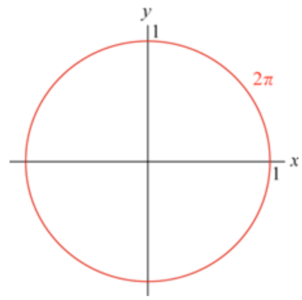
TOPIC 2: TRIGONOMETRY I

PART A: ANGLE MEASURES

Radian measure is more "mathematically natural" than degree measure, and it is typically assumed in calculus. In fact, **radian measure is assumed** if there are no units present.

There are 2π radians in a full (counterclockwise) revolution, because the entire unit circle (which has circumference 2π) is intercepted exactly once by such an angle.



There are 360° (360 degrees) in a full (counterclockwise) revolution.
(This is something of a cultural artifact; ancient Babylonians operated on a base-60 number system.)

2π radians is equivalent to 360°. Therefore, π radians is equivalent to 180°.
Either relationship may be used to construct conversion factors.

In any unit conversion, we effectively multiply by 1 in such a way that the old unit is canceled out.

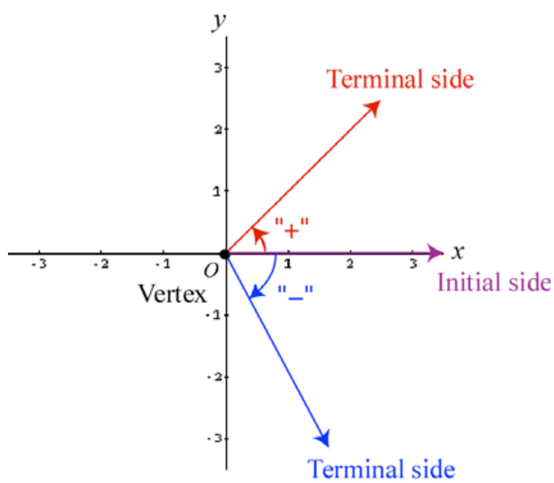
For example, to convert 45° into radians:

$$45^\circ = \left(45^\circ\right)\overbrace{\left(\frac{\pi\left[\text{radians}\right]}{180^\circ}\right)}^{=(1)} = \overbrace{\left(\cancel{45^\circ}\right)}^{(1)}\left(\frac{\pi\left[\text{radians}\right]}{\underbrace{\cancel{180^\circ}}_{(4)}}\right) = \frac{\pi}{4}\left[\text{radians}\right].$$

PART B: QUADRANTS AND QUADRANTAL ANGLES

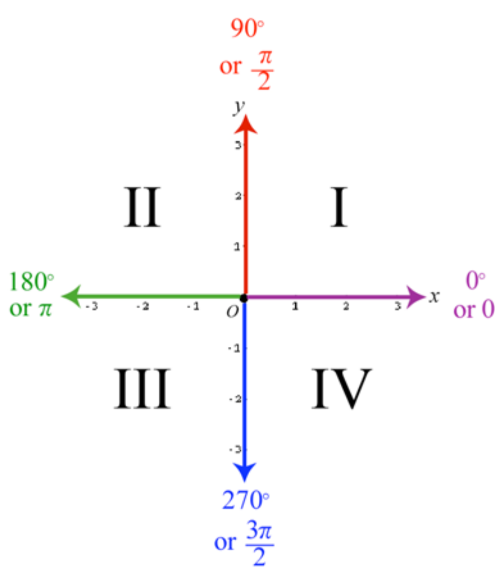
The x- and y-axes divide the xy-plane into 4 quadrants.
Quadrant I is the upper right quadrant; the others are numbered in counterclockwise order.

A standard angle in standard position has the positive [really, nonnegative] x-axis as its initial side and the origin as its vertex. We say that the angle lies in the quadrant that its terminal side shoots through. For example, in the figure below, the positive standard angle with the red terminal side is a Quadrant I angle:



A standard angle whose terminal side lies on the x- or y-axis is called a quadrantal angle. Quadrantal angles correspond to "integer multiples" of 90° or $\dfrac{\pi}{2}$ radians.

The quadrants and some quadrantal angles are below.
(For convenience, we may label a standard angle by labeling its terminal side.)



PART C: COTERMINAL ANGLES

Standard angles that share the same terminal side are called coterminal angles. They differ by an integer number of full revolutions counterclockwise or clockwise.

If the angle θ is measured in **radians**, then its **coterminal angles** are of the form: $\theta + 2\pi n$, where n is any integer $\left(n \in \mathbb{Z}\right)$.

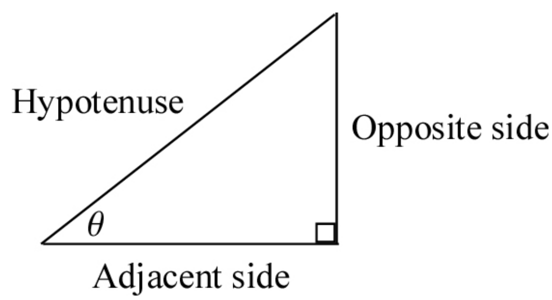
If the angle θ is measured in **degrees**, then its **coterminal angles** are of the form: $\theta + 360n^\circ$, where n is any integer $\left(n \in \mathbb{Z}\right)$.

Note: Since n could be negative, the "+" sign is sufficient in the above forms, as opposed to "$\pm$."

PART D: TRIGONOMETRIC FUNCTIONS: THE RIGHT TRIANGLE APPROACH

The Setup

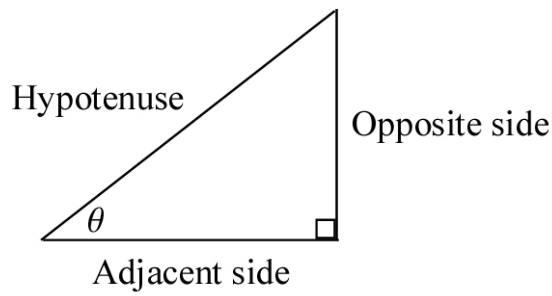
The **acute** angles of a right triangle are **complementary**. Consider such an angle, θ. Relative to θ, we may label the sides as follows:



The hypotenuse always faces the right angle, and it is always the longest side.

The other two sides are the legs. The opposite side (relative to θ) faces the θ angle. The other leg is the adjacent side (relative to θ).

Defining the Six Basic Trig Functions (where θ is acute)



"SOH-CAH-TOA"

$$\text{Sine } \theta = \sin\theta = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\text{Cosine } \theta = \cos\theta = \frac{\text{Adj.}}{\text{Hyp.}}$$

$$\text{Tangent } \theta = \tan\theta = \frac{\text{Opp.}}{\text{Adj.}}$$

Reciprocal Identities

$$\text{Cosecant } \theta = \csc\theta = \frac{1}{\sin\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Opp.}} \right)$$

$$\text{Secant } \theta = \sec\theta = \frac{1}{\cos\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Adj.}} \right)$$

$$\text{Cotangent } \theta = \cot\theta = \frac{1}{\tan\theta} \quad \left(= \frac{\text{Adj.}}{\text{Opp.}} \right)$$

WARNING 1: Remember that the reciprocal of $\sin\theta$ is $\csc\theta$, not $\sec\theta$.

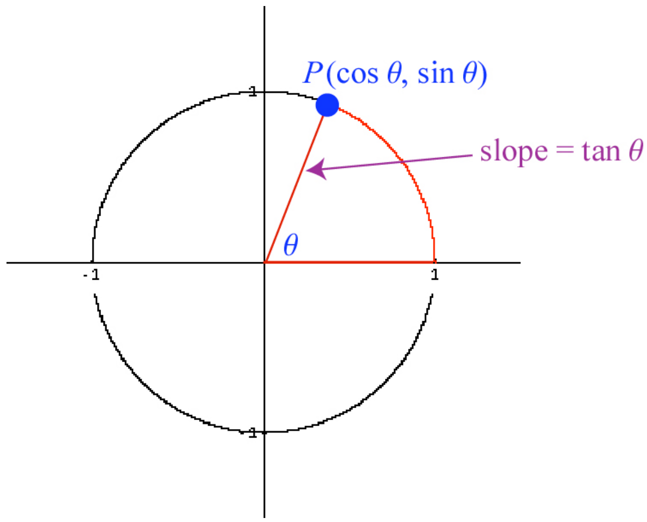
TIP 1: We informally treat "0" and "undefined" as **reciprocals** when we are dealing with basic **trigonometric** functions. Your algebra teacher will not want to hear this, though!

PART E: TRIGONOMETRIC FUNCTIONS: THE UNIT CIRCLE APPROACH

The Setup

Consider a standard angle θ measured in radians (or, equivalently, let θ represent a real number).

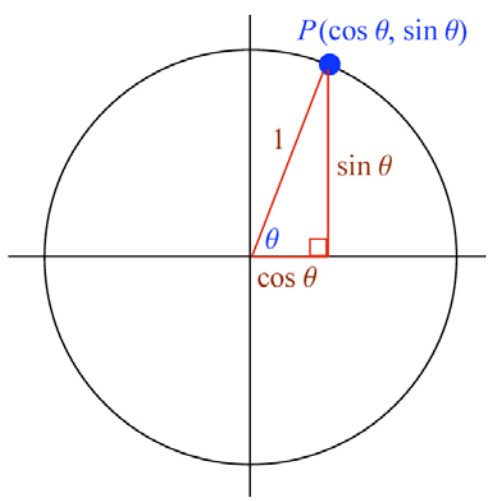
The point $P(\cos\theta, \sin\theta)$ is the **intersection point** between the **terminal side** of the angle and the **unit circle** centered at the origin. The **slope** of the terminal side is, in fact, $\tan\theta$.



Note: The intercepted arc along the circle (in red) has arc length θ.

The figure below demonstrates how this is consistent with the SOH-CAH-TOA (or Right Triangle) approach. Observe:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{rise}}{\text{run}} = \text{slope of terminal side}$$



"THE Table"

We will use our knowledge of the 30°-60°-90° and 45°-45°-90° **special triangles** to construct "THE Table" below. The **unit circle** approach is used to find the trigonometric values for **quadrantal** angles such as 0° and 90°.

Key Angles θ: Degrees, (Radians)	$\sin\theta$	$\cos\theta$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	Intersection Point $P(\cos\theta, \sin\theta)$
0°, (0)	$\dfrac{\sqrt{0}}{2} = \mathbf{0}$	$\mathbf{1}$	$\dfrac{0}{1} = \mathbf{0}$	$(1, 0)$
30°, $\left(\dfrac{\pi}{6}\right)$	$\dfrac{\sqrt{1}}{2} = \dfrac{\mathbf{1}}{\mathbf{2}}$	$\dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{3}}$	$\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$
45°, $\left(\dfrac{\pi}{4}\right)$	$\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\dfrac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\dfrac{\sqrt{2}/2}{\sqrt{2}/2} = \mathbf{1}$	$\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$
60°, $\left(\dfrac{\pi}{3}\right)$	$\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\dfrac{\mathbf{1}}{\mathbf{2}}$	$\dfrac{\sqrt{3}/2}{1/2} = \sqrt{\mathbf{3}}$	$\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$
90°, $\left(\dfrac{\pi}{2}\right)$	$\dfrac{\sqrt{4}}{2} = \mathbf{1}$	$\mathbf{0}$	$\dfrac{1}{0}$ is **undefined**	$(0, 1)$

WARNING 2: $\dfrac{\pi}{5}$ is **not** a "special" angle.

WARNING 3: Always make sure what **mode** your **calculator** is in (DEG vs. RAD) whenever you evaluate trigonometric functions.

The values for the **reciprocals**, $\csc\theta$, $\sec\theta$, and $\cot\theta$, are then readily found. Remember that it is sometimes better to take a trigonometric value where the denominator is **not rationalized** before taking its **reciprocal**. For example, because $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$, we know immediately that $\cot 30^\circ = \sqrt{3}$.

Observe:

• The pattern in the $\sin\theta$ column

Technical Note: An explanation for this pattern appears in the Sept. 2004 issue of the College Mathematics Journal (p.302).

• The fact that the $\sin\theta$ column is **reversed** to form the $\cos\theta$ column. This is due to the Cofunction Identities (or the Pythagorean Identities).

• As θ increases from 0° to 90° (i.e., from 0 to $\frac{\pi}{2}$ radians),

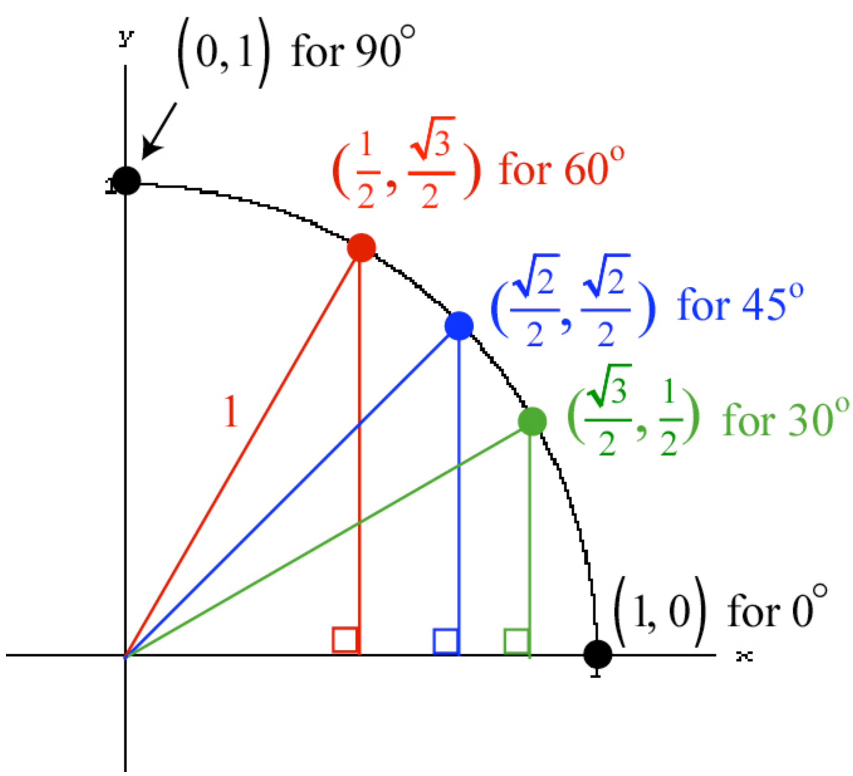
•• $\sin\theta$ (the ***y*-coordinate** of P) **increases** from 0 to 1.

Note: This is more obvious using the **Unit Circle** approach instead of the Right Triangle approach.

•• $\cos\theta$ (the ***x*-coordinate** of P) **decreases** from 1 to 0.

•• $\tan\theta$ (the **slope** of the terminal side) starts at 0, **increases**, and approaches ∞.

• Here is the "Big Picture." Remember that each **intersection point** is of the form $P(\cos\theta, \sin\theta)$.



PART F: EXTENDING FROM QUADRANT I TO OTHER QUADRANTS

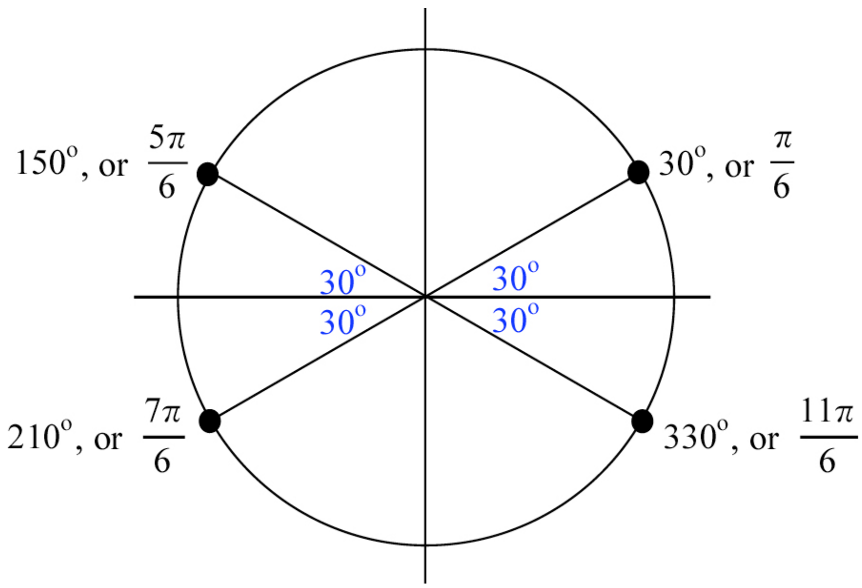
Reference angles

The reference angle for a non-quadrantal standard angle is the **acute** angle that its **terminal side** makes with the ***x*-axis**.

We will informally call angles that have the **same reference angle** "coreference angles," which is not a standard term.

• We may informally think of "coreference angles" as "brothers" and coterminal angles as "twins" (although an angle has infinitely many of them).

For example, the "coreference angles" below share the same reference angle, namely 30°, or $\frac{\pi}{6}$ radians.



Coterminal angles are also "coreference angles." For example, -30° (or $-\frac{\pi}{6}$ radians) is a coterminal "twin" for the 330° (or $\frac{11\pi}{6}$ radian) angle.

We will extend the following patterns for "coreference angles" of $\frac{\pi}{6}$ to other "families" of radian measures:

Quadrant II: $\frac{5\pi}{6}$; observe that 5 is **1 less than** 6.

Quadrant III: $\frac{7\pi}{6}$; observe that 7 is **1 more than** 6.

Quadrant IV: $\frac{11\pi}{6}$; observe that 11 is **1 less than twice** 6.

Key "coreference angles" of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ are below.

We've already seen some for $\frac{\pi}{6}$:
(The boxes correspond to Quadrants.)

$\frac{5\pi}{6}$	$\frac{\pi}{6}$
$\frac{7\pi}{6}$	$\frac{11\pi}{6}$

Now, $\frac{\pi}{4}$:

$\frac{3\pi}{4}$	$\frac{\pi}{4}$
$\frac{5\pi}{4}$	$\frac{7\pi}{4}$

Now, $\frac{\pi}{3}$:

$\frac{2\pi}{3}$	$\frac{\pi}{3}$
$\frac{4\pi}{3}$	$\frac{5\pi}{3}$

Why are "Coreference Angles" Useful?

Coterminal angles have the **same** basic trigonometric values, including the signs.

"Coreference angles" have the **same** basic trigonometric values **up to** (except maybe for) the **signs**.

Signs of Basic Trigonometric Values in Quadrants

Remember that **reciprocal** values have the **same sign** (or one is 0 and the other is undefined).

"ASTC" Rule for Signs

Think: "**A**ll **S**tudents **T**ake **C**alculus"

Start in Quadrant I and progress **counterclockwise** through the Quadrants:

S	A
T	C

All six basic trigonometric functions are **positive** in Quadrant I.
(They are all positive for acute angles.)

Sine and its reciprocal, Cosecant, are **positive** in Quadrant II.
(The **other** four functions are **negative**.)

Tangent and its reciprocal, Cotangent, are **positive** in Quadrant III.

Cosine and its reciprocal, Secant, are **positive** in Quadrant IV.

For example, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$, because $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, and $\frac{7\pi}{6}$ is in Quadrant III.