CALCULUS

CHAPTER 1, CHAPTER 2, SECTIONS 3.1-3.6

KEN KUNIYUKI
SAN DIEGO MESA COLLEGE
JULY 12, 2012

COLOR CODING

WARNINGS are in red.
TIPS are in purple.

TECHNOLOGY USED

This work was produced on Macs with Microsoft Word, MathType, Mathematica (for most graphs) and Calculus WIZ, Adobe Acrobat, and Adobe Illustrator.

CONTACT INFORMATION

Ken Kuniyuki:
Email: kkuniyuk@yahoo.com or kkuniyuk@sdccd.edu (San Diego Mesa College)
Website: http://www.kkuniyuk.com
• You may download these and other course notes, exercises, and exams.
Feel free to send emails with suggestions, improvements, tricks, etc.

LICENSING

PARTIAL BIBLIOGRAPHY / SOURCES

Algebra: Blitzer, Lial, Tussy and Gustafson
Trigonometry: Lial
Precalculus: Axler, Larson, Stewart, Sullivan
Calculus: Larson, Stewart, Swokowski, Tan
Complex Variables: Churchill and Brown, Schaum's Outlines
Discrete Mathematics: Rosen
Online: *Britannica Online Encyclopedia*: http://www.britannica.com,
Wikipedia: http://www.wikipedia.org,
Wolfram MathWorld: http://mathworld.wolfram.com/
Other: *Harper Collins Dictionary of Mathematics*
People: Larry Foster, Laleh Howard, Terrie Teegarden, Tom Teegarden (especially for the Frame Method for graphing trigonometric functions), and many more.

TABLE OF CONTENTS

CHAPTER 1: REVIEW

CHAPTER 2: LIMITS AND CONTINUITY

CHAPTER 3: DERIVATIVES

See the website for more: http://www.kkuniyuk.com

ASSUMPTIONS and NOTATION

Unless otherwise specified, we assume that:

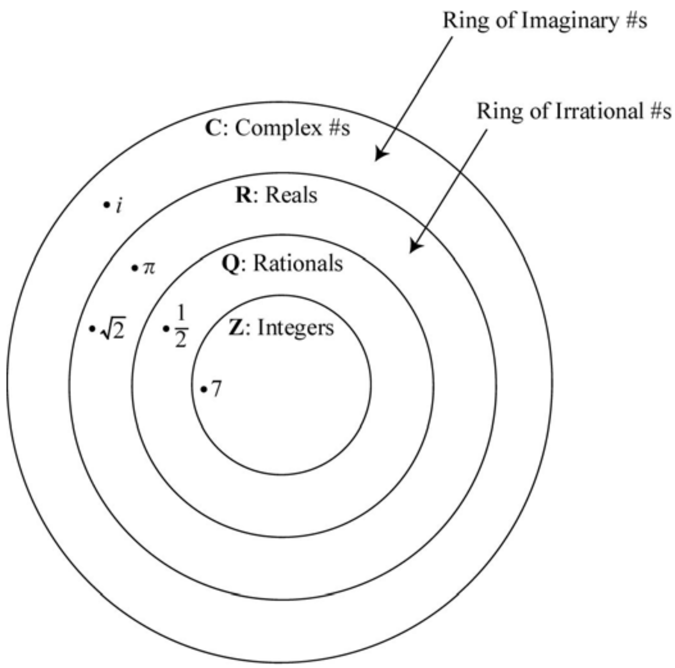
- f and g denote **functions**.
    - •• g sometimes denotes Earth's gravitational constant.
    - •• h may denote a function, or it may denote the "run" in some difference quotients in Chapter 3.
- a, b, c, k, and n denote **real constants** (or simply **real numbers**).
    - •• c sometimes denotes the speed of light in a vacuum.
    - •• d may denote a constant or a distance function.
    - •• e denotes a mathematical constant defined in Chapter 7. $e \approx 2.718$.
    - •• n might be restricted to be an integer $(n \in \mathbb{Z})$.
- The **domain** of a function, which we will denote by $\text{Dom}(f)$ for a function f (though this is nonstandard), is its implied (or mathematical) domain.
    - •• This might not be the case in applied "word problems."
    - •• In single variable calculus (in which a function is of only one variable), we assume that the **domain** and the **range** of a function only consist of **real** numbers, as opposed to imaginary numbers. That is, $\text{Dom}(f) \subseteq \mathbb{R}$, and $\text{Range}(f) \subseteq \mathbb{R}$. ($\subseteq$ means "is a subset of.")
- **Graphs extend** beyond the scope of a figure in an expected manner, unless **endpoints** are clearly shown. Arrowheads may help to make this clearer.
- In single variable calculus, "real constants" are "real constant **scalars**," as opposed to vectors.
    - •• This will change in multivariable calculus and linear algebra.

MORE NOTATION

Sets of Numbers

Notation	*Meaning*	*Comments*
$\mathbb{Z}^+$, $\mathbf{Z}^+$	the set of positive integers	This is the set (collection) $\{1, 2, 3, ...\}$. “Zahlen” is a related German word. $\mathbb{Z}$ is in blackboard bold typeface; it is more commonly used than **Z**.
$\mathbb{Z}$, $\mathbf{Z}$	the set of integers	This set consists of the positive integers, the negative integers ($-1, -2, -3, \ldots$), and 0.
$\mathbb{Q}$, $\mathbf{Q}$	the set of rational numbers	This set includes the integers and numbers such as $\frac{1}{3}$, $-\frac{9}{4}$, 7.13, and $14.3\overline{587}$. $\mathbb{Q}$ comes from “Quotient.”
$\mathbb{R}$, $\mathbf{R}$	the set of real numbers	This set includes the rational numbers and irrational numbers such as $\sqrt{2}$, π, *e*, and $0.1010010001\ldots$.
$\mathbb{C}$, $\mathbf{C}$	the set of complex numbers	This set includes the real numbers and imaginary numbers such as *i* and $2+3i$.

The Venn diagram below indicates the (proper) subset relations: $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. For example, every integer is a rational number, so $\mathbb{Z} \subset \mathbb{Q}$. ($\subseteq$ permits equality.) Each disk is contained within each larger disk.



Set Notation

Notation	*Meaning*	*Comments*
$\in$	in, is in	This denotes set membership. Example: $7 \in \mathbb{Z}$.
$\notin$	not in, is not in	Example: $1.7 \notin \mathbb{Z}$.
$\ni$	such that	
$\mid$ or :	such that (in set-builder form)	Example: $\left\{ x \in \mathbb{R} \mid x > 3 \right\}$, or $\left\{ x \in \mathbb{R} : x > 3 \right\}$, is the set of all real numbers greater than 3.
$\forall$	for all, for any	This is called the <u>universal quantifier</u>.
$\exists$	there is, there exists	This is called the <u>existential quantifier</u>.
$\exists!$	there exists a unique, there is one and only one	This is called the <u>unique quantifier</u>. Example: $\exists! x \in \mathbb{R} \ni x = 3$, which states that there exists a unique real number equal to 3.
$\forall x \in \mathbb{R}$	for every real number (denoted by x)	More precisely: for any arbitrary element of the set of real numbers; this element will be denoted by x. Example: $\forall x \in \mathbb{R},\ x < x + 1$; that is, every real number is less than one added to itself.
$\forall x, y \in \mathbb{R}$	for every pair of real numbers (denoted by x and y)	More precise notation: $\forall \left(x, y \right) \in \mathbb{R}^2$.
$\varnothing$ or $\{\}$	empty set (or null set)	This is the set consisting of no elements. Example: The solution set of the equation $x = x + 1$ is $\varnothing$. The symbol $\varnothing$ is not to be confused with the Greek letter phi (ϕ).
$\cup$	set union	Example: If $f\left(x \right) = \csc x$, then $\text{Dom}\left(f \right) = \left(-\infty, -1 \right] \cup \left[1, \infty \right)$. $\cup$ is used to indicate that one or more number(s) is/are being skipped over.
$\cap$	set intersection	Example: $\left[4, 6 \right] \cap \left[5, 7 \right] = \left[5, 6 \right]$. Think: “overlap.”
$\backslash$ or $-$	set difference, set complement	Example: If $f\left(x \right) = \dfrac{1}{x}$, then $\text{Dom}\left(f \right) = \mathbb{R} \setminus \left\{ 0 \right\}$, or $\mathbb{R} - \left\{ 0 \right\}$.

Logical Operators

Notation	*Meaning*	*Comments*
$\vee$	or, disjunction	Example: If $f(x)=\csc x$, then $\text{Dom}(f)=\{x\in\mathbb{R}\mid x\le -1\ \vee\ x\ge 1\}$.
$\wedge$	and, conjunction	Example: If $f(x)=\dfrac{\sqrt{x-3}}{x-4}$, then $\text{Dom}(f)=\{x\in\mathbb{R}\mid x\ge 3\ \wedge\ x\ne 4\}$.
$\sim$ or $\neg$	not, negation	Example: The statement $\sim(x=3)$ is equivalent to the statement $x\ne 3$.
$\Rightarrow$	implies	Example: $x=2\ \Rightarrow\ x^2=4$.
$\Leftrightarrow$	if and only if (iff)	Example: $x+1=3\ \Leftrightarrow\ x=2$.

Greek Letters

The lowercase Greek letters below (especially θ) often denote **angle measures**.

Notation	*Name*	*Comments*
α	alpha	This is the first letter of the Greek alphabet.
β	beta	This is the second letter of the Greek alphabet.
γ	gamma	This is the third letter of the Greek alphabet.
θ	theta	This is frequently used to denote angle measures.
ϕ or φ	phi	This is not to be confused with $\varnothing$, which denotes the empty set (or null set). ϕ also denotes the golden ratio, $\dfrac{1+\sqrt{5}}{2}$, which is about 1.618. Tau (τ) is also used.

The lowercase Greek letters below often denote (perhaps infinitesimally) **small positive quantities** in calculus, particularly when defining **limits**.

Notation	*Name*	*Comments*
δ	delta	This is the fourth letter of the Greek alphabet.
ε	epsilon	This is the fifth letter of the Greek alphabet. This is not be confused with $\in$, which denotes set membership.

Some other Greek letters of interest:

Notation	*Name*	*Comments*
Δ	(uppercase) delta	This denotes "change in" or increment. Example: slope is often written as $\frac{\Delta y}{\Delta x}$. It also denotes the discriminant, $b^2 - 4ac$, from the Quadratic Formula.
κ	(lowercase) kappa	This denotes the curvature of a curve.
λ	(lowercase) lambda	This denotes an eigenvalue (in linear algebra), a Lagrange multiplier (in multivariable optimization), and a wavelength (in physics).
π	(lowercase) pi	This is a famous mathematical constant. It is the ratio of a circle's circumference to its diameter. $\pi \approx 3.14159$. It is irrational.
Π	(uppercase) pi	This is the product operator.
ρ	(lowercase) rho	This denotes mass density and also the distance between a point in 3-space and the origin (ρ is a spherical coordinate).
Σ	(uppercase) sigma	This is the summation operator.
τ	(lowercase) tau	This denotes the golden ratio, though phi (ϕ) is more commonly used.
ω	(lowercase) omega	This is the last letter of the Greek alphabet. It denotes angular velocity.
Ω	(uppercase) omega	This denotes ohm, a unit of electrical resistance.

More lowercase Greek letters:

zeta (ζ), eta (η), iota (ι), mu (μ), nu (ν), xi (ξ), omicron (o), sigma (σ), upsilon (υ), chi (χ), psi (ψ)

Geometry

Notation	*Meaning*	*Comments*
$\angle$	angle	
$\parallel$	is parallel to	
$\perp$	is perpendicular to, is orthogonal to, is normal to	

Vector Operators

Notation	*Meaning*	*Comments*
$\bullet$	dot product, Euclidean inner product	See Precalculus notes, Section 6.4.
$\times$	cross product, vector product	See Precalculus notes, Section 8.4.

Other Notations

Notation	*Meaning*	*Comments*
$\therefore$	therefore	This is placed before a concluding statement.
Q.E.D., or $\square$	end of proof	Q.E.D. stands for "quod erat demonstrandum," which is Latin for "which was to be demonstrated / proven / shown."
$\approx$, $\cong$	is approximately	
$\lfloor \ \rfloor$ or $[\![\ ]\!]$	floor, greatest integer	Think: "round down." Examples: $\lfloor 2.9 \rfloor = 2$, $\lfloor -2.9 \rfloor = -3$
∞	infinity	
min	minimum	The least of …
max	maximum	The greatest of …
$\mathrm{Dom}(f)$	domain of a function f	The set of legal (real) input values for f
$\deg\big(f(x)\big)$	degree of a polynomial $f(x)$	
$\circ$	composition of functions	Example: $(f \circ g)(x) = f\big(g(x)\big)$.

CHAPTER 1:

Review

(See also the Precalculus notes at http://www.kkuniyuk.com)

TOPIC 1: FUNCTIONS

PART A: AN EXAMPLE OF A FUNCTION

Consider a function f whose rule is given by $f(x) = x^2$; $f(u) = u^2$ also works.

WARNING 1: $f(x)$ is read as "f **of** x" or "f **at** x." It does **not** mean "f **times** x."

x is the input (or argument) for f, and x^2 is the output or function value.

$$x \rightarrow \boxed{f} \rightarrow x^2$$

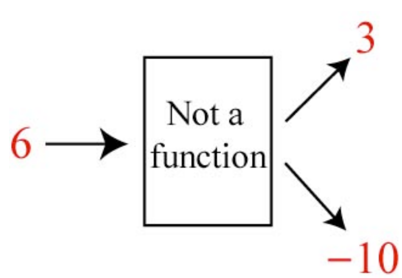
This function **squares its input**, and the result is its output.
For example, $f(3) = (3)^2 = 9$.

$$3 \rightarrow \boxed{f} \rightarrow 9$$

Think of a function as a **calculator button**. In fact, your calculator should have a "squaring" button labeled x^2.

f is a function, because no "legal" input yields more than one output.

A function button on a calculator **never** outputs two or more values at the same time. We never get: "I don't know. The answer could be 3 or -10."



• A **function** is a special type of relation. Relations that are not functions permit multiple outputs for a legal input.

PART B: POLYNOMIAL FUNCTIONS

A polynomial in x can be written in the form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer called the degree of the polynomial,
the a_k coefficients are typically real numbers, and the leading coefficient $a_n \neq 0$.

A polynomial function has a rule that can be written as: $f(x) =$ polynomial in x.

For example:

$4x^3 - \frac{5}{2}x^2 + 1$ is a 3rd-degree **polynomial** in x with leading coefficient 4.

The rule $f(x) = 4x^3 - \frac{5}{2}x^2 + 1$ corresponds to a **polynomial function** f.

PART C: RATIONAL FUNCTIONS

A rational expression in x can be written in the form: $\dfrac{\text{polynomial in } x}{\text{nonzero polynomial in } x}$.

Examples include: $\dfrac{1}{x}$, $\dfrac{5x^3 - 1}{x^2 + 7x - \sqrt{2}}$, and $x^7 + x$ $\left(\text{which equals } \dfrac{x^7 + x}{1}\right)$.

- Observe in the **second** example that irrational numbers such as $\sqrt{2}$ are permissible.
- The **last** example correctly suggests that **all polynomials are rational expressions**.

A rational function has a rule that can be written as:
$f(x) =$ rational expression in x.

PART D: ALGEBRAIC FUNCTIONS

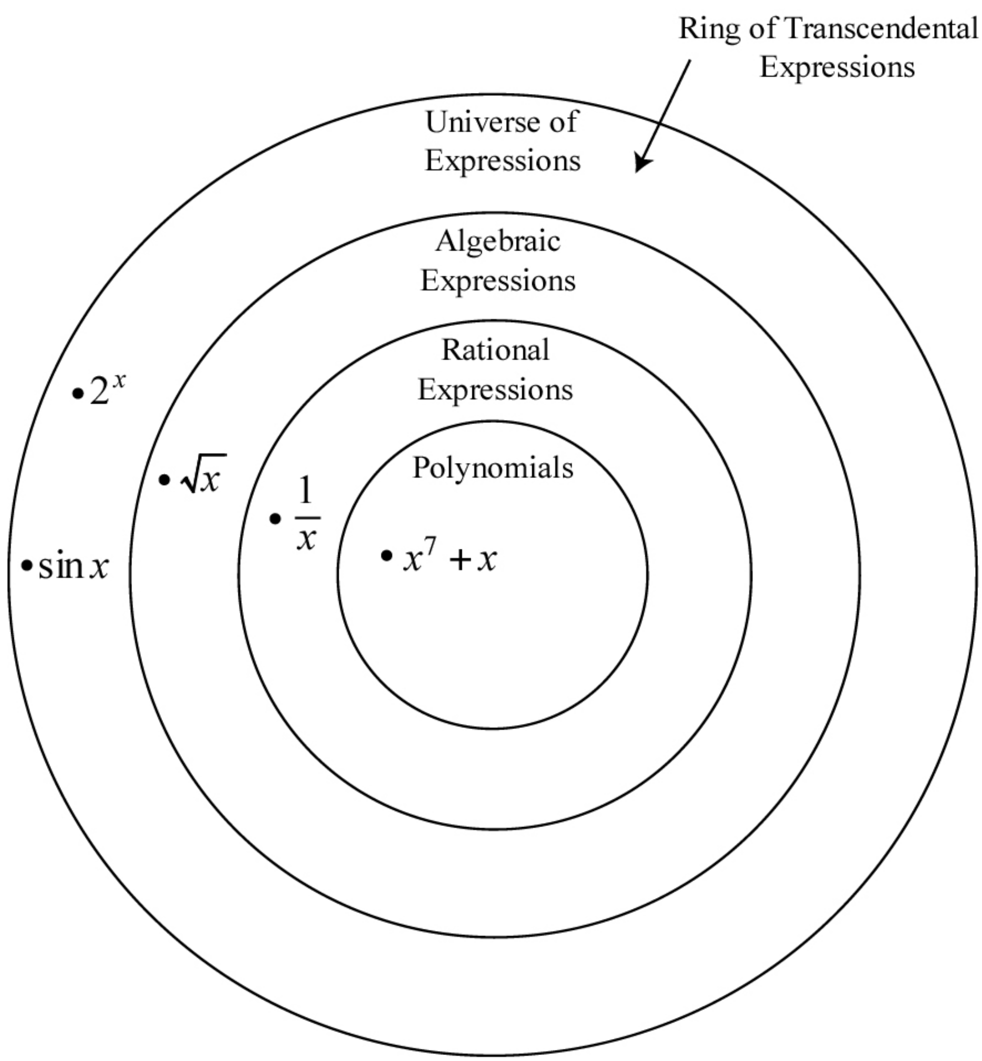
An algebraic expression in x resembles a **rational expression**, except that **radicals** and **exponents** that are noninteger **rational numbers** $\left(\text{such as } \dfrac{5}{7}\right)$ are also permitted, even when x appears in a radicand or in a base (but not in an exponent).

Examples include: $\sqrt{x}$ and $\dfrac{x^3 + 7x^{5/7}}{x - \sqrt[3]{x+5} + 4}$.

All rational expressions are algebraic. Although sources such as *MathWorld* allow only algebraic numbers (such as rational numbers and $\sqrt{2}$) to be coefficients in an algebraic expression, we will typically allow all real numbers (including π, for instance) in this work.

An algebraic function has a rule that can be written as:
$f(x) =$ algebraic expression in x.

A Venn diagram for expressions in x corresponding to **functions** is below.
Each disk represents a **subset** of every larger disk; for example, every polynomial is a rational expression and an algebraic expression (based on the definition in this work).



PART E: DOMAIN and RANGE

The domain of a function f, which we will denote by $\text{Dom}(f)$ (though this is not standard), is the set of all "legal" **inputs**.

The range of f, which we will denote by $\text{Range}(f)$, is then the set of all resulting **outputs**.

Unless otherwise specified (or in the context of a "word problem"), we typically assume that the **domain** of a function is the set of **all real** input values that yield an output that is a **real** number. This set is the implied (or natural) domain.

The implied domain of an **algebraic** function consists of all real numbers **except** those that lead to (the equivalent of):

1) a zero denominator $\left(\text{Think: } \frac{-}{0}\right)$, or

2) an even root of a negative-valued radicand $\left(\text{Think: } \sqrt[\text{even}]{-}\right)$.

As we study more types of functions, the list of **restrictions** will grow.
We will also exclude real numbers that lead to:

3) logarithms of nonpositive values $\left(\text{Think: } \log_b(\leq 0)\right)$, or

4) arguments of trigonometric functions that correspond to vertical asymptotes.

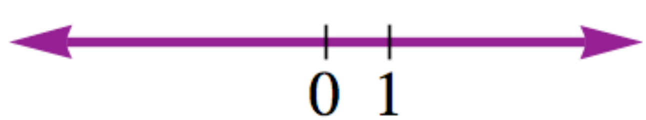
"Word problems" may imply other restrictions: nonnegativity, integer values, etc.

Example 1 (Domain and Range of the Squaring Function)

Let $f(x) = x^2$. Find the **domain** and the **range** of f.

§ Solution

The implied domain of a **polynomial** function (such as this f) is $\mathbb{R}$, the set of all real numbers. In interval form, $\mathbb{R}$ is $(-\infty, \infty)$. Its graph is the entire real number line:



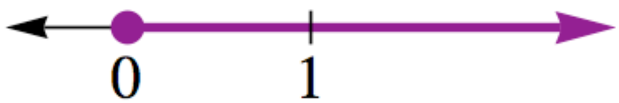
WARNING 2: We use parentheses in the interval form, because ∞ ("infinity") and $-\infty$ ("negative infinity") are **not** real numbers and are therefore **excluded** from the set of numbers. We will discuss **infinity** further in Chapter 2. If x approaches ∞, it (generally) **increases without bound**. If x approaches $-\infty$, it (generally) **decreases without bound**.

Note: It is debatable whether an expression like $\dfrac{x^2+x}{x}$ is a polynomial. It simplifies to $x+1$, but its domain excludes 0.

The resulting **range** of f is the set of all **nonnegative** real numbers (all real numbers that are **greater than or equal to 0**), because **every** such number is the **square** of some real number, and **only** those numbers are.

WARNING 3: Squares of real numbers are never negative.

The graph of the range is:



The **filled-in** circle indicates that 0 is **included** in the range. We could also use a left bracket ("[") at 0; the bracket opens towards the shading. The graph helps us figure out the interval form.

In interval form, the range is $[0, \infty)$. The **bracket** next to the 0 indicates that 0 is **included** in the range.

In set-builder form, the range is: $\{y \in \mathbb{R} \mid y \geq 0\}$, or $\{y \in \mathbb{R} : y \geq 0\}$, which is read "the set of all real numbers y such that $y \geq 0$." Using y instead of x is more consistent with our graphing conventions in the xy-plane (since we typically associate **function values** in the range with ***y*-coordinates**), and it helps us avoid confusion with the domain. $\in$ denotes set membership. §

Example 2 (Domain of a Function)

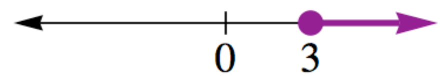
Let $f(x)=\sqrt{x-3}$, find $\text{Dom}(f)$, the domain of f.

§ Solution

$f(x)$ is **real** $\Leftrightarrow\ x-3\geq 0\ \Leftrightarrow\ x\geq 3$.

WARNING 4: We solve the **weak inequality** $x-3\geq 0$, **not** the **strict inequality** $x-3>0$. Observe that $\sqrt{0}=0$, a real number.

The domain of f …

… in set-builder form is:	$\{x\in\mathbb{R}\mid x\geq 3\}$, or $\{x\in\mathbb{R}:\ x\geq 3\}$
… in graphical form is:	
… in interval form is:	$[3,\infty)$

Note: $\text{Range}(f)=[0,\infty)$. Consider the graph of $y=f(x)$. §

Example 3 (Domain of a Function)

Let $f(x)=\sqrt[4]{3-x}$. Find $\text{Dom}(f)$.

§ Solution

Solve the weak inequality: $3-x\geq 0$.

Method 1

$3-x\geq 0$ Now subtract 3 from both sides.

$-x\geq -3$ Now multiply or divide both sides by -1.

WARNING 5: We must then **reverse** the direction of the inequality symbol.

$x\leq 3$

Method 2

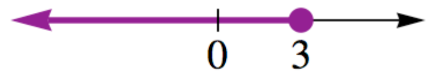
$3-x\geq 0$ Now add x to both sides.

$3\geq x$ Now switch the left side and the right side.

WARNING 6: We must then **reverse** the direction of the inequality symbol.

$x\leq 3$

The domain of f …

… in set-builder form is:	$\{x \in \mathbb{R} \mid x \le 3\}$, or $\{x \in \mathbb{R} : x \le 3\}$
… in graphical form is:	
… in interval form is:	$(-\infty, 3]$

§

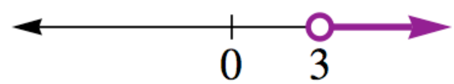
Example 4 (Domain of a Function)

Let $f(x) = \dfrac{1}{\sqrt{x-3}}$. Find $\mathrm{Dom}(f)$.

§ Solution

This is similar to Example 2, but we must **avoid a zero denominator**. We solve the **strict inequality** $x - 3 > 0$, which gives us $x > 3$.

The domain of f …

… in set-builder form is:	$\{x \in \mathbb{R} \mid x > 3\}$, or $\{x \in \mathbb{R} : x > 3\}$
… in graphical form is:	
… in interval form is:	$(3, \infty)$

The hollow circle on the **graph** indicates that 3 is **excluded** from the domain. We could also use a left parenthesis ("(") here; the parenthesis opens towards the shading. Likewise, we have a **parenthesis** next to the 3 in the interval form, because 3 is **excluded** from the domain. §

Types of Intervals

$(5, 7)$ and $(3, \infty)$ are examples of open intervals, because they **exclude** their endpoints. $(5, 7)$ is a bounded interval, because it is trapped between two real numbers.

$(3, \infty)$ is an unbounded interval.

$[5, 7]$ is a closed interval, because it **includes** its endpoints, and it is bounded.

Example 5 (Domain of a Function)

Let $f(x) = \sqrt[3]{x-3}$. Find $\mathrm{Dom}(f)$.

§ Solution

$\mathrm{Dom}(f) = \mathbb{R}$, because:

• The radicand, $x-3$, is a **polynomial**, and

• **WARNING 7:** The taking of **odd** roots (such as cube roots) does **not** impose any new restrictions on the domain. Remember that the cube root of a negative real number is a negative real number. §

Example 6 (Domain of a Function)

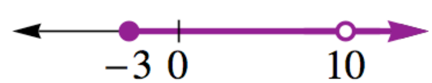
Let $g(t) = \dfrac{\sqrt{t+3}}{t-10}$. Find $\mathrm{Dom}(g)$.

§ Solution

The **square root** operation requires: $t+3 \ge 0 \iff t \ge -3$.

We **forbid zero denominators**, so we also require: $t-10 \ne 0 \iff t \ne 10$.

The domain of g …

… in set-builder form is:	$\{t \in \mathbb{R} \mid t \ge -3 \text{ and } t \ne 10\}$, or $\{t \in \mathbb{R} : t \ge -3 \text{ and } t \ne 10\}$
… in graphical form is:	
… in interval form is:	$[-3, 10) \cup (10, \infty)$

• We **include** -3 but **exclude** 10. (Some instructors believe that 0 should also be indicated on the number line.)

• The <u>union symbol</u> $(\cup)$ is used to separate intervals in the event that a number or numbers need to be skipped. §

PART F: GRAPHS OF FUNCTIONS

The graph of $y = f(x)$, or the graph of f, in the standard xy-plane consists of all points [representing ordered pairs] of the form $(x, f(x))$, where x is in the domain of f.

In a sense, the graph of $f = \{(x, f(x)) \mid x \in \mathrm{Dom}(f)\}$.

We typically assume …

x is the independent variable, because it is the **input** variable.

y is the dependent variable, because it is the **output** variable.
Its value (the **function value**) typically "depends" on the value of the input x.

• Then, it is customary to say that y is a function of x, even though y is a variable here. The form $y = f(x)$ implies this.

A "brute force" graphing method follows.

Point-Plotting Method for Graphing a Function f in the xy-Plane

• **Choose** several x values in $\mathrm{Dom}(f)$.

• For each chosen x value, find $f(x)$, its corresponding **function value**.

• **Plot** the corresponding points $(x, f(x))$ in the xy-plane.

• Try to **interpolate** (connect the points, though often not with line segments) and **extrapolate** (go beyond the scope of the points) as necessary, ideally based on some apparent pattern.

•• Ensure that the set of ***x*-coordinates** of the points on the graph is, in fact, $\mathrm{Dom}(f)$.

Example 7 (Graph of the Squaring Function)

Let $f(x)=\sqrt{x}$. Graph $y=f(x)$.

§ Solution

TIP 1: As usual, we associate ***y*-coordinates** with **function values**.

When **point-plotting**, observe that: $\text{Dom}(f)=[0,\infty)$.

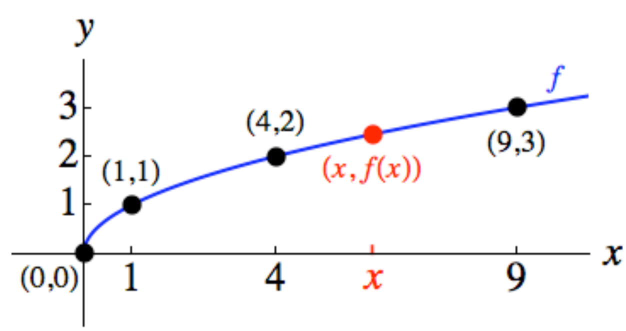
• For instance, if we choose $x=9$, we find that $f(9)=\sqrt{9}=3$, which means that the **point** $(9,\ f(9))$, or $(9,3)$, lies on the graph.

• On the other hand, $f(-9)$ is **undefined** (as a real number), because $9\notin \text{Dom}(f)$. Therefore, there is **no** corresponding point on the graph with $x=-9$.

A (partial) table can help:

x	$f(x)$	Point
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$
9	3	$(9,3)$

Below, we sketch the graph of f, or $y=f(x)$.



WARNING 8: Clearly indicate any **endpoints** on a graph, such as the origin here.

The lack of a clearly indicated right endpoint on our sketch implies that the graph **extends** beyond the edge of our figure. We want to draw graphs in such a way that these extensions are "as one would expect."

WARNING 9: Sketches of graphs produced by **graphing utilities** might not extend as expected. The user must still understand the math involved. Point-plotting may be insufficient. §

PART G: THE VERTICAL LINE TEST (VLT)

The Vertical Line Test (VLT)

A curve in a coordinate plane passes the Vertical Line Test (VLT) $\Leftrightarrow$ there is **no** vertical line that intersects the curve more than once.

An **equation** in x and y describes y as a **function** of x $\Leftrightarrow$ its **graph** in the xy-plane **passes the VLT**.

- Then, there is **no** input x that yields more than one output y.
- Then, we can write $y = f(x)$, where f is a function.

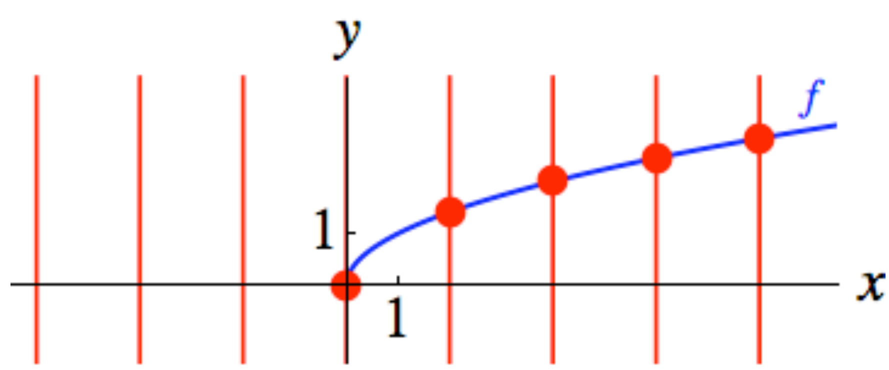
Example 8 (Square Root Function and the VLT; Revisiting Example 7)

The equation $y = \sqrt{x}$ explicitly describes y as a **function** of x, since it is of the form $y = f(x)$, where f is the square root function from Example 7.

Observe that the graph of $y = \sqrt{x}$ **passes** the VLT.
Each **vertical line** in the xy-plane either …

• … **misses the graph entirely**, meaning that the corresponding x value is **not** in $\text{Dom}(f)$, or

• … **intersects the graph in exactly one point**, meaning that the corresponding x value yields exactly one y value as its output.



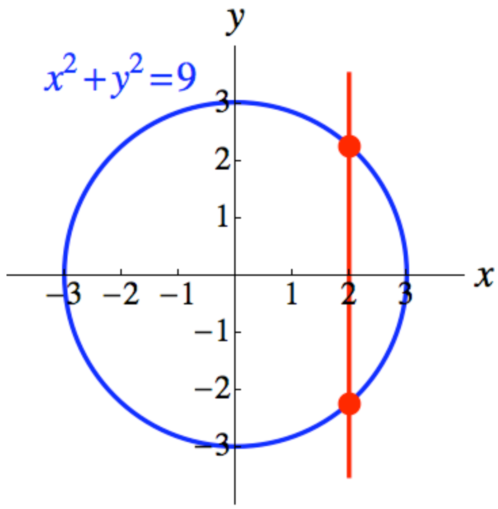
§

Example 9 (An Equation that Does Not Describe a Function)

Show that the equation $x^2 + y^2 = 9$ does **not** describe y as a function of x.

§ Solution (Method 1: VLT)

The circular graph of $x^2 + y^2 = 9$ below **fails** the VLT, because there exists a vertical line that intersects the graph **more than once**. For example, we can take the red line ($x = 2$) below:



Therefore, $x^2 + y^2 = 9$ does **not** describe y as a function of x. §

§ Solution (Method 2: Solve for y)

This is also evident if we solve $x^2 + y^2 = 9$ for y:

$$x^2 + y^2 = 9$$
$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$

• Any input value for x in the interval $(-3, 3)$ yields two different y outputs.

• For example, $x = 2$ yields the outputs $y = \sqrt{5}$ and $y = -\sqrt{5}$. §

PART H: ESTIMATING DOMAIN, RANGE, and FUNCTION VALUES FROM A GRAPH

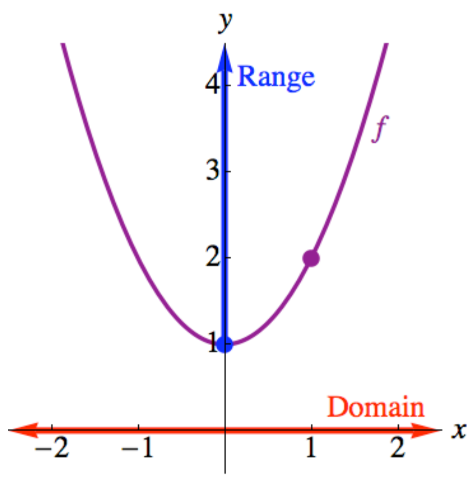
The **domain** of f is the set of all ***x*-coordinates** of points on the graph of $y = f(x)$. (Think of projecting the graph onto the x-axis.)

The **range** of f is the set of all ***y*-coordinates** of points on the graph of $y = f(x)$. (Think of projecting the graph onto the y-axis.)

$\boxed{\begin{array}{c}\text{Domain}\\ \text{Think: } x\end{array}} \rightarrow \boxed{f} \rightarrow \boxed{\begin{array}{c}\text{Range}\\ \text{Think: } y\end{array}}$

Example 10 (Estimating Domain, Range, and Function Values from a Graph)

Let $f(x) = x^2 + 1$. Estimate the **domain** and the **range** of f based on the graph of $y = f(x)$ below. Also, estimate $f(1)$.



§ Solution

Apparently, $\text{Dom}(f) = \mathbb{R}$, or $(-\infty, \infty)$, and $\text{Range}(f) = [1, \infty)$.

- We will learn more about determining ranges from the graphing techniques in Chapter 4.

It also appears that the point $(1, 2)$ lies on the graph, and thus $f(1) = 2$.

WARNING 10: Graph analyses can be imprecise. The point $(1, 2.001)$, for example, may be hard to identify on a graph. Not all coordinates are integers. §

PART I: FUNCTIONS THAT ARE EVEN / ODD / NEITHER; SYMMETRY

A function f is even $\Leftrightarrow$ $f(-x) = f(x), \ \forall x \in \text{Dom}(f)$
$\Leftrightarrow$ The graph of $y = f(x)$ is **symmetric about the y-axis**.

$\forall$ means "for all" or "for every."

Example 11 (Proving that a Function is Even)

Let $f(x) = x^2$. Prove that f is an **even** function.

§ Solution

$\text{Dom}(f) = \mathbb{R}$. $\forall x \in \mathbb{R}$,

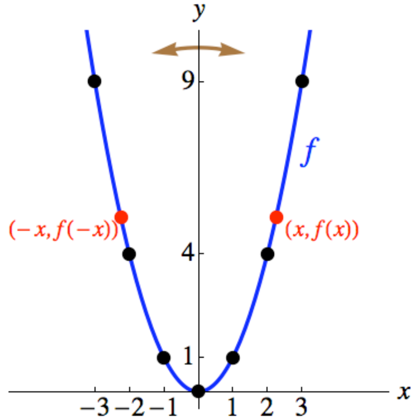
$$\begin{aligned} f(-x) &= (-x)^2 \\ &= x^2 \\ &= f(x) \end{aligned}$$

Q.E.D. (Latin: Quod Erat Demonstrandum)

• This signifies the end of a proof. It means "which was to be demonstrated / proven / shown."

TIP 2: Think: If we replace x with $(-x)$ as the input, we obtain **equivalent (y) outputs**. The point (x, y) lies on the graph if and only if $(-x, y)$ does.

Observe that the graph of $y = f(x)$ below is symmetric about the y-axis, meaning that the parts of the graph to the right and to the left of the y-axis are mirror images (or reflections) of each other.



§

The term "even function" may have come from the following fact:

If $f(x) = x^n$, where n is an **even integer**, then f is an **even function**.

• These are the functions for: $\ldots,\ x^{-4},\ x^{-2},\ x^0,\ x^2,\ x^4,\ \ldots.$

• The graph of $y = x^2$ is called a parabola. The graphs of $y = x^4$, $y = x^6$, etc. are similarly bowl-shaped but are **not** parabolas.

A function f is odd $\Leftrightarrow$ $f(-x) = -f(x),\quad \forall x \in \mathrm{Dom}(f)$

$\Leftrightarrow$ The graph of $y = f(x)$ is **symmetric about the origin**.

• In other words, if the graph is rotated 180° about the origin, we obtain the same graph.

Example 12 (Proving that a Function is Odd)

Let $f(x) = x^3$. Prove that f is an **odd** function.

§ Solution

$\mathrm{Dom}(f) = \mathbb{R}$. $\forall x \in \mathbb{R}$,

$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -(x^3) \\ &= -f(x) \end{aligned}$$

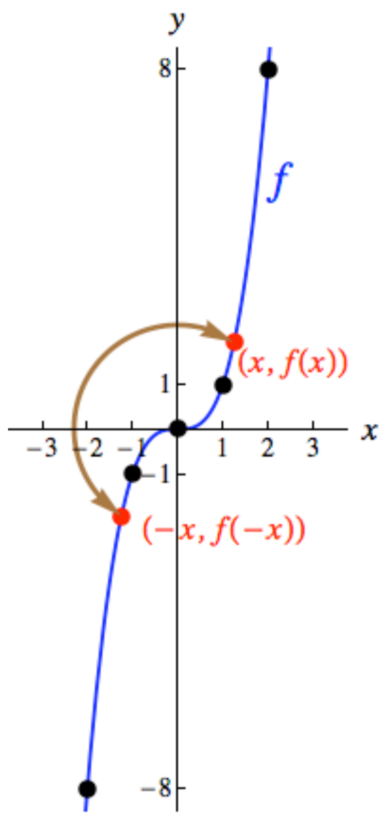
Q.E.D.

TIP 3: Think: If we replace x with $(-x)$ as the input, we obtain **opposite (y) outputs**. The point (x, y) lies on the graph if and only if $(-x, -y)$ does. §

The term "odd function" may have come from the following fact:

> If $f(x) = x^n$, where n is an **odd integer**, then f is an **odd function**.

- The graphs of $y = x^5$, $y = x^7$, etc. resemble the graph of $y = x^3$ below.



<u>WARNING 11</u>: <u>Zero functions</u> are functions that only output 0 (Think: $f(x) = 0$). Zero functions on domains that are symmetric about 0 on the real number line are the **only** functions that are **both even and odd**. (Can you show this?)

<u>WARNING 12</u>: Many functions are **neither** even nor odd.

PART J: ARITHMETIC COMBINATIONS OF FUNCTIONS

Let f and g be functions.

If their domains overlap, then the overlap (intersection) $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ is the **domain** of the following functions with the specified rules, with one possible exception (*):

$f+g$, where $(f+g)(x) = f(x)+g(x)$

$f-g$, where $(f-g)(x) = f(x)-g(x)$

fg, where $(fg)(x) = f(x)g(x)$

$\dfrac{f}{g}$, where $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

(*) **WARNING 13:** $\mathrm{Dom}\left(\dfrac{f}{g}\right) = \left\{x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \,\middle|\, \boldsymbol{g(x) \neq 0}\right\}$.

Example 13 (Subtracting Functions)

Let $f(x) = 4x$ and $g(x) = x + \dfrac{1}{x}$. Find $(f-g)(x)$ and $\mathrm{Dom}(f-g)$.

§ Solution

$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= (4x) - \left(x + \frac{1}{x}\right)\end{aligned}$$

WARNING 14: Use **grouping symbols** when expanding $g(x)$ here, since we are **subtracting** an expression with **more than one term**.

$$\begin{aligned}&= 4x - x - \frac{1}{x}\\ &= 3x - \frac{1}{x}\end{aligned}$$

$\mathrm{Dom}(f) = \mathbb{R}$. We **omit only 0** from $\mathrm{Dom}(g)$ and also $\mathrm{Dom}(f-g)$.

$\mathrm{Dom}(f-g) = \mathbb{R} \setminus \{0\} = \{x \in \mathbb{R} \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$. §

PART K: COMPOSITIONS OF FUNCTIONS

We compose functions when we apply them **in sequence**.

> Let f and g be functions. The composite function $f \circ g$ is defined by:
>
> $$(f \circ g)(x) = f(g(x))$$
>
> Its domain is $\{x \in \mathbb{R} \mid x \in \text{Dom}(g) \textbf{ and } g(x) \in \text{Dom}(f)\}$.
>
> • The domain consists of the "legal" **inputs** to g that yield **outputs** that are "legal" **inputs** to f.

$$x \to \underbrace{\boxed{g} \to g(x) \to \boxed{f}}_{\boxed{f \circ g}} \to f(g(x))$$

Think of $f \circ g$ as a "merged" function.

WARNING 15: The function $f \circ g$ applies g **first** and then f. Think of pressing a g button on a calculator followed by an f button.

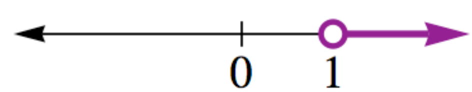
WARNING 16: $f \circ g$ **may or may not** represent the same function as $g \circ f$ (in which f is applied first). Composition of functions is **not commutative** the way that, say, addition is. Think About It: Try to think of examples where $f \circ g$ and $g \circ f$ represent the same function.

Example 14 (Composition of Functions)

Let $f(u) = \dfrac{1}{u}$ and $g(x) = \sqrt{x-1}$. Find $(f \circ g)(x)$ and $\text{Dom}(f \circ g)$.

§ Solution

$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x-1}\right) = \dfrac{1}{\sqrt{x-1}}$. In fact, $\text{Dom}(f \circ g)$ …

… in set-builder form is:	$\{x \in \mathbb{R} \mid x > 1\}$, or $\{x \in \mathbb{R} : x > 1\}$
… in graphical form is:	
… in interval form is:	$(1, \infty)$

§

Example 15 (Decomposing a Composite Function)

Find component functions f and g such that $(f \circ g)(x) = \sqrt{3x+1}$.
We want to "decompose" $f \circ g$.

- Neither f nor g may be an identity function.

  For example, do **not** use: $g(x) = x$ and $f(u) = \sqrt{3u+1}$.
  This would **not** truly be a decomposition. f does all the work!

$$x \to \underbrace{\boxed{\begin{array}{c} g: \\ g(x) = x \end{array}} \to \underbrace{x\text{, our } u}_{g(x)} \to \boxed{\begin{array}{c} f: \\ f(u) = \sqrt{3u+1} \end{array}}}_{\boxed{f \circ g}} \to \underbrace{\sqrt{3x+1}}_{f(g(x))}$$

§ Solution

- We need: $f(g(x)) = \sqrt{3x+1}$.

- We can think of f and g as **buttons** we are designing on a calculator.
We need to set up f and g so that, if x is an **initial input** to $\text{Dom}(f \circ g)$, and if the g button and **then** the f button are pressed, then the output is $\sqrt{3x+1}$.

$$x \to \underbrace{\boxed{\begin{array}{c} g: \\ g(x) = ? \end{array}} \to u = ? \to \boxed{\begin{array}{c} f: \\ f(u) = ?? \end{array}}}_{\boxed{f \circ g}} \to \underbrace{\sqrt{3x+1}}_{f(g(x))}$$

- A common strategy is to let $g(x)$, or u, be an **"inside" expression** (for example, a radicand, an exponent, a base of a power, a denominator, an argument, or something being repeated) whose replacement simplifies the overall expression.

- Here, we will let $g(x) = 3x+1$.

• We then need f to apply the square root operation. We will let $f(u)=\sqrt{u}$.

The use of u is more helpful in calculus, but $f(x)=\sqrt{x}$ is also acceptable. However, $f(u)=\sqrt{x}$ is **not** acceptable.

Possible Answer: Let $g(x)=3x+1$ and $f(u)=\sqrt{u}$.

$$x \rightarrow \underbrace{\boxed{\begin{array}{c} g: \\ g(x)=3x+1 \end{array}} \rightarrow \underbrace{3x+1,\text{ our } u}_{g(x)} \rightarrow \boxed{\begin{array}{c} f: \\ f(u)=\sqrt{u} \end{array}}}_{\boxed{f\circ g}} \rightarrow \underbrace{\sqrt{3x+1}}_{f(g(x))}$$

There are infinitely many possible answers.
For example, we could let $g(x)=3x$ and $f(u)=\sqrt{u+1}$.

$$x \rightarrow \underbrace{\boxed{\begin{array}{c} g: \\ g(x)=3x \end{array}} \rightarrow \underbrace{3x,\text{ our } u}_{g(x)} \rightarrow \boxed{\begin{array}{c} f: \\ f(u)=\sqrt{u+1} \end{array}}}_{\boxed{f\circ g}} \rightarrow \underbrace{\sqrt{3x+1}}_{f(g(x))}$$

§

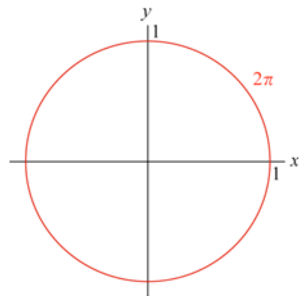
These ideas will be critical to the Chain Rule of Differentiation in Section 3.6 and the u-substitution technique of integration in Section 5.2.

TOPIC 2: TRIGONOMETRY I

PART A: ANGLE MEASURES

Radian measure is more "mathematically natural" than degree measure, and it is typically assumed in calculus. In fact, **radian measure is assumed** if there are no units present.

There are 2π radians in a full (counterclockwise) revolution, because the entire unit circle (which has circumference 2π) is intercepted exactly once by such an angle.



There are 360° (360 degrees) in a full (counterclockwise) revolution.
(This is something of a cultural artifact; ancient Babylonians operated on a base-60 number system.)

2π radians is equivalent to 360°. Therefore, π radians is equivalent to 180°. Either relationship may be used to construct conversion factors.

In any unit conversion, we effectively multiply by 1 in such a way that the old unit is canceled out.

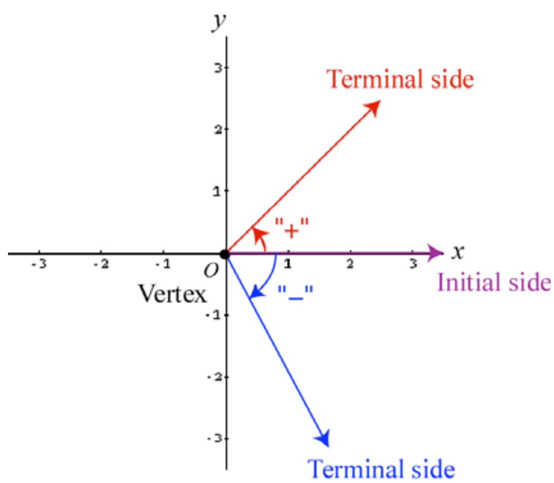
For example, to convert 45° into radians:

$$45^\circ = (45^\circ)\overbrace{\left(\frac{\pi\,[\text{radians}]}{180^\circ}\right)}^{=(1)} = \overbrace{(\cancel{45^\circ})}^{(1)}\left(\frac{\pi\,[\text{radians}]}{\underbrace{\cancel{180^\circ}}_{(4)}}\right) = \frac{\pi}{4}[\text{radians}].$$

PART B: QUADRANTS AND QUADRANTAL ANGLES

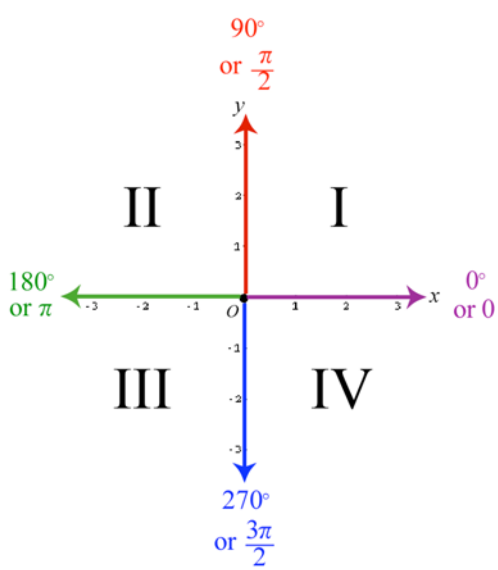
The x- and y-axes divide the xy-plane into 4 quadrants.
Quadrant I is the upper right quadrant; the others are numbered in counterclockwise order.

A standard angle in standard position has the positive [really, nonnegative] x-axis as its initial side and the origin as its vertex. We say that the angle lies in the quadrant that its terminal side shoots through. For example, in the figure below, the positive standard angle with the red terminal side is a Quadrant I angle:



A standard angle whose terminal side lies on the x- or y-axis is called a quadrantal angle. Quadrantal angles correspond to "integer multiples" of 90° or $\dfrac{\pi}{2}$ radians.

The quadrants and some quadrantal angles are below.
(For convenience, we may label a standard angle by labeling its terminal side.)



PART C: COTERMINAL ANGLES

Standard angles that share the same terminal side are called coterminal angles. They differ by an integer number of full revolutions counterclockwise or clockwise.

If the angle θ is measured in **radians**, then its **coterminal angles** are of the form: $\theta + 2\pi n$, where n is any integer $(n \in \mathbb{Z})$.

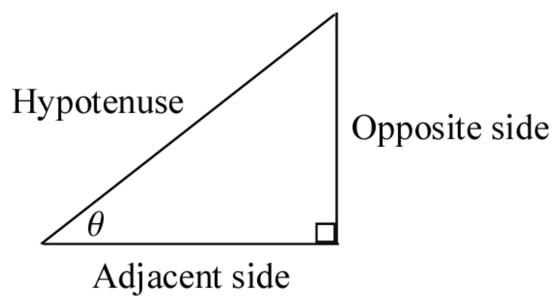
If the angle θ is measured in **degrees**, then its **coterminal angles** are of the form: $\theta + 360n^\circ$, where n is any integer $(n \in \mathbb{Z})$.

Note: Since n could be negative, the "+" sign is sufficient in the above forms, as opposed to "$\pm$."

PART D: TRIGONOMETRIC FUNCTIONS: THE RIGHT TRIANGLE APPROACH

The Setup

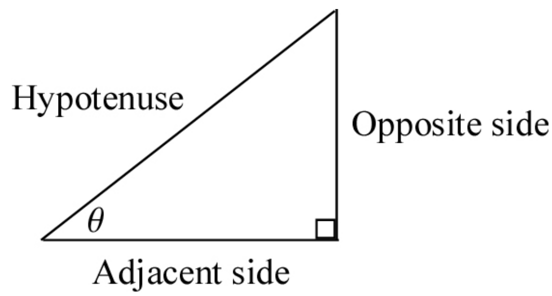
The **acute** angles of a right triangle are **complementary**. Consider such an angle, θ. Relative to θ, we may label the sides as follows:



The hypotenuse always faces the right angle, and it is always the longest side.

The other two sides are the legs. The opposite side (relative to θ) faces the θ angle. The other leg is the adjacent side (relative to θ).

Defining the Six Basic Trig Functions (where θ is acute)



"SOH-CAH-TOA"

$$\text{Sine } \theta = \sin\theta = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\text{Cosine } \theta = \cos\theta = \frac{\text{Adj.}}{\text{Hyp.}}$$

$$\text{Tangent } \theta = \tan\theta = \frac{\text{Opp.}}{\text{Adj.}}$$

Reciprocal Identities

$$\text{Cosecant } \theta = \csc\theta = \frac{1}{\sin\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Opp.}} \right)$$

$$\text{Secant } \theta = \sec\theta = \frac{1}{\cos\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Adj.}} \right)$$

$$\text{Cotangent } \theta = \cot\theta = \frac{1}{\tan\theta} \quad \left(= \frac{\text{Adj.}}{\text{Opp.}} \right)$$

WARNING 1: Remember that the reciprocal of $\sin\theta$ is $\csc\theta$, not $\sec\theta$.

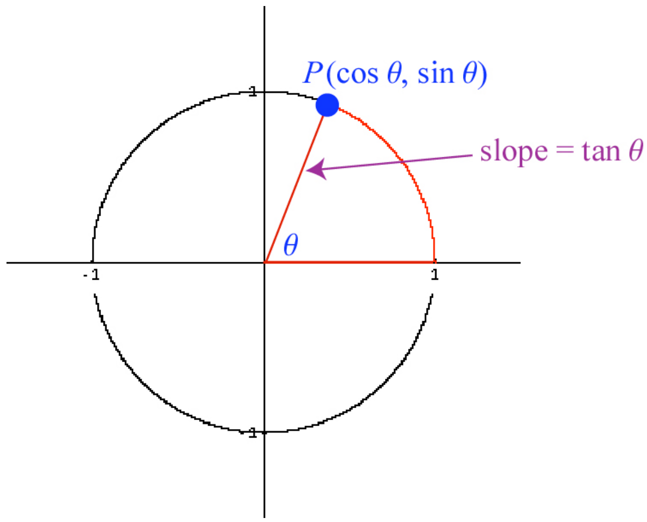
TIP 1: We informally treat "0" and "undefined" as **reciprocals** when we are dealing with basic **trigonometric** functions. Your algebra teacher will not want to hear this, though!

PART E: TRIGONOMETRIC FUNCTIONS: THE UNIT CIRCLE APPROACH

The Setup

Consider a standard angle θ measured in radians (or, equivalently, let θ represent a real number).

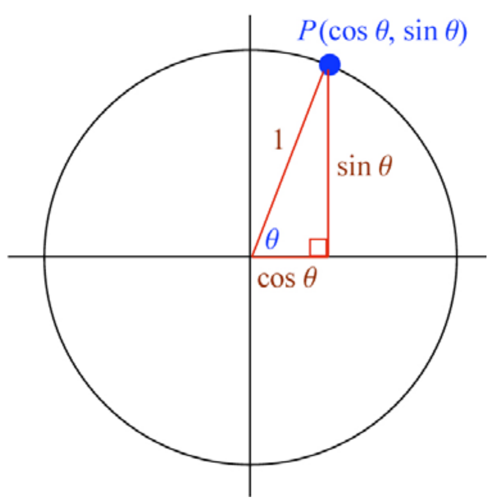
The point $P(\cos\theta, \sin\theta)$ is the **intersection point** between the **terminal side** of the angle and the **unit circle** centered at the origin. The **slope** of the terminal side is, in fact, $\tan\theta$.



Note: The intercepted arc along the circle (in red) has arc length θ.

The figure below demonstrates how this is consistent with the SOH-CAH-TOA (or Right Triangle) approach. Observe:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{rise}}{\text{run}} = \text{slope of terminal side}$$



"THE Table"

We will use our knowledge of the 30°-60°-90° and 45°-45°-90° **special triangles** to construct "THE Table" below. The **unit circle** approach is used to find the trigonometric values for **quadrantal** angles such as 0° and 90°.

Key Angles θ: Degrees, (Radians)	$\sin\theta$	$\cos\theta$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	Intersection Point $P(\cos\theta, \sin\theta)$
0°, (0)	$\dfrac{\sqrt{0}}{2} = \mathbf{0}$	$\mathbf{1}$	$\dfrac{0}{1} = \mathbf{0}$	$(1, 0)$
30°, $\left(\dfrac{\pi}{6}\right)$	$\dfrac{\sqrt{1}}{2} = \mathbf{\dfrac{1}{2}}$	$\mathbf{\dfrac{\sqrt{3}}{2}}$	$\dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}} = \mathbf{\dfrac{\sqrt{3}}{3}}$	$\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$
45°, $\left(\dfrac{\pi}{4}\right)$	$\dfrac{\sqrt{2}}{2} = \mathbf{\dfrac{\sqrt{2}}{2}}$	$\mathbf{\dfrac{\sqrt{2}}{2}}$	$\dfrac{\sqrt{2}/2}{\sqrt{2}/2} = \mathbf{1}$	$\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$
60°, $\left(\dfrac{\pi}{3}\right)$	$\dfrac{\sqrt{3}}{2} = \mathbf{\dfrac{\sqrt{3}}{2}}$	$\mathbf{\dfrac{1}{2}}$	$\dfrac{\sqrt{3}/2}{1/2} = \mathbf{\sqrt{3}}$	$\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$
90°, $\left(\dfrac{\pi}{2}\right)$	$\dfrac{\sqrt{4}}{2} = \mathbf{1}$	$\mathbf{0}$	$\dfrac{1}{0}$ is **undefined**	$(0, 1)$

WARNING 2: $\dfrac{\pi}{5}$ is **not** a "special" angle.

WARNING 3: Always make sure what **mode** your **calculator** is in (DEG vs. RAD) whenever you evaluate trigonometric functions.

The values for the **reciprocals**, $\csc\theta$, $\sec\theta$, and $\cot\theta$, are then readily found. Remember that it is sometimes better to take a trigonometric value where the denominator is **not rationalized** before taking its **reciprocal**. For example, because $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$, we know immediately that $\cot 30^\circ = \sqrt{3}$.

Observe:

• The pattern in the $\sin\theta$ column

Technical Note: An explanation for this pattern appears in the Sept. 2004 issue of the College Mathematics Journal (p.302).

• The fact that the $\sin\theta$ column is **reversed** to form the $\cos\theta$ column. This is due to the Cofunction Identities (or the Pythagorean Identities).

• As θ increases from 0° to 90° (i.e., from 0 to $\frac{\pi}{2}$ radians),

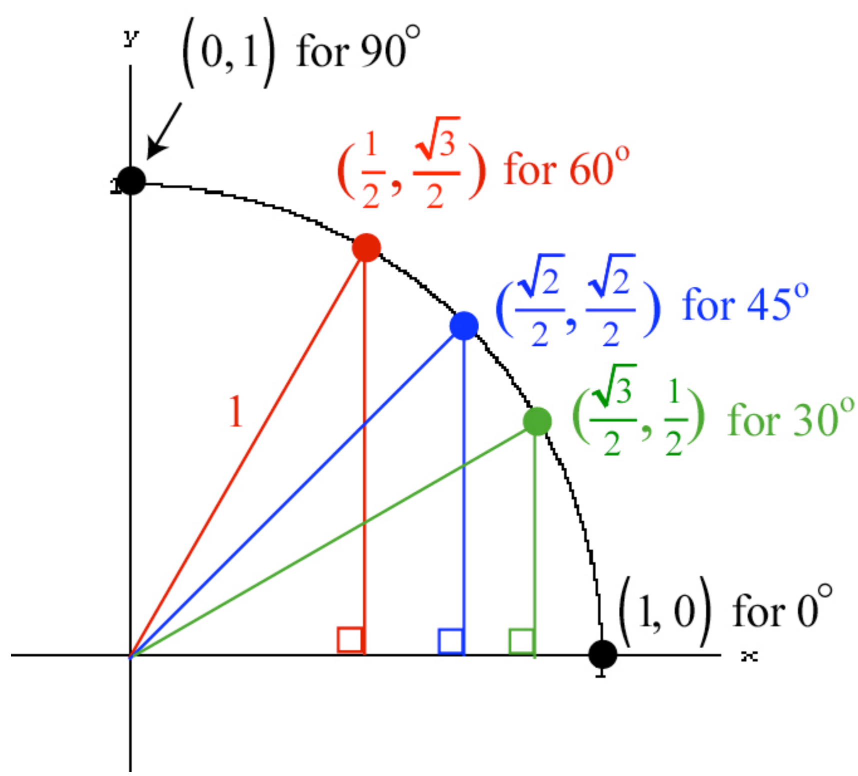
•• $\sin\theta$ (the ***y*-coordinate** of P) **increases** from 0 to 1.

Note: This is more obvious using the **Unit Circle** approach instead of the Right Triangle approach.

•• $\cos\theta$ (the ***x*-coordinate** of P) **decreases** from 1 to 0.

•• $\tan\theta$ (the **slope** of the terminal side) starts at 0, **increases**, and approaches ∞.

• Here is the "Big Picture." Remember that each **intersection point** is of the form $P(\cos\theta, \sin\theta)$.



PART F: EXTENDING FROM QUADRANT I TO OTHER QUADRANTS

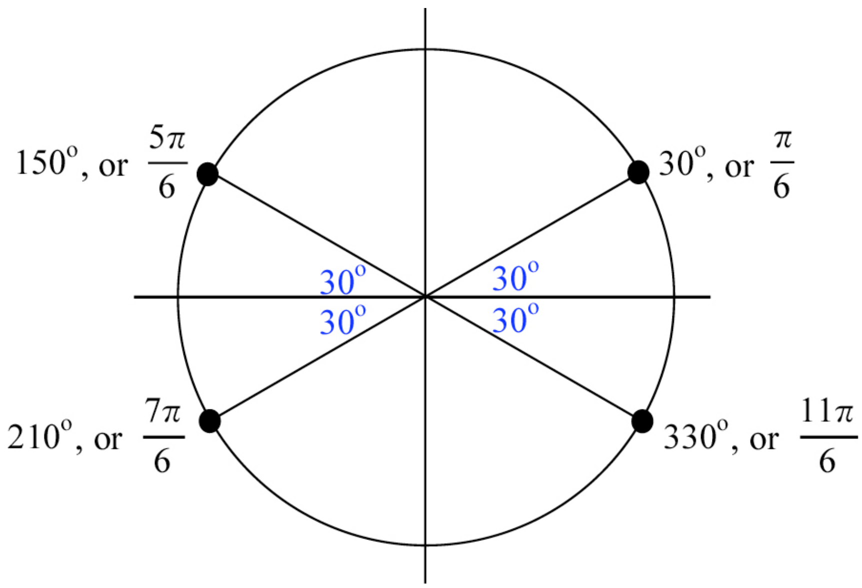
Reference angles

The reference angle for a non-quadrantal standard angle is the **acute** angle that its **terminal side** makes with the ***x*-axis**.

We will informally call angles that have the **same reference angle** "coreference angles," which is not a standard term.

• We may informally think of "coreference angles" as "brothers" and coterminal angles as "twins" (although an angle has infinitely many of them).

For example, the "coreference angles" below share the same reference angle, namely 30°, or $\dfrac{\pi}{6}$ radians.



Coterminal angles are also "coreference angles." For example, -30° (or $-\dfrac{\pi}{6}$ radians) is a coterminal "twin" for the 330° (or $\dfrac{11\pi}{6}$ radian) angle.

We will extend the following patterns for "coreference angles" of $\frac{\pi}{6}$ to other "families" of radian measures:

Quadrant II: $\frac{5\pi}{6}$; observe that 5 is **1 less than** 6.

Quadrant III: $\frac{7\pi}{6}$; observe that 7 is **1 more than** 6.

Quadrant IV: $\frac{11\pi}{6}$; observe that 11 is **1 less than twice** 6.

Key "coreference angles" of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ are below.

We've already seen some for $\frac{\pi}{6}$:

(The boxes correspond to Quadrants.)

$\frac{5\pi}{6}$	$\frac{\pi}{6}$
$\frac{7\pi}{6}$	$\frac{11\pi}{6}$

Now, $\frac{\pi}{4}$:

$\frac{3\pi}{4}$	$\frac{\pi}{4}$
$\frac{5\pi}{4}$	$\frac{7\pi}{4}$

Now, $\frac{\pi}{3}$:

$\frac{2\pi}{3}$	$\frac{\pi}{3}$
$\frac{4\pi}{3}$	$\frac{5\pi}{3}$

Why are "Coreference Angles" Useful?

Coterminal angles have the **same** basic trigonometric values, including the signs.

"Coreference angles" have the **same** basic trigonometric values **up to** (except maybe for) the **signs**.

Signs of Basic Trigonometric Values in Quadrants

Remember that **reciprocal** values have the **same sign** (or one is 0 and the other is undefined).

"ASTC" Rule for Signs

Think: "**A**ll **S**tudents **T**ake **C**alculus"

Start in Quadrant I and progress **counterclockwise** through the Quadrants:

S	A
T	C

All six basic trigonometric functions are **positive** in Quadrant I.
(They are all positive for acute angles.)

Sine and its reciprocal, Cosecant, are **positive** in Quadrant II.
(The **other** four functions are **negative**.)

Tangent and its reciprocal, Cotangent, are **positive** in Quadrant III.

Cosine and its reciprocal, Secant, are **positive** in Quadrant IV.

For example, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$, because $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, and $\frac{7\pi}{6}$ is in Quadrant III.

TOPIC 3: TRIGONOMETRY II

PART A: FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Memorize these in both "directions" (i.e., left-to-right and right-to-left).

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sin x = \frac{1}{\csc x}$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x = \frac{1}{\sec x}$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

WARNING 1: Remember that the reciprocal of $\sin x$ is $\csc x$, not $\sec x$.

TIP 1: We informally treat "0" and "undefined" as **reciprocals** when we are dealing with basic **trigonometric** functions. Your algebra teacher will not want to hear this, though!

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

TIP 2: The second and third Pythagorean Identities can be obtained from the first by dividing both of its sides by $\sin^2 x$ and $\cos^2 x$, respectively.

TIP 3: The squares of $\csc x$ and $\sec x$, which have "Up-U, Down-U" graphs, are all alone on the right sides of the last two identities. They can never be 0 in value. (Why is that? Look at the left sides.)

Cofunction Identities

If x is measured in radians, then:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

We have analogous relationships for tangent and cotangent, and for secant and cosecant; remember that they are sometimes undefined.

Think: **Cofunctions of complementary angles are equal.**

Even / Odd (or Negative Angle) Identities

Among the six basic trigonometric functions, only cosine (and its reciprocal, secant) are **even**:

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x$$

However, the other four are **odd**:

$$\sin(-x) = -\sin x$$
$$\csc(-x) = -\csc x$$
$$\tan(-x) = -\tan x$$
$$\cot(-x) = -\cot x$$

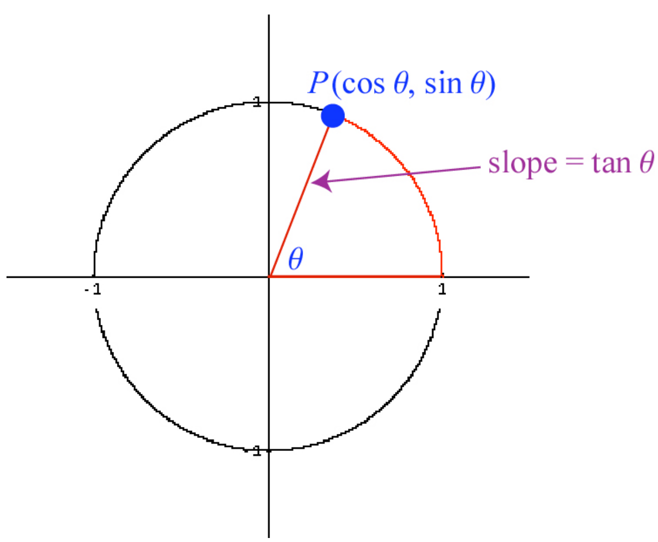
• If f is an **even** function, then the graph of $y = f(x)$ is symmetric about the ***y*-axis**.

• If f is an **odd** function, then the graph of $y = f(x)$ is symmetric about the **origin**.

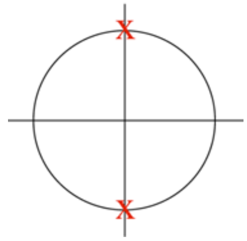
PART B: DOMAINS AND RANGES OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$f(x)$	Domain	Range
$\sin x$	$(-\infty, \infty)$	$[-1, 1]$
$\cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \dfrac{\pi}{2} + \pi n \ \ (n \in \mathbb{Z}) \right\}$	$(-\infty, \infty)$
$\csc x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \pi n \ \ (n \in \mathbb{Z}) \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \dfrac{\pi}{2} + \pi n \ \ (n \in \mathbb{Z}) \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \pi n \ \ (n \in \mathbb{Z}) \right\}$	$(-\infty, \infty)$

• The **unit circle** approach explains the <u>domain and range</u> for <u>sine and cosine</u>, as well as the <u>range</u> for <u>tangent</u> (since any real number can be a **slope**).

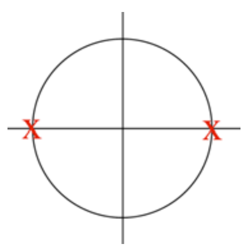


• <u>Domain for tangent</u>: The "**X**"s on the unit circle below correspond to an **undefined slope**. Therefore, the corresponding real numbers (the corresponding angle measures in radians) are **excluded** from the domain.



• <u>Domain for tangent and secant</u>: The "**X**"s on the unit circle above also correspond to a cosine value of 0. By the Quotient Identity for tangent $\left(\tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$ and the Reciprocal Identity for secant $\left(\sec\theta = \dfrac{1}{\cos\theta}\right)$, we **exclude** the corresponding radian measures from the domains of both functions.

• <u>Domain for cotangent and cosecant</u>: The "**X**"s on the unit circle below correspond to a sine value of 0. By the Quotient Identity for cotangent $\left(\cot\theta = \dfrac{\cos\theta}{\sin\theta}\right)$ and the Reciprocal Identity for cosecant $\left(\csc\theta = \dfrac{1}{\sin\theta}\right)$, we **exclude** the corresponding radian measures from the domains of both functions.



• <u>Range for cosecant and secant</u>: We turn "inside out" the range for both sine and cosine, which is $[-1, 1]$.

• <u>Range for cotangent</u>: This is explained by the fact that the range for tangent is $(-\infty, \infty)$ and the Reciprocal Identity for cotangent: $\left(\cot\theta = \dfrac{1}{\tan\theta}\right)$. $\cot\theta$ is 0 in value $\Leftrightarrow$ $\tan\theta$ is undefined.

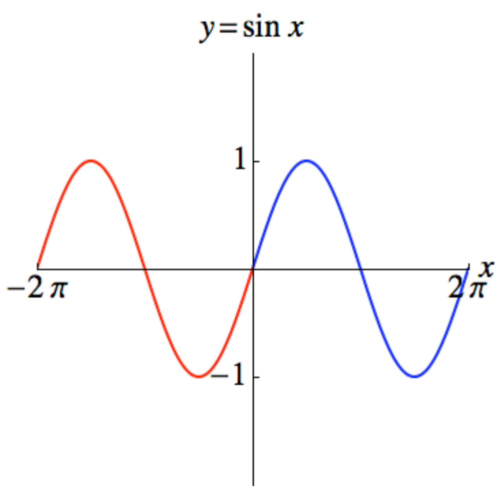
PART C: GRAPHS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

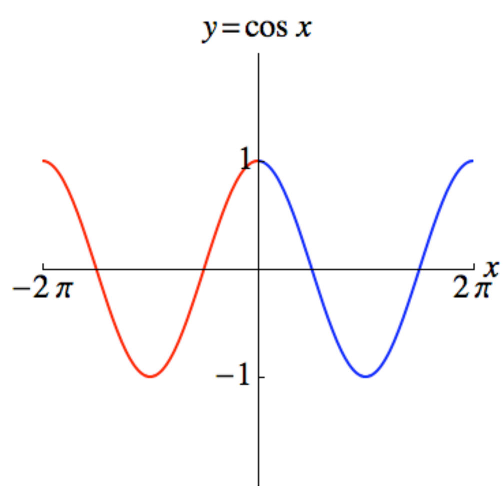
• The six basic trigonometric functions are <u>periodic</u>, so their graphs can be decomposed into cycles that repeat like wallpaper patterns. The <u>period</u> for tangent and cotangent is π; it is 2π for the others.

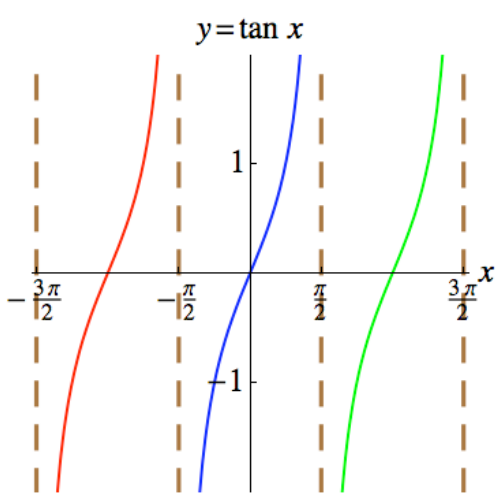
• A <u>vertical asymptote ("VA")</u> is a vertical line that a graph approaches in an "explosive" sense. (This idea will be made more precise in Section 2.4.) VAs on the graph of a basic trigonometric function correspond to **exclusions from the domain**. They are graphed as dashed lines.

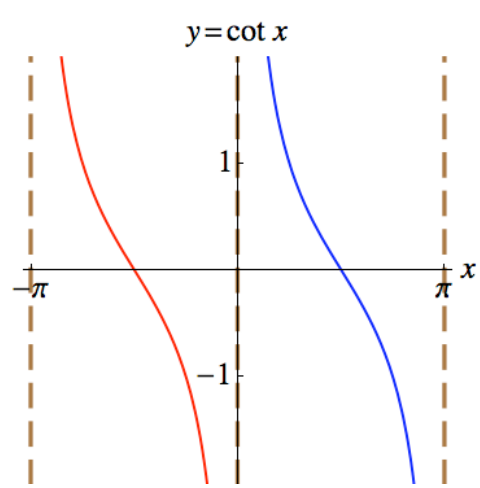
• Remember that the **domain** of a function f corresponds to the ***x*-coordinates** picked up by the graph of $y = f(x)$, and the **range** corresponds to the ***y*-coordinates**.

• Remember that cosine and secant are the only **even** functions among the six, so their graphs are **symmetric about the *y*-axis**. The other four are **odd**, so their graphs are **symmetric about the origin**.

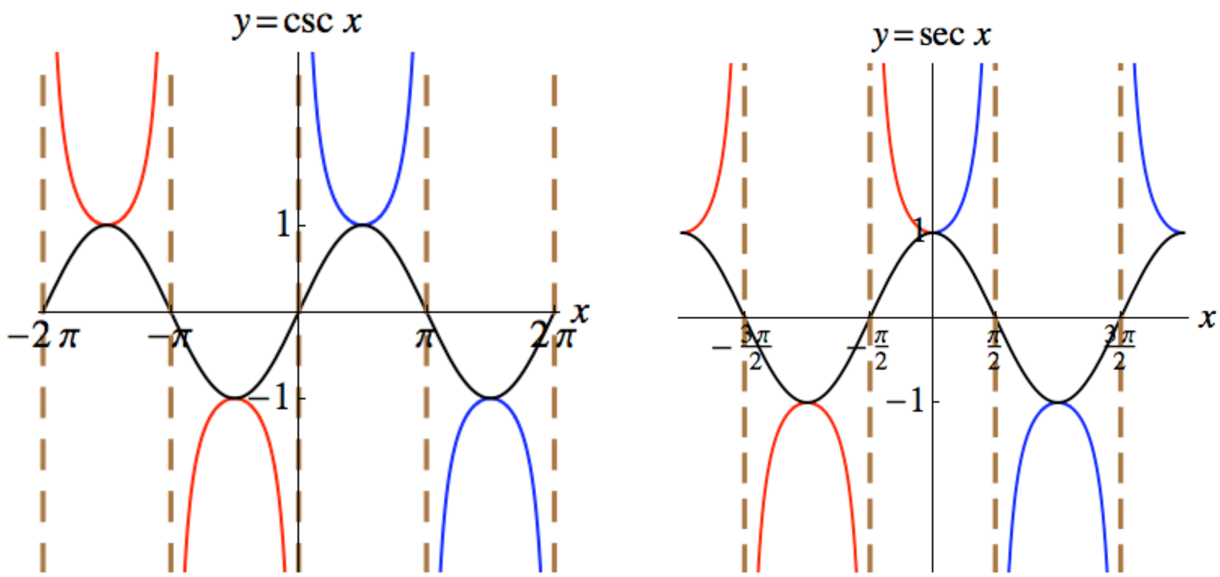








• We use the graphs of $y = \sin x$ and $y = \cos x$ (in black in the figures below) as **guide graphs** to help us graph $y = \csc x$ and $y = \sec x$.



Relationships between the graphs of $y = \csc x$ and $y = \sin x$ (and between the graphs of $y = \sec x$ and $y = \cos x$):

•• The **VAs** on the graph of $y = \csc x$ are drawn through the ***x*-intercepts** of the graph of $y = \sin x$. This is because $\csc x$ is undefined $\Leftrightarrow$ $\sin x = 0$.

•• The reciprocals of 1 and -1 are themselves, so $\csc x$ and $\sin x$ take on each of those values simultaneously. This explains how their graphs **intersect**.

•• Because sine and cosecant are **reciprocal** functions, we know that, between the VAs in the graph of $y = \csc x$, they share the **same sign**, and **one increases** $\Leftrightarrow$ **the other decreases**.

PART D: SOLVING TRIGONOMETRIC EQUATIONS

Example 1 (Solving a Trigonometric Equation)

Solve: $2\sin(4x) = -\sqrt{3}$

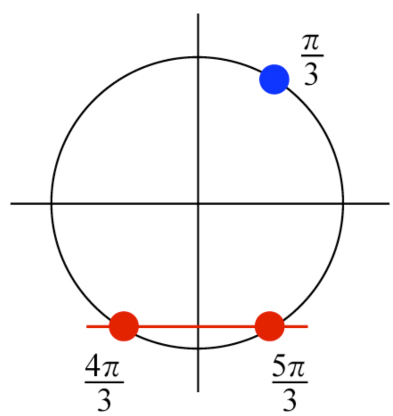
§ Solution

$2\sin(4x) = -\sqrt{3}$ Isolate the sine expression.

$\sin\underbrace{(4x)}_{=\theta} = -\dfrac{\sqrt{3}}{2}$ Substitution: Let $\theta = 4x$.

$\sin\theta = -\dfrac{\sqrt{3}}{2}$ We will now solve this equation for θ.

Observe that $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$, so $\dfrac{\pi}{3}$ will be the **reference angle** for our solutions for θ. Since $-\dfrac{\sqrt{3}}{2}$ is a **negative** sine value, we want **"coreference angles"** of $\dfrac{\pi}{3}$ in **Quadrants III and IV**.



Our solutions for θ are:

$$\theta = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

From this point on, it is a matter of algebra.

To find our solutions for *x*, replace θ with $4x$, and solve for *x*.

$$4x = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad 4x = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

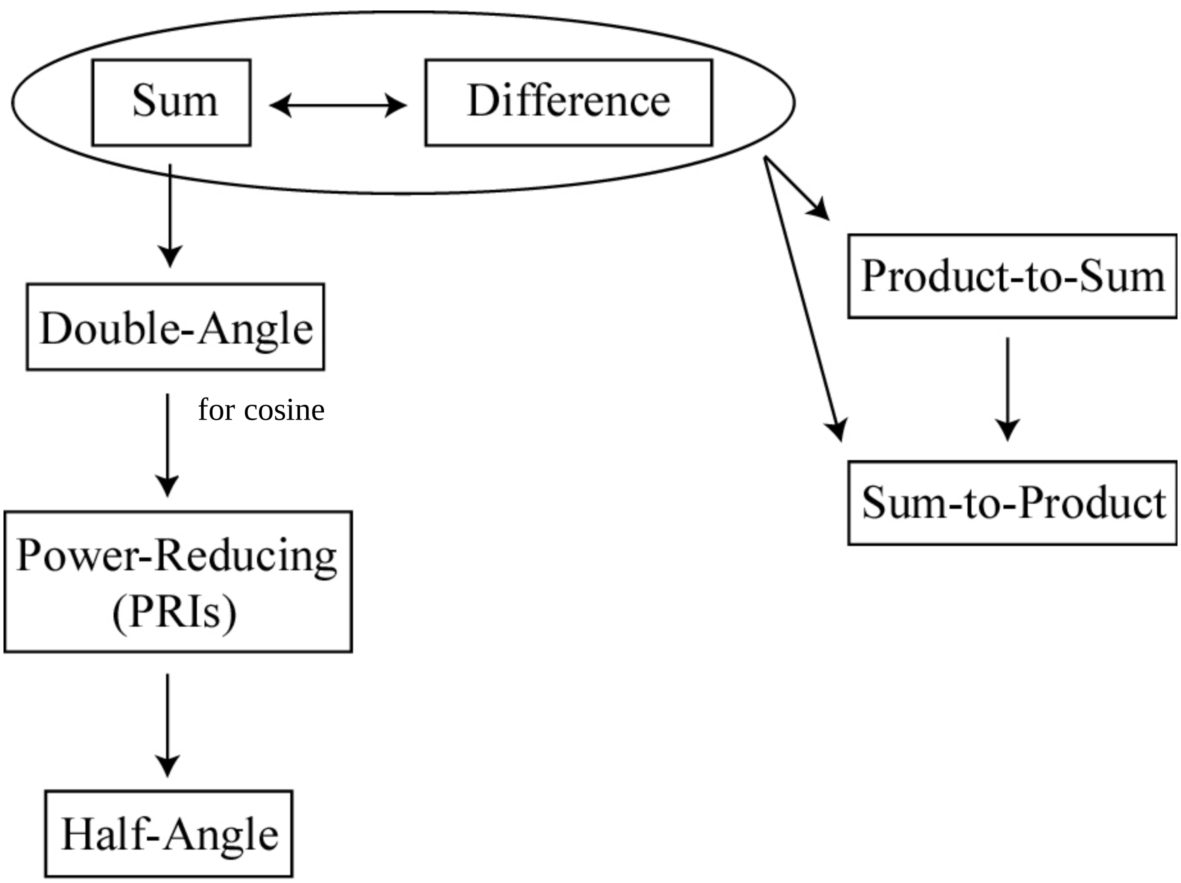
$$x = \frac{1}{4}\left(\frac{4\pi}{3}\right) + \frac{2\pi}{4}n, \quad \text{or} \quad x = \frac{1}{4}\left(\frac{5\pi}{3}\right) + \frac{2\pi}{4}n \quad (n \in \mathbb{Z})$$

$$x = \frac{\pi}{3} + \frac{\pi}{2}n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2}n \quad (n \in \mathbb{Z})$$

Solution set: $\left\{ x \in \mathbb{R} \,\middle|\, x = \frac{\pi}{3} + \frac{\pi}{2}n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2}n \quad (n \in \mathbb{Z}) \right\}$. §

PART E: ADVANCED TRIGONOMETRIC IDENTITIES

These identities may be derived according to the flowchart below.



GROUP 1: SUM IDENTITIES

Memorize:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

Think: "Sum of the mixed-up products"
(Multiplication and addition are commutative, but start with the $\sin u \cos v$ term in anticipation of the Difference Identities.)

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Think: "Cosines [product] – Sines [product]"

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

Think: $"\dfrac{\text{Sum}}{1-\text{Product}}"$

GROUP 2: DIFFERENCE IDENTITIES

Memorize:

Simply take the Sum Identities above and change every sign in sight!

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

(Make sure that the right side of your identity for $\sin(u+v)$ started with the $\sin u \cos v$ term!)

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Obtaining the Difference Identities from the Sum Identities:

Replace v with $(-v)$ and use the fact that sin and tan are odd, while cos is even.

For example,

$$\begin{aligned}\sin(u-v) &= \sin[u+(-v)] \\ &= \sin u \cos(-v) + \cos u \sin(-v) \\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

<u>GROUP 3a: DOUBLE-ANGLE (Think: Angle-Reducing, if *u* > 0) IDENTITIES</u>

Memorize:

(Also be prepared to recognize and know these "right-to-left.")

$$\sin(2u) = 2\sin u\cos u$$

<u>Think</u>: "Twice the product"

Reading "right-to-left," we have:

$$2\sin u\cos u = \sin(2u)$$

(This is helpful when simplifying.)

$$\cos(2u) = \cos^2 u - \sin^2 u$$

<u>Think</u>: "Cosines – Sines" (again)

Reading "right-to-left," we have:

$$\cos^2 u - \sin^2 u = \cos(2u)$$

Contrast this with the Pythagorean Identity:

$$\cos^2 u + \sin^2 u = 1$$

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

(Hard to memorize; we'll show how to obtain it.)

Notice that these identities are "angle-reducing" (if $u > 0$) in that they allow you to go from trigonometric functions of $(2u)$ to trigonometric functions of simply u.

Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace v with u, and simplify.

$$\begin{aligned}\sin(2u) &= \sin(u+u) \\ &= \sin u \cos u + \cos u \sin u \quad \text{(From Sum Identity)} \\ &= \sin u \cos u + \sin u \cos u \quad \text{(Like terms!!)} \\ &= 2 \sin u \cos u\end{aligned}$$

$$\begin{aligned}\cos(2u) &= \cos(u+u) \\ &= \cos u \cos u - \sin u \sin u \quad \text{(From Sum Identity)} \\ &= \cos^2 u - \sin^2 u\end{aligned}$$

$$\begin{aligned}\tan(2u) &= \tan(u+u) \\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad \text{(From Sum Identity)} \\ &= \frac{2 \tan u}{1 - \tan^2 u}\end{aligned}$$

This is a “last resort” if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the $\tan(2u)$ identity is harder to remember, you may prefer to remember the Sum Identity for $\tan(u+v)$ and then derive the $\tan(2u)$ identity this way.

If you’re quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.

GROUP 3b: DOUBLE-ANGLE IDENTITIES FOR cos

Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$:

Let's begin with the version we've already seen:

Version 1: $\cos(2u) = \cos^2 u - \sin^2 u$

Also know these two, from "left-to-right," and from "right-to-left":

Version 2: $\cos(2u) = 1 - 2\sin^2 u$

Version 3: $\cos(2u) = 2\cos^2 u - 1$

Obtaining Versions 2 and 3 from Version 1

It's tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ written in different ways.

To obtain Version 2, which contains $\sin^2 u$, we replace $\cos^2 u$ with $\left(1 - \sin^2 u\right)$.

$$\begin{aligned}
\cos\left(2u\right) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \underbrace{\left(1 - \sin^2 u\right)}_{\text{from Pythagorean Identity}} - \sin^2 u \\
&= 1 - \sin^2 u - \sin^2 u \\
&= 1 - 2\sin^2 u && (\Rightarrow \text{Version 2})
\end{aligned}$$

To obtain Version 3, which contains $\cos^2 u$, we replace $\sin^2 u$ with $\left(1 - \cos^2 u\right)$.

$$\begin{aligned}
\cos\left(2u\right) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \cos^2 u - \underbrace{\left(1 - \cos^2 u\right)}_{\text{from Pythagorean Identity}} \\
&= \cos^2 u - 1 + \cos^2 u \\
&= 2\cos^2 u - 1 && (\Rightarrow \text{Version 3})
\end{aligned}$$

GROUP 4: POWER-REDUCING IDENTITIES ("PRIs")

(These are called the "Half-Angle Formulas" in some books.)

Memorize:

$$\sin^2 u = \frac{1-\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2}-\frac{1}{2}\cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2}+\frac{1}{2}\cos(2u)$$

Then,

$$\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1-\cos(2u)}{1+\cos(2u)}$$

Actually, you just need to memorize one of the $\sin^2 u$ or $\cos^2 u$ identities and then switch the visible sign to get the other. Think: "sin" is "bad" or "negative"; this is a reminder that the minus sign belongs in the $\sin^2 u$ formula.

Obtaining the Power-Reducing Identities from the Double-Angle Identities for $\cos(2u)$

To obtain the identity for $\sin^2 u$, start with Version 2 of the $\cos(2u)$ identity:

$$\cos(2u) = 1 - 2\sin^2 u$$

Now, solve for $\sin^2 u$.

$$2\sin^2 u = 1-\cos(2u)$$

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

To obtain the identity for $\cos^2 u$, start with Version 3 of the $\cos(2u)$ identity:

$$\cos(2u) = 2\cos^2 u - 1$$

Now, switch sides and solve for $\cos^2 u$.

$$2\cos^2 u - 1 = \cos(2u)$$

$$2\cos^2 u = 1+\cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

GROUP 5: HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution $\theta = 2u$. (See **Obtaining …** below.)

The Identities:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

For a given θ, the choices among the $\pm$ signs depend on the Quadrant that $\frac{\theta}{2}$ lies in.

Here, the $\pm$ symbols indicate incomplete knowledge; unlike when we handle the Quadratic Formula, we do **not** take both signs for any of the above formulas for a given θ. There are no $\pm$ symbols in the last two $\tan\left(\frac{\theta}{2}\right)$ formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two $\tan\left(\frac{\theta}{2}\right)$ formulas: Keep either the numerator or the denominator of the radicand of the first formula, place $\sin\theta$ in the other part of the fraction, and remove the radical sign and the $\pm$ symbol.

Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):

For the $\sin\left(\frac{\theta}{2}\right)$ identity, we begin with the PRI:

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

Let $u = \frac{\theta}{2}$, or $\theta = 2u$.

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \text{(by the Square Root Method)}$$

Again, the choice among the $\pm$ signs depends on the Quadrant that $\frac{\theta}{2}$ lies in.

The story is similar for the $\cos\left(\frac{\theta}{2}\right)$ and the $\tan\left(\frac{\theta}{2}\right)$ identities.

What about the last two formulas for $\tan\left(\frac{\theta}{2}\right)$? The key trick is multiplication by trigonometric conjugates. For example:

$$\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}\cdot\frac{(1-\cos\theta)}{(1-\cos\theta)}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\
&= \pm\sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2} \\
&= \pm\left|\frac{1-\cos\theta}{\sin\theta}\right| \qquad \left(\text{because } \sqrt{a^2} = |a|\right)
\end{aligned}$$

Now, $1-\cos\theta \geq 0$ for all real θ, and $\tan\left(\frac{\theta}{2}\right)$ has the same sign as $\sin\theta$ (can you see why?), so …

$$= \frac{1-\cos\theta}{\sin\theta}$$

To get the third formula, use the numerator's (instead of the denominator's) trigonometric conjugate, $1+\cos\theta$, when multiplying into the numerator and the denominator of the radicand in the first few steps.

GROUP 6: PRODUCT-TO-SUM IDENTITIES

These can be verified from right-to-left using the Sum and Difference Identities.

The Identities:

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$
$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$
$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$
$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

GROUP 7: SUM-TO-PRODUCT IDENTITIES

These can be verified from right-to-left using the Product-To-Sum Identities.

The Identities:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$