rule that can be written as:

$$f(x) = \text{algebraic expression in } x.$$

Here’s a Venn diagram for standard symbolic mathematical expressions:



PART E: DOMAIN AND RANGE

The domain of a function f, abbreviated $\text{Dom}(f)$, is the set of all “legal” **inputs**.

The range of f is then the set of all resulting **outputs**.

Unless otherwise specified, we typically assume that the domain of a function is the set of **all real** input values that yield an output that is a **real** number.

This set is the implied (or natural) domain.

The implied domain of an algebraic function consists of all real numbers **except** those that:

1) lead to zero denominators $\left(\text{Think: } \frac{-}{0}\right)$, or

2) lead to negative radicands of even roots $\left(\text{Think: } \sqrt[\text{even}]{-}\right)$.

As we study more types of functions, the list of restrictions will grow.
For example, as we will see later, we also exclude real numbers that lead to

3) logs of nonpositive values $\left(\text{Think: } \log_b\left(\le 0\right)\right)$, or

4) arguments of trig functions that correspond to vertical asymptotes.

Word problems may imply other restrictions: nonnegativity, integer values, etc.

Example 1

$$f\left(x\right) = x^2$$

The implied domain of a polynomial function (such as this f) is **R**, the set of all real numbers. In interval form, **R** is $\left(-\infty, \infty\right)$. Its graph is the entire real number line:

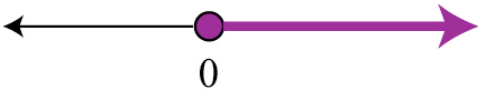
Warning: We use parentheses in the interval form, because ∞ ("infinity") and $-\infty$ ("negative infinity") are **not** real numbers and are therefore **excluded** from the set.

Note: It is debatable whether an expression like $\frac{x^2 + x}{x}$ is a polynomial. It simplifies to $x + 1$, but its domain excludes 0.

The resulting range of f is the set of all nonnegative real numbers $\mathbf{R}_{\geq 0}$ (i.e., all real numbers that are greater than or equal to 0), because every such number is the square of some real number.

> Warning: Squares of real numbers are never negative. This fact comes in very handy throughout math.

The graph of the range is:

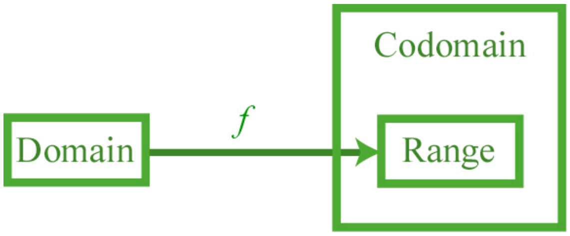


> The filled-in circle serves to **include** 0 in the range. We could also use a left bracket ("[") here instead of a filled-in circle; the bracket opens towards the shading. The graph helps us figure out the interval form

In interval form, the range is $[0, \infty)$. We have a bracket next to the 0, because 0 is **included** in the range.

In set-builder form, the range is: $\{y \in \mathbf{R} \mid y \geq 0\}$, or $\{y \in \mathbf{R} : y \geq 0\}$, which is read "the set of all real values y such that $y \geq 0$." Using y instead of x is more consistent with our graphing conventions, and it helps us avoid confusion with the domain. Note: $\in$ denotes set membership.

Technical Note: We say that f maps the domain **R** to the codomain **R**, or that f maps **R** to itself. Using notation, we write "$f : \mathbf{R} \rightarrow \mathbf{R}$." This is because f assigns a real number output (i.e., a member of the codomain) to **each** real number input in the domain. The range is a subset of the codomain. In fact, here, the range is a proper subset of the codomain, because not every real number in the codomain is assigned. In particular, the negative reals are not assigned.



Example 2

If $f(x)=\sqrt{x-3}$, find $\mathrm{Dom}(f)$, the domain of f.

Solution

$f(x)$ yields real outputs $\Leftrightarrow$ $x-3\geq 0$.
The solution set of this inequality is the domain of f.

Solve the inequality:

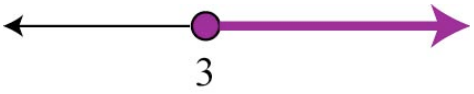
$$\begin{aligned} x-3&\geq 0\\ x&\geq 3\end{aligned}$$

The domain of f …

… in set-builder form is:

$\{x\in\mathbf{R}\,|\,x\geq 3\}$, or $\{x\in\mathbf{R}:\ x\geq 3\}$

… in graphical form is:



… in interval form is:

$[3,\infty)$

Note: If the rule for f had been given by $f(t)=\sqrt{t-3}$, we're still talking about the same function. In particular, the domain and the range are still the same. However, when writing the domain in set-builder form, it may be more appropriate to write $\{t\in\mathbf{R}\,|\,t\geq 3\}$, or $\{t\in\mathbf{R}:\ t\geq 3\}$. The notation is self-contained, so we still could have used *x*, especially if *x* is not used elsewhere in the problem.

Example 3

If $f(x)=\sqrt{3-x}$, find $\text{Dom}(f)$, the domain of f.

Solution

Solve the inequality: $3-x\geq 0$.

Method 1

$$3-x\geq 0$$

Subtract 3 from both sides.

$$-x\geq -3$$

Multiply or divide both sides by -1.

Warning: When we multiply or divide both sides of an inequality by the same negative real quantity, we must **reverse** the direction of the inequality symbol.

$$x\leq 3$$

Method 2

$$3-x\geq 0$$

Add x to both sides.

$$3\geq x$$

Switch the left side and the right side.

Warning: We must then **reverse** the direction of the inequality symbol.

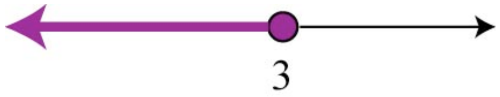
$$x\leq 3$$

The domain of f …

… in set-builder form is:

$$\left\{ x \in \mathbf{R} \,\middle|\, x \le 3 \right\}, \text{ or } \left\{ x \in \mathbf{R} : \; x \le 3 \right\}$$

… in graphical form is:



We could also use a right bracket ("]") here instead of a filled-in circle at 3.

… in interval form is:

$$\left(-\infty, 3 \right]$$

Example 4

If $f(x) = \dfrac{1}{\sqrt{x-3}}$, find $\text{Dom}(f)$, the domain of f.

Solution

We must forbid a zero denominator here, so instead of solving the weak inequality $x - 3 \geq 0$, as we did in Example 2, we must solve the strict inequality: $x - 3 > 0$.

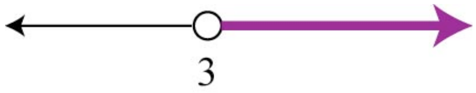
$$\begin{aligned} x - 3 &> 0 \\ x &> 3 \end{aligned}$$

The domain of f …

… in set-builder form is:

$$\{x \in \mathbf{R} \mid x > 3\}, \text{ or } \{x \in \mathbf{R} : x > 3\}$$

… in graphical form is:



The hollow circle serves to **exclude** 3 from the domain. We could also use a left parenthesis ("(") here instead of a hollow circle at 3; the parenthesis opens towards the shading.

… in interval form is:

$$(3, \infty)$$

We have a parenthesis next to the 3, because 3 is **excluded** from the domain.

Types of Intervals

$\left(5, 7\right)$ and $\left(3, \infty\right)$ are examples of open intervals, because they **exclude** their endpoints. $\left(5, 7\right)$ is a bounded interval, because it is trapped between two numbers.

$\left(3, \infty\right)$ is an unbounded interval.

$\left[5, 7\right]$ is an example of a closed interval, because it **includes** its endpoints, and it is bounded.

Example 5

If $f\left(x\right) = \sqrt[3]{x - 3}$, find $\text{Dom}\left(f\right)$, the domain of f.

Solution

The domain of f is **R**, because:

- $x - 3$ is a polynomial, and
- (Warning!) The taking of **odd** roots (such as cube roots) does **not** impose any new restrictions on the domain. Remember that the cube root of a negative real number is a negative real number. This is different from **even** roots (such as square roots); we do not permit even roots of negative numbers when we find a domain.

Example 6

If $f(x) = \dfrac{\sqrt{x+3}}{x-10}$, find $\mathrm{Dom}(f)$, the domain of f.

Solution

Because of the square root radical, we require:

$$\begin{aligned} x+3 &\geq 0 \\ x &\geq -3 \end{aligned}$$

Because we forbid zero denominators, we also require:

$$\begin{aligned} x-10 &\neq 0 \\ x &\neq 10 \end{aligned}$$

The domain of f …

… in set-builder form is:

$\left\{ x \in \mathbf{R} \,\middle|\, x \geq -3 \text{ and } x \neq 10 \right\}$, or
$\left\{ x \in \mathbf{R} : \; x \geq -3 \text{ and } x \neq 10 \right\}$

… in graphical form is:

We include -3 but exclude 10.

… in interval form is:

$$[-3, 10) \cup (10, \infty)$$

The union symbol, $\cup$, is used to separate intervals in the event that a number or numbers need to be skipped.

PART F: GRAPHS OF FUNCTIONS

The graph of f, or the graph of $y = f(x)$, in the standard *xy*-plane consists of all points [representing ordered pairs] of the form $(x,\ f(x))$, where *x* is in the domain of f.

In a sense: Graph of $f = \left\{ \left(x,\ f(x)\right) \;\middle|\; x \in \text{Dom}(f) \right\}$.

Here, as we typically assume, …

x is the independent variable, because it is the input variable.

y is the dependent variable, because it is the output variable.
Its value (the function value) typically "depends" on the value of the input *x*.

Technical Note: Even for a constant function f where, say, $f(x) = 3$, we refer to *y* as the **dependent** variable, even though (informally speaking) the value of *y* is always 3 and does not really "depend" on the value of *x* in the traditional sense. "3" does technically represent a function of *x*.

Example

Consider the graph of $\underbrace{f(x)}_{\text{or } y} = \sqrt{x}$.

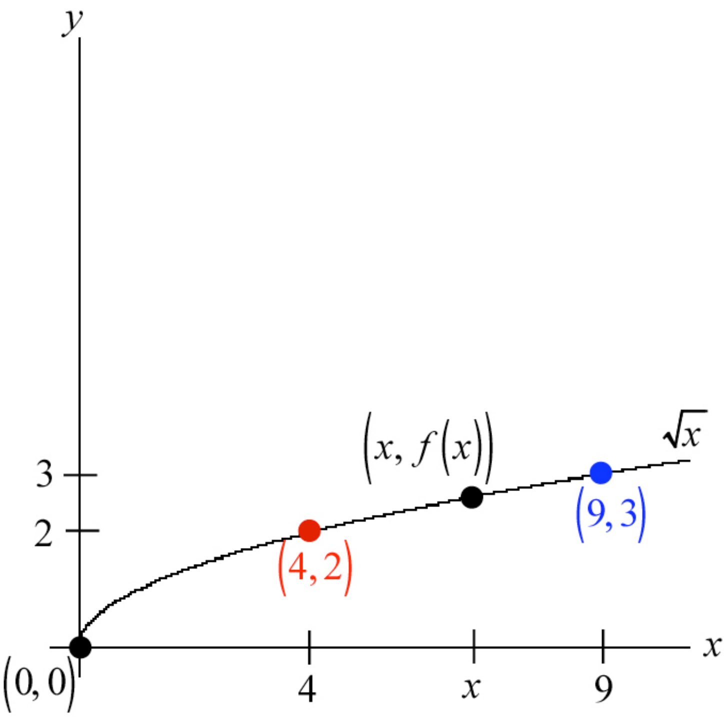
Note: *y* and $f(x)$ are interchangeable here.

The domain of f is $[0, \infty)$, the nonnegative reals.

The input $x = 9$ corresponds to the point $(9,\ f(9))$, or $(9, 3)$, on the graph.

The input $x = -9$ is not "legal" in that it does not lie in the domain of f.
There is no corresponding point on the graph.

Here is the graph of f (with a few points indicated):



Warning: Clearly indicate any endpoints on a graph, such as the origin here.

The lack of a right endpoint on our graph implies that the graph extends beyond the edge of our figure. We want to draw graphs in such a way that these extensions are "as one might expect."

Technical Note: The x between the 4 and the 9 on the x-axis represents a generic x-coordinate in the domain. Some may prefer to use x_0 (called "x sub zero" or the more British "x naught") to represent a particular or fixed x-coordinate. In these notes, we may sometimes be a bit sloppy and use x when perhaps x_0 would be more appropriate.

The Point-Plotting Method for Graphing a Function

This is when you choose a bunch of x values in the domain, find their corresponding $f(x)$ values, plot the corresponding points $(x, f(x))$, and connect the dots "nicely." We will generally avoid this method by appealing to properties of the function, but it is available as a last resort!

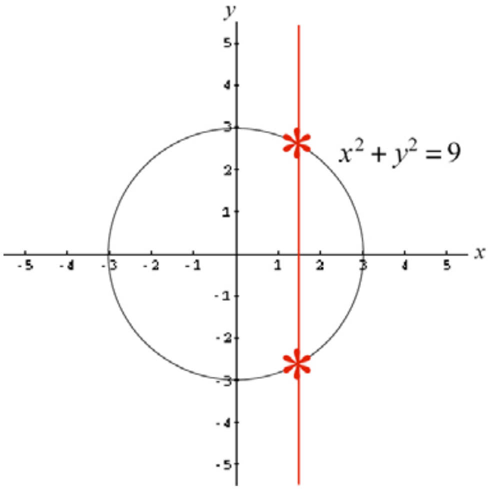
PART G: THE VERTICAL LINE TEST (VLT)

An equation in *x* and *y* describes *y* as a function of *x*, and we can then say $y = f(x)$ $\Leftrightarrow$

Its graph passes the VLT in the standard *xy*-plane, meaning that there is **no** vertical line that intersects the graph more than once (i.e., there is no input *x* that yields more than one output *y*).

Example

The graph of $x^2 + y^2 = 9$ is a circle of radius 3 centered at the origin. It fails the VLT, because there exists a vertical line that intersects the graph more than once. For example, we can take the red line below:



Therefore, $x^2 + y^2 = 9$ does **not** describe *y* as a function of *x*.

This is also evident if we solve $x^2 + y^2 = 9$ for *y*:

$$\begin{aligned} x^2 + y^2 &= 9 \\ y^2 &= 9 - x^2 \\ y &= \pm\sqrt{9 - x^2} \end{aligned}$$

Any input value *x* in the interval $(-3, 3)$ yields two *y* different outputs.

PART H: DOMAIN AND RANGE FROM GRAPHS

The domain of f, which is the set of all legal **inputs**, is the set of all ***x*-coordinates** picked up by the graph of f. (We assume x is the independent variable.)

(Think of crushing, or "projecting," the graph of f onto the x-axis.)

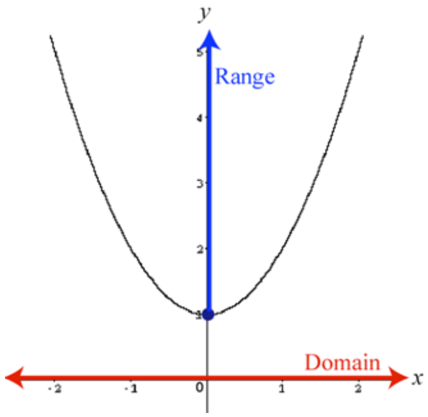
The range of f, which is the set of all resulting **outputs**, is the set of all ***y*-coordinates** picked up by the graph of f. (We assume y is the dependent variable.)

(Think of crushing, or "projecting," the graph of f onto the y-axis.)

$$\boxed{\begin{array}{c}\text{Domain}\\ \text{Think: } x\end{array}} \to \boxed{f} \to \boxed{\begin{array}{c}\text{Range}\\ \text{Think: } y\end{array}}$$

Example

The graph of $f(x) = x^2 + 1$ is given below.
Find the domain and the range of f.



The domain of f is **R**, or $(-\infty, \infty)$.

The range of f is $[1, \infty)$.

In Calculus: As you learn more about how to graph functions, you will be better able to determine the ranges of functions based on these principles.

PART I: FUNCTIONS THAT ARE EVEN / ODD / NEITHER; SYMMETRY

A function f is **even** $\Leftrightarrow$ $f(-x) = f(x)$ $\underbrace{\forall x \in \mathrm{Dom}(f)}_{\text{for every } x \text{ in the domain of } f}$

$\Leftrightarrow$ The graph of $y = f(x)$ is symmetric about the ***y*** **- axis**.

Example

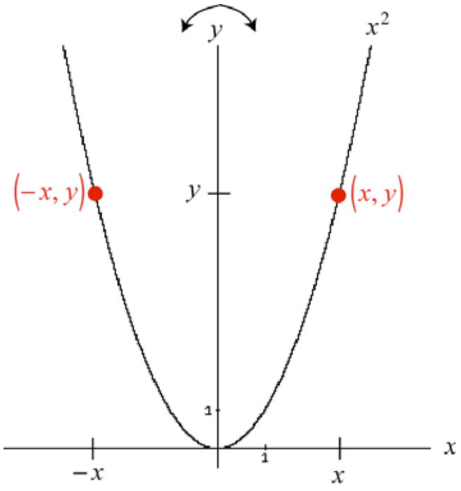
If $f(x) = x^2$, then f is even, because $\forall x \in \mathbf{R}$,

$$\begin{aligned} f(-x) &= (-x)^2 \\ &= x^2 \\ &= f(x) \end{aligned}$$

The "bowl" graph of $\underbrace{f(x)}_{y} = x^2$ below is <u>symmetric about the *y*-axis</u>.

This means that the parts of the graph to the right and to the left of the *y*-axis are mirror images (or <u>reflections</u>) of each other.

More formally, the point (x, y) lies on the graph if and only if the point $(-x, y)$ does.



The term “even function” may have come from the following fact:

If $f(x) = x^n$, where n is an even integer, then f is an even function.

These are the functions for: $\ldots,\ x^{-4},\ x^{-2},\ x^{0},\ x^{2},\ x^{4},\ \ldots.$

The graph for the x^2 function on the previous page is called a parabola. However, the graphs for x^4, x^6, etc. are **not** parabolas.

The reciprocal of a nonzero even function is even.

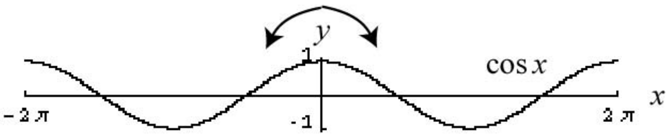
Example

The functions for both x^2 and x^{-2} $\left(\text{which equals } \frac{1}{x^2}\right)$ are even.

Example

The graph of the even function f, where $f(x) = \cos x$, is below.

Its reciprocal function, for $\sec x$, is also even.



A function f is **odd** $\Leftrightarrow$ $f(-x) = -f(x) \quad \forall x \in \mathrm{Dom}(f)$

$\Leftrightarrow$ The graph of $y = f(x)$ is symmetric about the **origin**.

Example

If $f(x) = x^3$, then f is odd, because $\forall x \in \mathbf{R}$,

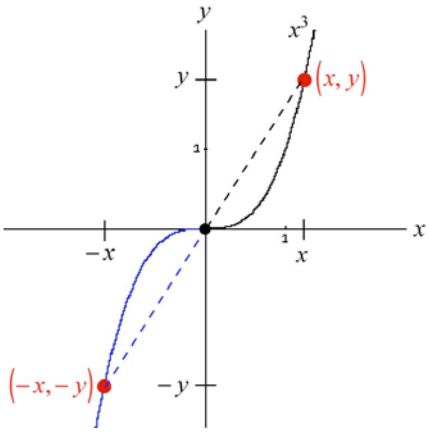
$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x) \end{aligned}$$

The "snake" graph of $\underbrace{f(x)}_{y} = x^3$ below is symmetric about the origin.

This means that, if the entire graph is rotated 180° about the origin, we obtain it again.

If the graph has a *y*-intercept, it must be at the origin.

More formally, the point (x, y) lies on the graph if and only if the point $(-x, -y)$ does. In the graph below, imagine rotating the black dashed line segment 180° about the origin until it coincides with the blue dashed line segment.



The term "odd function" may have come from the following fact:

If $f(x) = x^n$, where n is an odd integer, then f is an odd function.

These are the functions for: $\ldots,\ x^{-3},\ x^{-1},\ x^{1},\ x^{3},\ x^{5},\ \ldots.$

The reciprocal of a nonzero odd function is odd.

Example

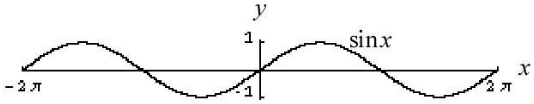
The functions for both x^1 (which equals x) and x^{-1} $\left(\text{which equals } \dfrac{1}{x}\right)$ are odd.

Example

The graph of the odd function f, where $f(x) = \sin x$, is below.

Its reciprocal function, for $\csc x$, is also odd.

Note: $\tan x$ and $\cot x$ are also odd.



Note: The zero function (over various domains that are symmetric about 0) is the **only** function that is both even **and** odd. (Can you show this?)

Note: Many functions are **neither** even nor odd.

PART J: ARITHMETIC COMBINATIONS OF FUNCTIONS

Let f and g be functions. If their domains overlap, then the overlap is the domain of the following functions, with one exception (*):

$f+g$, where $(f+g)(x)=f(x)+g(x)$

$f-g$, where $(f-g)(x)=f(x)-g(x)$

fg, where $(fg)(x)=f(x)g(x)$

$\frac{f}{g}$, where $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

(*) Values of x that make $g(x)=0$ must be excluded from the domain of $\frac{f}{g}$.

PART K: COMPOSITIONS OF FUNCTIONS

These arise when we apply a **sequence** of functions.

Let f and g be functions. The composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

Its domain is $\{x \in \mathbf{R} \mid x \in \text{Dom}(g) \text{ and } g(x) \in \text{Dom}(f)\}$.

The following input-output model may help:

$$x \rightarrow \underbrace{\boxed{g} \rightarrow g(x) \rightarrow \boxed{f}}_{\boxed{f \circ g}} \rightarrow f(g(x))$$

Think of $f \circ g$ as a "merged" function.

Warning: The function $f \circ g$ applies g first and then f. Think of pressing a g button on a calculator followed by an f button.

Warning: $f \circ g$ **may or may not** represent the same function as $g \circ f$ (in which f is applied first). Composition of functions is not commutative the way that, say, addition is.

Think About It: Try to think of examples where $f \circ g$ and $g \circ f$ represent the same function.

Example

Find component functions f and g such that $\left(f \circ g\right)\left(x\right) = \sqrt{3x+1}$.
We want to "decompose" $f \circ g$.

Neither f nor g may be the identity function.

(For example, you may not use: $g\left(x\right) = x$ and $f\left(x\right) = \sqrt{3x+1}$; this would not truly be a decomposition.)

Solution

We need $f\left(g\left(x\right)\right) = \sqrt{3x+1}$.

We can think of f and g as buttons we are designing on a calculator. We need to set up f and g so that, if x is an initial input in the domain of $f \circ g$, and if the g button and then the f button are pressed, then the output is $\sqrt{3x+1}$.

Here's the input-output model:

$$x \;\rightarrow\; \underbrace{\boxed{\begin{array}{c} g: \\ g\left(x\right)=? \end{array}} \;\rightarrow\; u=? \;\rightarrow\; \boxed{\begin{array}{c} f: \\ f\left(u\right)=?? \end{array}}}_{\boxed{f\circ g}} \;\rightarrow\; f\left(g\left(x\right)\right)=\sqrt{3x+1}$$

A common strategy is to let $g\left(x\right)$, or u, be an "inside" expression (for example, an argument, a radicand, an exponent, a base of a power, or a denominator).

Here, let's let $g\left(x\right) = 3x+1$.

We then need f to apply the square root operation.

We will let $f(u)=\sqrt{u}$. The use of u is more helpful in Calculus, but $f(x)=\sqrt{x}$ is also acceptable. However, $f(u)=\sqrt{x}$ is not acceptable.

<u>Possible Answer</u>: $f(u)=\sqrt{u}$ and $\underbrace{g(x)}_{=u}=3x+1$.

We can check this to see that $f(g(x))=\sqrt{3x+1}$.

<u>Note</u>: Here's the complete input-output model:

$$x \to \underbrace{\boxed{\begin{matrix} g: \\ g(x)=3x+1 \end{matrix}} \to u=3x+1 \to \boxed{\begin{matrix} f: \\ f(u)=\sqrt{u} \end{matrix}}}_{\boxed{f\circ g}} \to f(g(x))=\sqrt{3x+1}$$

<u>Note</u>: There are infinitely many other possibilities. For example, we could let $g(x)=3x$ and let $f(u)=\sqrt{u+1}$.

$$x \to \underbrace{\boxed{\begin{matrix} g: \\ g(x)=3x \end{matrix}} \to u=3x \to \boxed{\begin{matrix} f: \\ f(u)=\sqrt{u+1} \end{matrix}}}_{\boxed{f\circ g}} \to f(g(x))=\sqrt{3x+1}$$

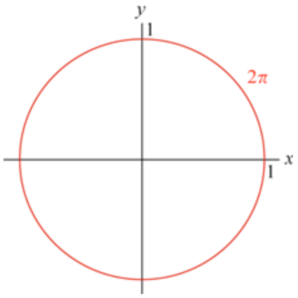
<u>In Calculus</u>: This concept will be critical to the Chain Rule of Differentiation and the *u*-substitution technique of integration.

TOPIC 2: TRIGONOMETRY I

PART A: ANGLE MEASURES

Radian measure is more “mathematically natural” than degree measure, and it is typically assumed in calculus. In fact, radian measure is assumed if there are no units present.

There are 2π radians in a full (counterclockwise) revolution, because the entire unit circle (which has circumference 2π) is intercepted exactly once by such an angle.



There are 360° (360 degrees) in a full (counterclockwise) revolution.
This is something of a cultural artifact; ancient Babylonians operated on a base-60 number system.

2π radians is equivalent to 360°. Therefore, π radians is equivalent to 180°. Either relationship may be used to construct conversion factors.

In any unit conversion, we effectively multiply by 1 in such a way that the old unit is canceled out.

Example

Convert 45° into radians.

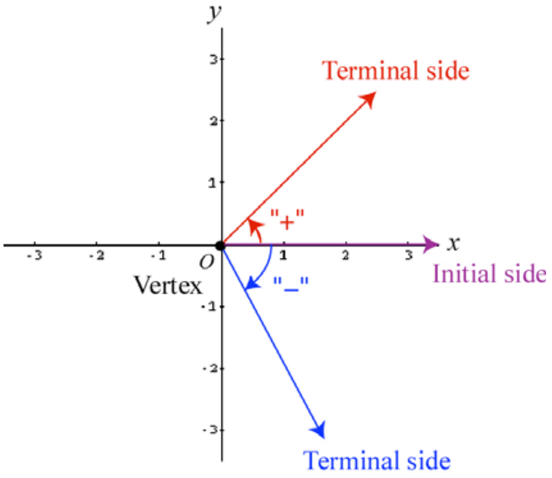
Solution

$$45^\circ = \left(45^\circ\right)\overbrace{\left(\frac{\pi\left[\text{rad}\right]}{180^\circ}\right)}^{=1} = \left(\cancel{45^\circ}\right)\left(\frac{\pi\left[\text{rad}\right]}{\cancel{180^\circ}_{\;4}}\right) = \frac{\boldsymbol{\pi}}{\mathbf{4}}\left[\textbf{rad}\right]$$

PART B: QUADRANTS AND QUADRANTAL ANGLES

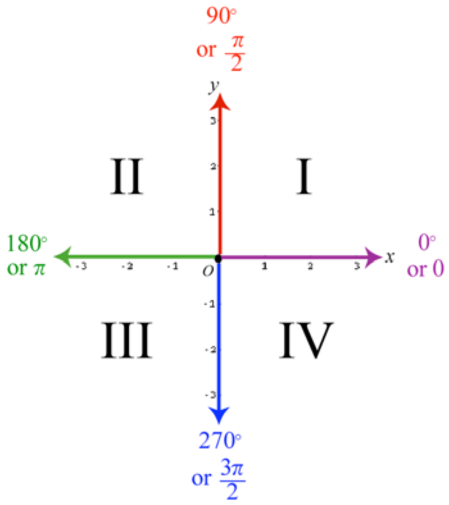
The *x*- and *y*-axes divide the *xy*-plane into 4 quadrants.
Quadrant I is the upper right quadrant; the others are numbered in counterclockwise order.

A standard angle in standard position has the positive [really, nonnegative] *x*-axis as its initial side and the origin as its vertex. We say that the angle lies in the quadrant that its terminal side shoots through. For example, in the figure below, the positive standard angle with the red terminal side is a Quadrant I angle:



A standard angle whose terminal side lies on the *x*- or *y*-axis is called a quadrantal angle. Quadrantal angles correspond to "integer multiples" of 90° or $\frac{\pi}{2}$ radians.

The quadrants and some quadrantal angles:
(For convenience, we may label a standard angle by labeling its terminal side.)



PART C: COTERMINAL ANGLES

Standard angles that share the same terminal side are called coterminal angles. They differ by an integer number of full revolutions counterclockwise or clockwise.

If the angle θ is measured in **radians**, then its coterminal angles are of the form: $\theta + 2\pi n$, where *n* is any integer.

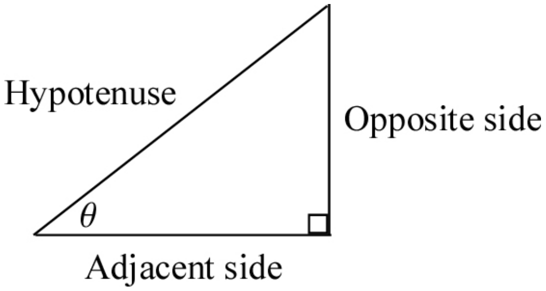
If the angle θ is measured in **degrees**, then its coterminal angles are of the form: $\theta + 360n^\circ$, where *n* is any integer.

Note: Since *n* could be negative, the "+" sign is sufficient in the above forms, as opposed to "$\pm$."

PART D: TRIG FUNCTIONS: THE RIGHT TRIANGLE APPROACH

The Setup

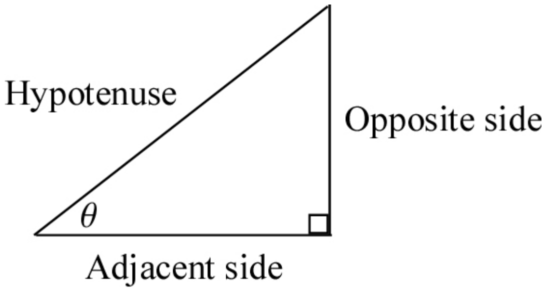
The acute angles of a right triangle are complementary. Consider such an angle, θ. Relative to θ, we may label the sides as follows:



The hypotenuse always faces the right angle, and it is always the longest side.

The other two sides are the legs. The opposite side (relative to θ) faces the θ angle. The other leg is the adjacent side (relative to θ).

Defining the Six Basic Trig Functions (where θ is acute)



The Ancient Curse (or "How to Define Trig Functions")

SOH-CAH-TOA

$$\text{Sine } \theta = \sin\theta = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\text{Cosine } \theta = \cos\theta = \frac{\text{Adj.}}{\text{Hyp.}}$$

$$\text{Tangent } \theta = \tan\theta = \frac{\text{Opp.}}{\text{Adj.}}$$

Reciprocal Identities (or "How to Define More Trig Functions")

$$\text{Cosecant } \theta = \csc\theta = \frac{1}{\sin\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Opp.}} \right)$$

$$\text{Secant } \theta = \sec\theta = \frac{1}{\cos\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Adj.}} \right)$$

$$\text{Cotangent } \theta = \cot\theta = \frac{1}{\tan\theta} \quad \left(= \frac{\text{Adj.}}{\text{Opp.}} \right)$$

Warning: Remember that the reciprocal of $\sin\theta$ is $\csc\theta$, not $\sec\theta$.

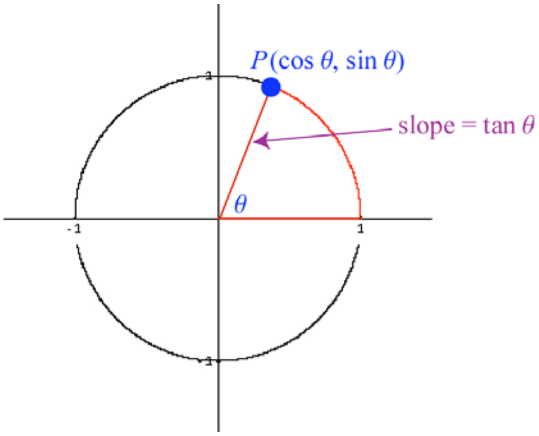
Note: We typically treat "0" and "undefined" as reciprocals when we are dealing with trig functions. Your algebra teacher will not want to hear this, though!

PART E: TRIG FUNCTIONS: THE UNIT CIRCLE APPROACH

The Setup

Consider a standard angle θ measured in radians (or, equivalently, let θ represent a real number).

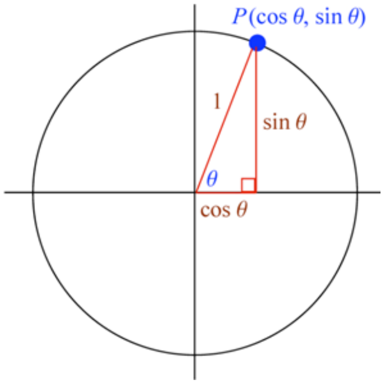
The point $P\left(\cos\theta, \sin\theta\right)$ is the intersection point between the terminal side of the angle and the unit circle. The slope of the terminal side is, in fact, $\tan\theta$.



Note: The intercepted arc along the circle (in red) has arc length θ.

The figure below demonstrates how this is consistent with the SOH-CAH-TOA (or Right Triangle) approach. Observe:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{rise}}{\text{run}} = \text{slope of terminal side}$$



“THE Table”

We will use our knowledge of the 30°-60°-90° and 45°-45°-90° special triangles to construct THE Table below. The unit circle approach is used to find the trig values for quadrantal angles such as 0° and 90°.

Key Angles θ: Degrees, (Radians)	$\sin\theta$	$\cos\theta$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	Intersection Point $P(\cos\theta, \sin\theta)$
0°, (0)	$\frac{\sqrt{0}}{2} = \mathbf{0}$	$\mathbf{1}$	$\frac{0}{1} = \mathbf{0}$	$(1, 0)$
30°, $\left(\frac{\pi}{6}\right)$	$\frac{\sqrt{1}}{2} = \mathbf{\frac{1}{2}}$	$\mathbf{\frac{\sqrt{3}}{2}}$	$\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \mathbf{\frac{\sqrt{3}}{3}}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
45°, $\left(\frac{\pi}{4}\right)$	$\frac{\sqrt{2}}{2} = \mathbf{\frac{\sqrt{2}}{2}}$	$\mathbf{\frac{\sqrt{2}}{2}}$	$\frac{\sqrt{2}/2}{\sqrt{2}/2} = \mathbf{1}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
60°, $\left(\frac{\pi}{3}\right)$	$\frac{\sqrt{3}}{2} = \mathbf{\frac{\sqrt{3}}{2}}$	$\mathbf{\frac{1}{2}}$	$\frac{\sqrt{3}/2}{1/2} = \mathbf{\sqrt{3}}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
90°, $\left(\frac{\pi}{2}\right)$	$\frac{\sqrt{4}}{2} = \mathbf{1}$	$\mathbf{0}$	$\frac{1}{0}$ is **undefined**	$(0, 1)$

Warning: $\frac{\pi}{5}$ is not a “special” angle.

Warning: Always make sure what mode your calculator is in (DEG vs. RAD) whenever you evaluate trig functions such as sin, cos, and tan.

The values for the reciprocal functions, $\csc\theta$, $\sec\theta$, and $\cot\theta$, are then readily found. Remember that it is sometimes better to take a trig value where the denominator is **not** rationalized before taking its reciprocal.

For example, because $\tan 30^\circ = \frac{1}{\sqrt{3}}$, we know immediately that

$\cot 30^\circ = \sqrt{3}$.

Observe:

- The pattern in the "sin" column

  Technical Note: An explanation for this pattern appears in the Sept. 2004 issue of the College Mathematics Journal (p.302).

- The fact that the "sin" column is flipped (or reversed) to form the "cos" column. This is due to the Cofunction Identities (or the Pythagorean Identities).

- As θ increases from 0° to 90° (i.e., from 0 to $\frac{\pi}{2}$ radians),

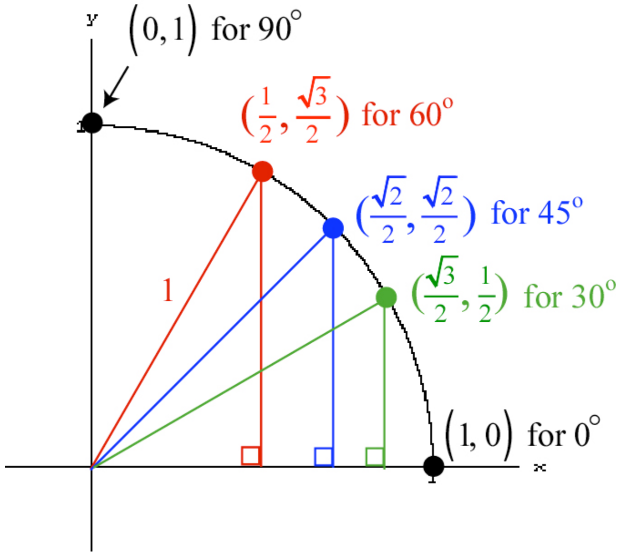
  - $\sin\theta$ (the *y*-coordinate of *P*) increases from 0 to 1.

    Note: This is more obvious using the Unit Circle approach instead of the Right Triangle approach.

  - $\cos\theta$ (the *x*-coordinate of *P*) decreases from 1 to 0.

  - $\tan\theta$ (the slope of the terminal side) starts at 0, increases, and approaches ∞.

- Here is the "Big Picture." Remember that each intersection point is of the form $P(\cos\theta, \sin\theta)$.



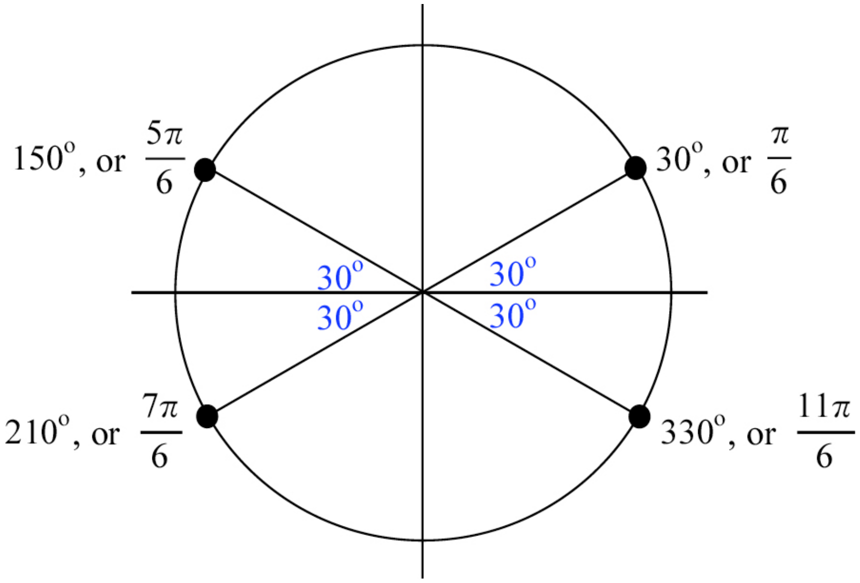
PART F: EXTENDING FROM QUADRANT I TO OTHER QUADRANTS

Reference angles

The reference angle for a non-quadrantal standard angle is the acute angle that its terminal side makes with the *x*-axis.

Brothers (the author's term) are angles that have the same reference angle.

For example, the angles below are brothers; they all have the same reference angle, namely 30°, or $\frac{\pi}{6}$ radians.



Brothers include coterminal "twins." For example, -30° (or $-\frac{\pi}{6}$ radians) is a "twin" for the 330° (or $\frac{11\pi}{6}$ radian) angle.

We will extend the following patterns for the $\frac{\pi}{6}$ brothers to other families of radian measures:

Quadrant II: $\frac{5\pi}{6}$; observe that 5 is 1 less than 6.
Quadrant III: $\frac{7\pi}{6}$; observe that 7 is 1 more than 6.
Quadrant IV: $\frac{11\pi}{6}$; observe that 11 is 1 less than twice 6.

Here are the "famous" positive brothers of the $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ angles:

We've already seen the situation with $\frac{\pi}{6}$:
(The boxes correspond to Quadrants.)

$\frac{5\pi}{6}$	$\frac{\pi}{6}$
$\frac{7\pi}{6}$	$\frac{11\pi}{6}$

Now, $\frac{\pi}{4}$:

$\frac{3\pi}{4}$	$\frac{\pi}{4}$
$\frac{5\pi}{4}$	$\frac{7\pi}{4}$

Now, $\frac{\pi}{3}$:

$\frac{2\pi}{3}$	$\frac{\pi}{3}$
$\frac{4\pi}{3}$	$\frac{5\pi}{3}$

Why is it useful to deal with brothers?

Coterminal "twins" have the same trig values, including the signs.

Brothers have the same basic trig values up to (i.e., except maybe for) the signs.

In other words, the basic trig values between two brothers are the same in magnitude, or absolute value.

How do signs of trig values differ between quadrants?

Remember that reciprocal values have the same sign.

"ASTC"

Think: "**A**ll **S**tudents **T**ake **C**alculus"

Start in Quadrant I and progress counterclockwise through the Quadrants:

S	A
T	C

All of the six basic trig functions are **positive** in Quadrant I.
(They are all positive for acute angles.)
Sin and its reciprocal, Csc, are positive in Quadrant II.
(The other four functions are negative.)
Tan and its reciprocal, Cot, are positive in Quadrant III.
Cos and its reciprocal, Sec, are positive in Quadrant IV.

Example

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}, \text{ because } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \text{ and } \frac{7\pi}{6} \text{ is in Quadrant III.}$$

TOPIC 3: TRIGONOMETRY II

PART A: FUNDAMENTAL TRIG IDENTITIES (IDs)

Memorize these in both "directions" (i.e., left-to-right and right-to-left).

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sin x = \frac{1}{\csc x}$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x = \frac{1}{\sec x}$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

Warning: Remember that the reciprocal of $\sin x$ is $\csc x$, not $\sec x$.

Note: We typically treat "0" and "undefined" as reciprocals when we are dealing with trig functions. Your algebra teacher will not want to hear this, though!

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ 1 + \cot^2 x &= \csc^2 x \\ \tan^2 x + 1 &= \sec^2 x \end{aligned}$$

Tip: The 2nd and 3rd IDs can be obtained by dividing both sides of the 1st ID by $\sin^2 x$ and $\cos^2 x$, respectively.

Tip: The squares of $\csc x$ and $\sec x$, which have the "Up-U, Down-U" graphs, are all alone on the right sides of the last two IDs. They can never be 0 in value. (Why is that? Look at the left sides.)

Cofunction Identities

If x is measured in radians, then:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

We have analogous relationships for tan and cot, and for sec and csc; remember that they are sometimes undefined.

Think: Cofunctions of complementary angles are equal.

Even / Odd (or Negative Angle) Identities

Among the six basic trig functions, cos (and its reciprocal, sec) are even:

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x, \text{ when both sides are defined}$$

However, the other four (sin and csc, tan and cot) are odd:

$$\sin(-x) = -\sin x$$
$$\csc(-x) = -\csc x, \text{ when both sides are defined}$$

$$\tan(-x) = -\tan x, \text{ when both sides are defined}$$
$$\cot(-x) = -\cot x, \text{ when both sides are defined}$$

Note: If f is an even function (such as cos), then the graph of $y = f(x)$ is symmetric about the y-axis.

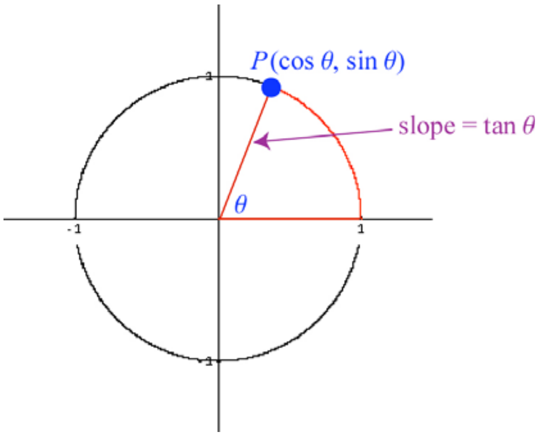
Note: If f is an odd function (such as sin), then the graph of $y = f(x)$ is symmetric about the origin.

PART B: DOMAINS AND RANGES OF THE SIX BASIC TRIG FUNCTIONS

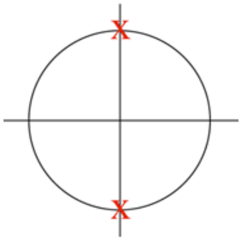
$f(x)$	Domain	Range
$\sin x$	$(-\infty, \infty)$	$[-1, 1]$
$\cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	Set-builder form: $\left\{ x \in \mathbf{R} \,\middle|\, x \neq \frac{\pi}{2} + \pi n \text{ for all integers } n \right\}$	$(-\infty, \infty)$
$\csc x$	Set-builder form: $\left\{ x \in \mathbf{R} \,\middle|\, x \neq \pi n \text{ for all integers } n \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec x$	Set-builder form: $\left\{ x \in \mathbf{R} \,\middle|\, x \neq \frac{\pi}{2} + \pi n \text{ for all integers } n \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	Set-builder form: $\left\{ x \in \mathbf{R} \,\middle|\, x \neq \pi n \text{ for all integers } n \right\}$	$(-\infty, \infty)$

COMMENTS

- The unit circle approach explains the domain and range for sin and cos, as well as the range for tan (since any real number can be a slope).

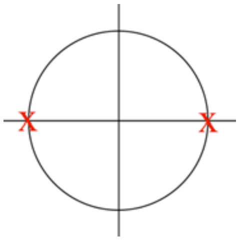


- Domain for tan: The "**X**"s on the unit circle below correspond to an undefined slope. Therefore, the corresponding real numbers (i.e., the corresponding angle measures in radians) are excluded from the domain.



- Domain for tan and sec: The "**X**"s on the unit circle above also correspond to a cosine value of 0. By the Quotient ID for tan $\left(\tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$ and the Reciprocal ID for sec $\left(\sec\theta = \dfrac{1}{\cos\theta}\right)$, we exclude the corresponding radian measures from the domains of both functions.

- Domain for cot and csc: The "**X**"s on the unit circle below correspond to a sine value of 0. By the Quotient ID for cot $\left(\cot\theta = \dfrac{\cos\theta}{\sin\theta}\right)$ and the Reciprocal ID for csc $\left(\csc\theta = \dfrac{1}{\sin\theta}\right)$, we exclude the corresponding radian measures from the domains of both functions.



- Range for csc and sec: We turn "inside out" the range for both sin and cos, which is $[-1,1]$.

- Range for cot: This is explained by the fact that the range for tan is $(-\infty,\infty)$ and the Reciprocal ID for cot: $\left(\cot\theta = \dfrac{1}{\tan\theta}\right)$. Cot is 0 in value when tan is undefined.

PART C: GRAPHS OF THE SIX BASIC TRIG FUNCTIONS

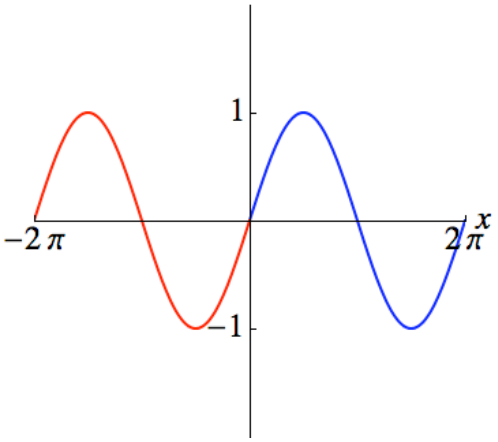
- These functions are periodic, so their graphs can be decomposed into cycles that repeat like wallpaper patterns. The period for tan and cot is π; it is 2π for the others.

- A vertical asymptote (VA) is a vertical line that a graph approaches. (This idea will be made more precise in Calculus.) VAs in the graph of a basic trig function correspond to exclusions from the domain. They are graphed as dashed lines.

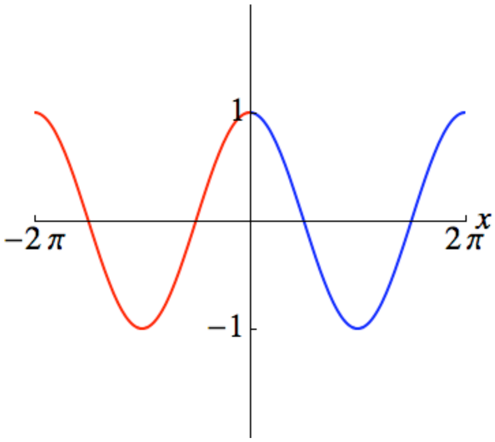
- Remember that the domain of a function f corresponds to the x-coordinates picked up by the graph of $y = f\left(x\right)$, and the range corresponds to the y-coordinates.

- Remember that cos and sec are the only even functions among the six, so their graphs are symmetric about the y-axis. The other four are odd, so their graphs are symmetric about the origin.

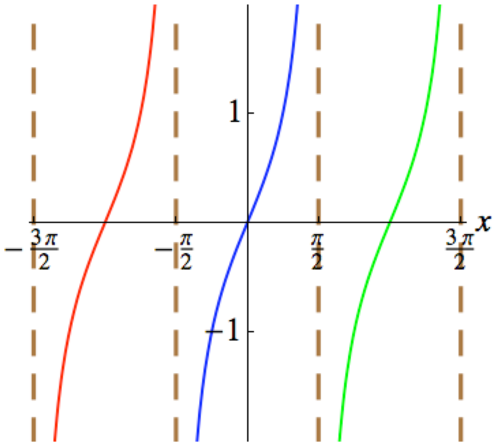
$y = \sin x$



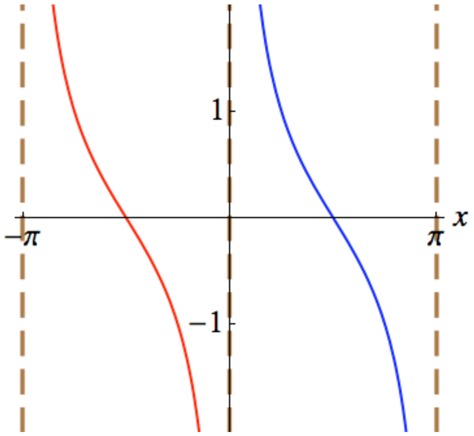
$y = \cos x$



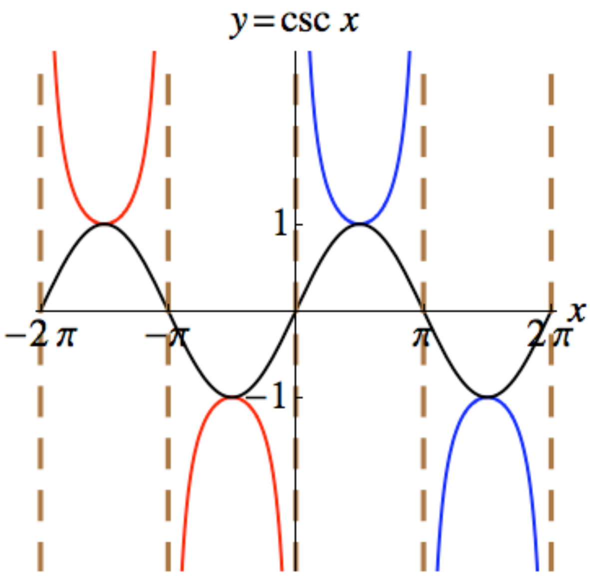
$y = \tan x$

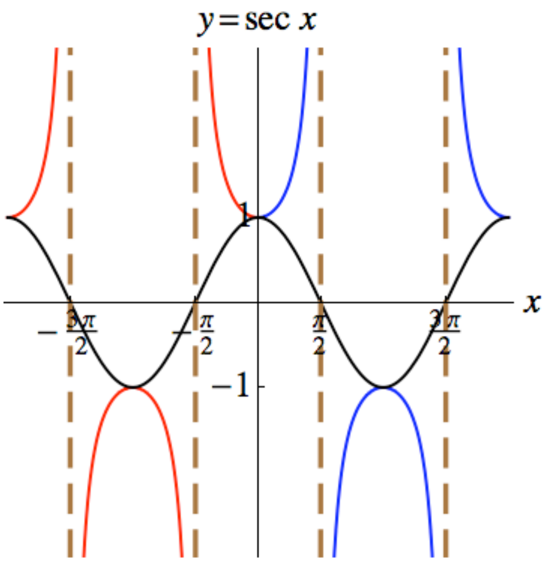


$y = \cot x$



- We use the graphs of $y = \sin x$ and $y = \cos x$ (in black in the figures below) as guide graphs to help us graph $y = \csc x$ and $y = \sec x$.





Relationships between the graphs of $y = \csc x$ and $y = \sin x$ (and between the graphs of $y = \sec x$ and $y = \cos x$):

- •• The VAs in the graph of $y = \csc x$ are drawn through the x-intercepts of the graph of $y = \sin x$. This is because $\csc x$ is undefined $\Leftrightarrow$ $\sin x = 0$.

- •• The reciprocals of 1 and -1 are themselves, so $\csc x$ and $\sin x$ take on each of those values simultaneously. This explains how their graphs intersect.

- •• Because $\sin$ and $\csc$ are reciprocal trig functions, we know that, between the VAs in the graph of $y = \csc x$, they share the same sign, and one increases $\Leftrightarrow$ the other decreases.

PART D: SOLVING TRIG EQUATIONS

Example

Solve: $2\sin\left(4x\right) = -\sqrt{3}$

Solution

$$2\sin\left(4x\right) = -\sqrt{3}$$

Isolate the sin expression.

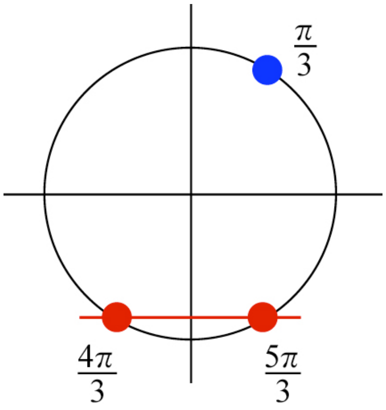
$$\sin\underbrace{\left(4x\right)}_{=\theta} = -\frac{\sqrt{3}}{2}$$

Substitution: Let $\theta = 4x$.

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

We will now solve this equation for θ.

Observe that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so $\frac{\pi}{3}$ will be the reference angle for our solutions for θ. Since $-\frac{\sqrt{3}}{2}$ is a negative sin value, we want brothers of $\frac{\pi}{3}$ in Quadrants III and IV.



Our solutions for θ are:

$$\theta = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2\pi n \quad (n \text{ integer})$$

From this point on, it is a matter of Algebra.

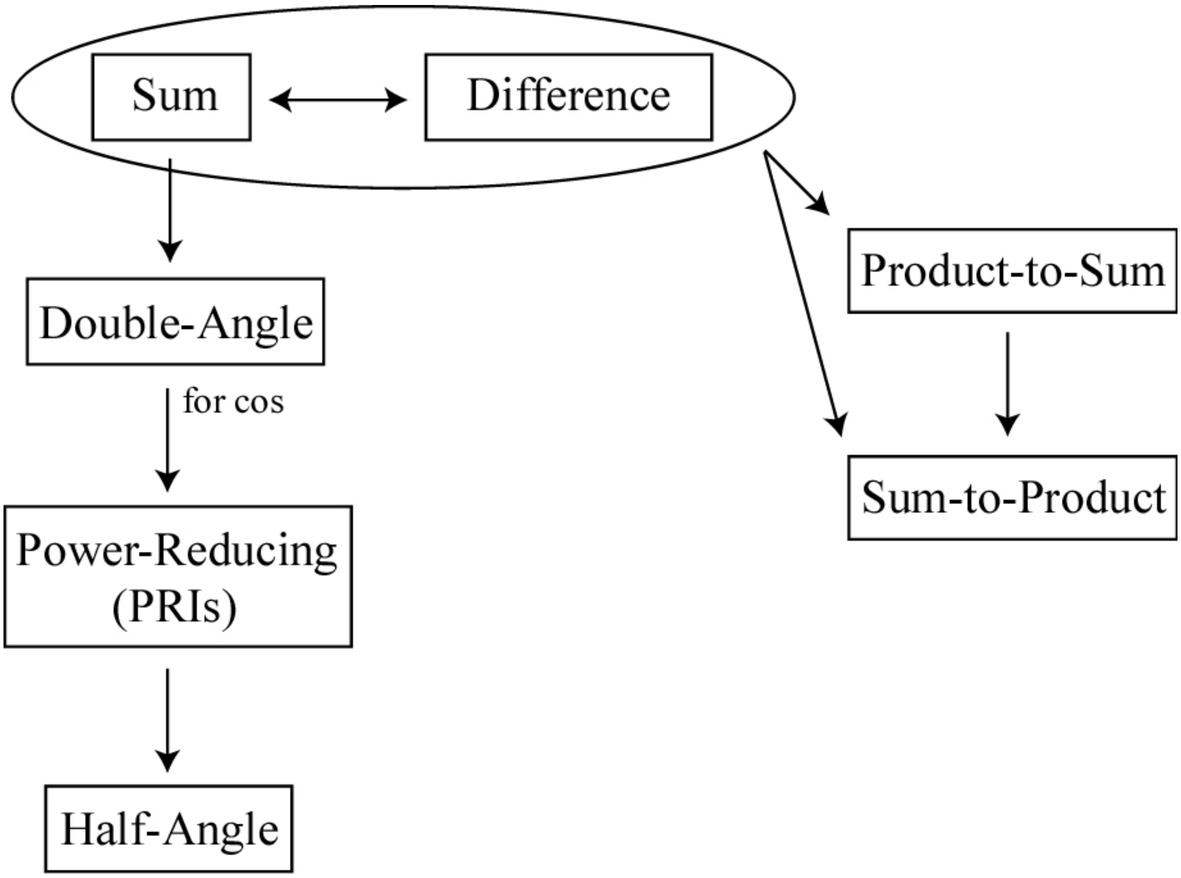
To find our solutions for x, replace θ with $4x$, and solve for x.

$$\begin{aligned}
4x &= \frac{4\pi}{3} + 2\pi n, && \text{or} && 4x = \frac{5\pi}{3} + 2\pi n && (n \text{ integer}) \\
x &= \frac{\frac{4\pi}{3}}{4} + \frac{2\pi}{4} n, && \text{or} && x = \frac{\frac{5\pi}{3}}{4} + \frac{2\pi}{4} n && (n \text{ integer}) \\
x &= \frac{\pi}{3} + \frac{\pi}{2} n, && \text{or} && x = \frac{5\pi}{12} + \frac{\pi}{2} n && (n \text{ integer})
\end{aligned}$$

Solution set: $\left\{ x \in \mathbf{R} \,\middle|\, x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \text{ integer}) \right\}$

PART E: ADVANCED TRIG IDENTITIES (IDs)

These identities (IDs) may be derived according to this flowchart:



GROUP 1: SUM IDENTITIES

Memorize:

$$\sin(u+v)=\sin u\cos v+\cos u\sin v$$

Think: "Sum of the mixed-up products"
(Multiplication and addition are commutative, but start with the $\sin u\cos v$ term in anticipation of the Difference Identities.)

$$\cos(u+v)=\cos u\cos v-\sin u\sin v$$

Think: "Cosines [product] – Sines [product]"

$$\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$$

Think: $"\frac{\text{Sum}}{1-\text{Product}}"$

GROUP 2: DIFFERENCE IDENTITIES

Memorize:

Simply take the Sum Identities above and change every sign in sight!

$$\sin(u-v)=\sin u\cos v-\cos u\sin v$$

(Make sure that the right side of your identity for $\sin(u+v)$ started with the $\sin u\cos v$ term!)

$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$

$$\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$$

Obtaining the Difference Identities from the Sum Identities:

Replace *v* with (–*v*) and use the fact that sin and tan are odd, while cos is even.

For example,

$$\begin{aligned}\sin(u-v)&=\sin\left[u+(-v)\right]\\&=\sin u\cos(-v)+\cos u\sin(-v)\\&=\sin u\cos v-\cos u\sin v\end{aligned}$$

GROUP 3a: DOUBLE-ANGLE (Think: Angle-Reducing, if *u* > 0) IDENTITIES

Memorize:

(Also be prepared to recognize and know these "right-to-left.")

$$\sin(2u) = 2\sin u\cos u$$

Think: "Twice the product"

Reading "right-to-left," we have:

$$2\sin u\cos u = \sin(2u)$$

(This is helpful when simplifying.)

$$\cos(2u) = \cos^2 u - \sin^2 u$$

Think: "Cosines – Sines" (again)

Reading "right-to-left," we have:

$$\cos^2 u - \sin^2 u = \cos(2u)$$

Contrast this with the Pythagorean Identity:

$$\cos^2 u + \sin^2 u = 1$$

$$\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$$

(Hard to memorize; we'll show how to obtain it.)

Notice that these identities are "angle-reducing" (if $u > 0$) in that they allow you to go from trig functions of $(2u)$ to trig functions of simply u.

Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace v with u, and simplify.

$$\begin{aligned}\sin(2u) &= \sin(u+u)\\ &= \sin u \cos u + \cos u \sin u \quad \text{(From Sum Identity)}\\ &= \sin u \cos u + \sin u \cos u \quad \text{(Like terms!!)}\\ &= 2 \sin u \cos u\end{aligned}$$

$$\begin{aligned}\cos(2u) &= \cos(u+u)\\ &= \cos u \cos u - \sin u \sin u \quad \text{(From Sum Identity)}\\ &= \cos^2 u - \sin^2 u\end{aligned}$$

$$\begin{aligned}\tan(2u) &= \tan(u+u)\\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad \text{(From Sum Identity)}\\ &= \frac{2 \tan u}{1 - \tan^2 u}\end{aligned}$$

This is a "last resort" if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the $\tan(2u)$ identity is harder to remember, you may prefer to remember the Sum Identity for $\tan(u+v)$ and then derive the $\tan(2u)$ identity this way.

If you're quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.

GROUP 3b: DOUBLE-ANGLE IDENTITIES FOR cos

Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$:

Let's begin with the version we've already seen:

Version 1: $\cos(2u) = \cos^2 u - \sin^2 u$

Also know these two, from "left-to-right," and from "right-to-left":

Version 2: $\cos(2u) = 1 - 2\sin^2 u$

Version 3: $\cos(2u) = 2\cos^2 u - 1$

Obtaining Versions 2 and 3 from Version 1

It's tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ written in different ways.

To obtain Version 2, which contains $\sin^2 u$, we replace $\cos^2 u$ with $\left(1 - \sin^2 u\right)$.

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \underbrace{\left(1 - \sin^2 u\right)}_{\text{from Pythagorean Identity}} - \sin^2 u \\
&= 1 - \sin^2 u - \sin^2 u \\
&= 1 - 2\sin^2 u && (\Rightarrow \text{Version 2})
\end{aligned}$$

To obtain Version 3, which contains $\cos^2 u$, we replace $\sin^2 u$ with $\left(1 - \cos^2 u\right)$.

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \cos^2 u - \underbrace{\left(1 - \cos^2 u\right)}_{\text{from Pythagorean Identity}} \\
&= \cos^2 u - 1 + \cos^2 u \\
&= 2\cos^2 u - 1 && (\Rightarrow \text{Version 3})
\end{aligned}$$

GROUP 4: POWER-REDUCING IDENTITIES ("PRIs")

(These are called the "Half-Angle Formulas" in some books.)

Memorize: **Then,**

$$\sin^2 u = \frac{1-\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} - \frac{1}{2}\cos(2u) \qquad \tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1-\cos(2u)}{1+\cos(2u)}$$

$$\cos^2 u = \frac{1+\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} + \frac{1}{2}\cos(2u)$$

Actually, you just need to memorize one of the $\sin^2 u$ or $\cos^2 u$ identities and then switch the visible sign to get the other. Think: "sin" is "bad" or "negative"; this is a reminder that the minus sign belongs in the $\sin^2 u$ formula.

Obtaining the Power-Reducing Identities from the Double-Angle Identities for $\cos(2u)$

To obtain the identity for $\sin^2 u$, start with Version 2 of the $\cos(2u)$ identity:

$$\cos(2u) = 1 - 2\sin^2 u$$

Now, solve for $\sin^2 u$.

$$2\sin^2 u = 1 - \cos(2u)$$

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

To obtain the identity for $\cos^2 u$, start with Version 3 of the $\cos(2u)$ identity:

$$\cos(2u) = 2\cos^2 u - 1$$

Now, switch sides and solve for $\cos^2 u$.

$$2\cos^2 u - 1 = \cos(2u)$$

$$2\cos^2 u = 1 + \cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

GROUP 5: HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution $\theta = 2u$. (See **Obtaining …** below.)

The Identities:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

For a given θ, the choices among the ± signs depend on the Quadrant that $\frac{\theta}{2}$ lies in.

Here, the ± symbols indicate incomplete knowledge; unlike when we deal with the Quadratic Formula, we do not take both signs for any of the above formulas for a given θ. There are no ± symbols in the last two $\tan\left(\frac{\theta}{2}\right)$ formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two $\tan\left(\frac{\theta}{2}\right)$ formulas: Keep either the numerator or the denominator of the radicand of the first formula, stick $\sin\theta$ in the other part of the fraction, and remove the radical sign and the ± symbol.

Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):

For the $\sin\left(\frac{\theta}{2}\right)$ identity, we begin with the PRI:

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

Let $u = \frac{\theta}{2}$, or $\theta = 2u$.

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \text{(by the Square Root Method)}$$

Again, the choice among the ± signs depends on the Quadrant that $\frac{\theta}{2}$ lies in.

The story is similar for the $\cos\left(\frac{\theta}{2}\right)$ and the $\tan\left(\frac{\theta}{2}\right)$ identities.

What about the last two formulas for $\tan\left(\frac{\theta}{2}\right)$? The key trick is multiplication by trig conjugates. For example:

$$\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}\cdot\frac{(1-\cos\theta)}{(1-\cos\theta)}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\
&= \pm\sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2} \\
&= \pm\left|\frac{1-\cos\theta}{\sin\theta}\right| \qquad \left(\text{because } \sqrt{blah^2} = |blah|\right)
\end{aligned}$$

Now, $1-\cos\theta \ge 0$ for all real θ, and $\tan\left(\frac{\theta}{2}\right)$ has the same sign as $\sin\theta$ (can you see why?), so …

$$= \frac{1-\cos\theta}{\sin\theta}$$

To get the third formula, use the numerator's (instead of the denominator's) trig conjugate, $1+\cos\theta$, when multiplying into the numerator and the denominator of the radicand in the first few steps.

GROUP 6: PRODUCT-TO-SUM IDENTITIES

These can be verified from right-to-left using the Sum and Difference Identities.

The Identities:

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

GROUP 7: SUM-TO-PRODUCT IDENTITIES

These can be verified from right-to-left using the Product-To-Sum Identities.

The Identities:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$