TOPIC 3: TRIGONOMETRY II

PART A: FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Memorize these in both "directions" (i.e., left-to-right and right-to-left).

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sin x = \frac{1}{\csc x}$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x = \frac{1}{\sec x}$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

WARNING 1: Remember that the reciprocal of $\sin x$ is $\csc x$, not $\sec x$.

TIP 1: We informally treat "0" and "undefined" as **reciprocals** when we are dealing with basic **trigonometric** functions. Your algebra teacher will not want to hear this, though!

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

TIP 2: The second and third Pythagorean Identities can be obtained from the first by dividing both of its sides by $\sin^2 x$ and $\cos^2 x$, respectively.

TIP 3: The squares of $\csc x$ and $\sec x$, which have "Up-U, Down-U" graphs, are all alone on the right sides of the last two identities. They can never be 0 in value. (Why is that? Look at the left sides.)

Cofunction Identities

If x is measured in radians, then:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

We have analogous relationships for tangent and cotangent, and for secant and cosecant; remember that they are sometimes undefined.

Think: **Cofunctions of complementary angles are equal.**

Even / Odd (or Negative Angle) Identities

Among the six basic trigonometric functions, only cosine (and its reciprocal, secant) are **even**:

$$\cos(-x) = \cos x$$

$$\sec(-x) = \sec x$$

However, the other four are **odd**:

$$\sin(-x) = -\sin x$$

$$\csc(-x) = -\csc x$$

$$\tan(-x) = -\tan x$$

$$\cot(-x) = -\cot x$$

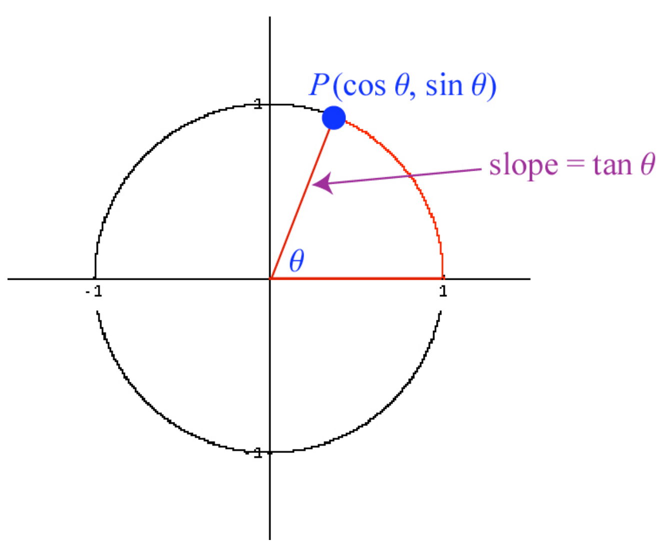
• If f is an **even** function, then the graph of $y = f(x)$ is symmetric about the ***y*-axis**.

• If f is an **odd** function, then the graph of $y = f(x)$ is symmetric about the **origin**.

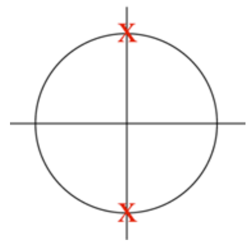
PART B: DOMAINS AND RANGES OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$f(x)$	Domain	Range
$\sin x$	$(-\infty, \infty)$	$[-1, 1]$
$\cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z}) \right\}$	$(-\infty, \infty)$
$\csc x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \pi n \ (n \in \mathbb{Z}) \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z}) \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	Set-builder form: $\left\{ x \in \mathbb{R} \,\middle|\, x \neq \pi n \ (n \in \mathbb{Z}) \right\}$	$(-\infty, \infty)$

• The **unit circle** approach explains the domain and range for sine and cosine, as well as the range for tangent (since any real number can be a **slope**).

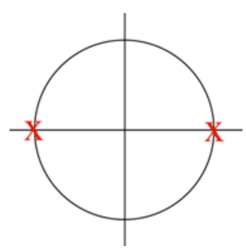


• Domain for tangent: The "**X**"s on the unit circle below correspond to an **undefined slope**. Therefore, the corresponding real numbers (the corresponding angle measures in radians) are **excluded** from the domain.



• Domain for tangent and secant: The "**X**"s on the unit circle above also correspond to a cosine value of 0. By the Quotient Identity for tangent $\left(\tan\theta = \frac{\sin\theta}{\cos\theta} \right)$ and the Reciprocal Identity for secant $\left(\sec\theta = \frac{1}{\cos\theta} \right)$, we **exclude** the corresponding radian measures from the domains of both functions.

• Domain for cotangent and cosecant: The "**X**"s on the unit circle below correspond to a sine value of 0. By the Quotient Identity for cotangent $\left(\cot\theta = \frac{\cos\theta}{\sin\theta} \right)$ and the Reciprocal Identity for cosecant $\left(\csc\theta = \frac{1}{\sin\theta} \right)$, we **exclude** the corresponding radian measures from the domains of both functions.



• Range for cosecant and secant: We turn "inside out" the range for both sine and cosine, which is $\left[-1, 1\right]$.

• Range for cotangent: This is explained by the fact that the range for tangent is $\left(-\infty, \infty\right)$ and the Reciprocal Identity for cotangent: $\left(\cot\theta = \frac{1}{\tan\theta} \right)$. $\cot\theta$ is 0 in value $\Leftrightarrow$ $\tan\theta$ is undefined.

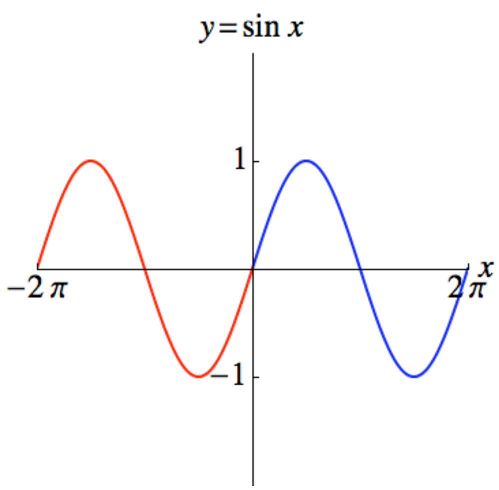
PART C: GRAPHS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

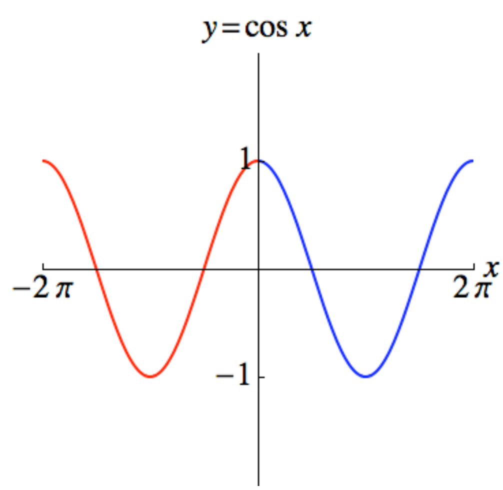
• The six basic trigonometric functions are periodic, so their graphs can be decomposed into cycles that repeat like wallpaper patterns. The period for tangent and cotangent is π; it is 2π for the others.

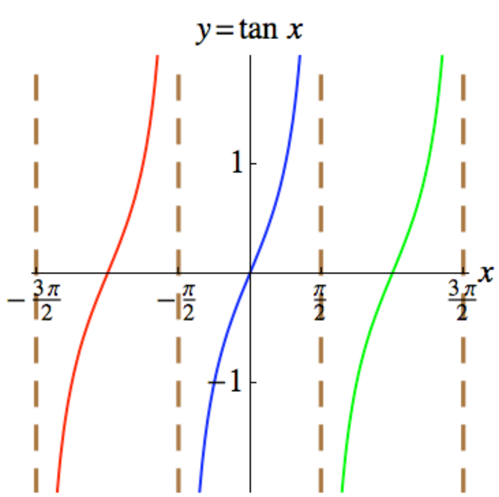
• A vertical asymptote ("VA") is a vertical line that a graph approaches in an "explosive" sense. (This idea will be made more precise in Section 2.4.) VAs on the graph of a basic trigonometric function correspond to **exclusions from the domain**. They are graphed as dashed lines.

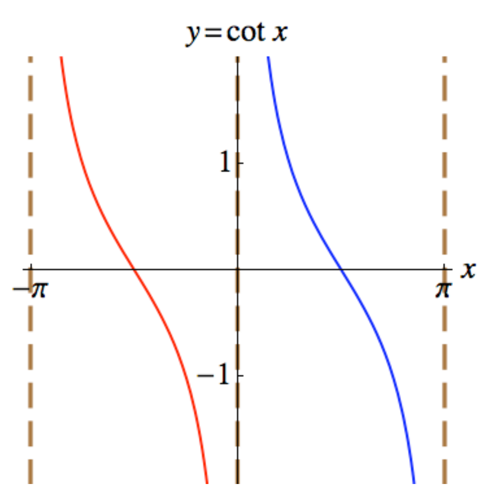
• Remember that the **domain** of a function f corresponds to the ***x*-coordinates** picked up by the graph of $y = f(x)$, and the **range** corresponds to the ***y*-coordinates**.

• Remember that cosine and secant are the only **even** functions among the six, so their graphs are **symmetric about the *y*-axis**. The other four are **odd**, so their graphs are **symmetric about the origin**.

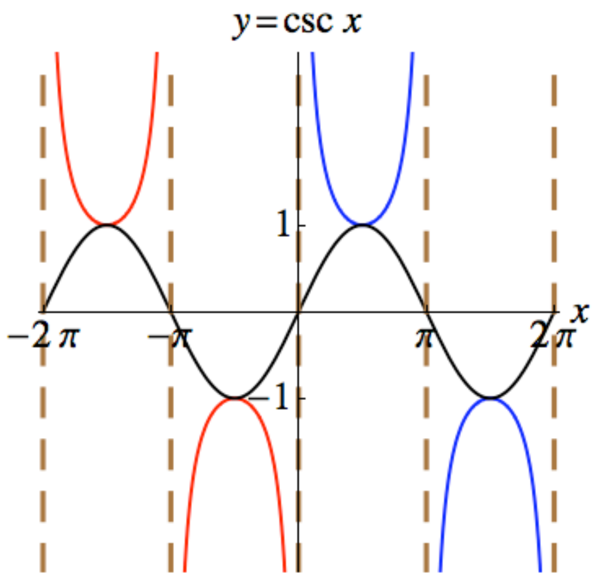


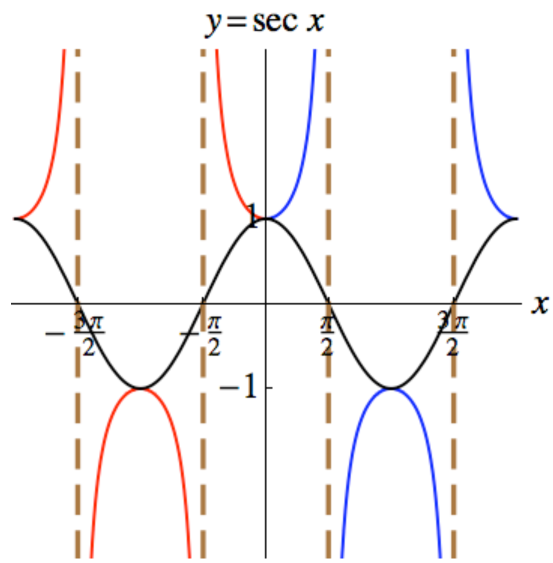






• We use the graphs of $y = \sin x$ and $y = \cos x$ (in black in the figures below) as **guide graphs** to help us graph $y = \csc x$ and $y = \sec x$.





Relationships between the graphs of $y = \csc x$ and $y = \sin x$ (and between the graphs of $y = \sec x$ and $y = \cos x$):

•• The **VAs** on the graph of $y = \csc x$ are drawn through the ***x*-intercepts** of the graph of $y = \sin x$. This is because $\csc x$ is undefined $\Leftrightarrow$ $\sin x = 0$.

•• The reciprocals of 1 and -1 are themselves, so $\csc x$ and $\sin x$ take on each of those values simultaneously. This explains how their graphs **intersect**.

•• Because sine and cosecant are **reciprocal** functions, we know that, between the VAs in the graph of $y = \csc x$, they share the **same sign**, and **one increases** $\Leftrightarrow$ **the other decreases**.

PART D: SOLVING TRIGONOMETRIC EQUATIONS

Example 1 (Solving a Trigonometric Equation)

Solve: $2\sin(4x) = -\sqrt{3}$

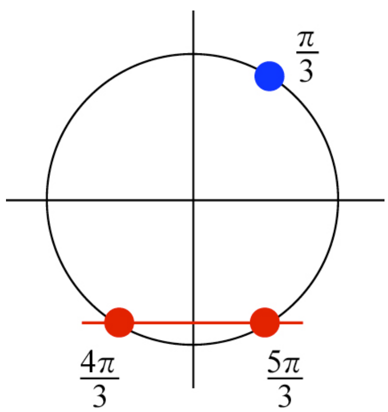
§ Solution

$2\sin(4x) = -\sqrt{3}$ Isolate the sine expression.

$\sin\underbrace{(4x)}_{=\theta} = -\dfrac{\sqrt{3}}{2}$ Substitution: Let $\theta = 4x$.

$\sin\theta = -\dfrac{\sqrt{3}}{2}$ We will now solve this equation for θ.

Observe that $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$, so $\dfrac{\pi}{3}$ will be the **reference angle** for our solutions for θ. Since $-\dfrac{\sqrt{3}}{2}$ is a **negative** sine value, we want **"coreference angles"** of $\dfrac{\pi}{3}$ in **Quadrants III and IV**.



Our solutions for θ are:

$$\theta = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

From this point on, it is a matter of algebra.

To find our solutions for *x*, replace θ with $4x$, and solve for *x*.

$$4x = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad 4x = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

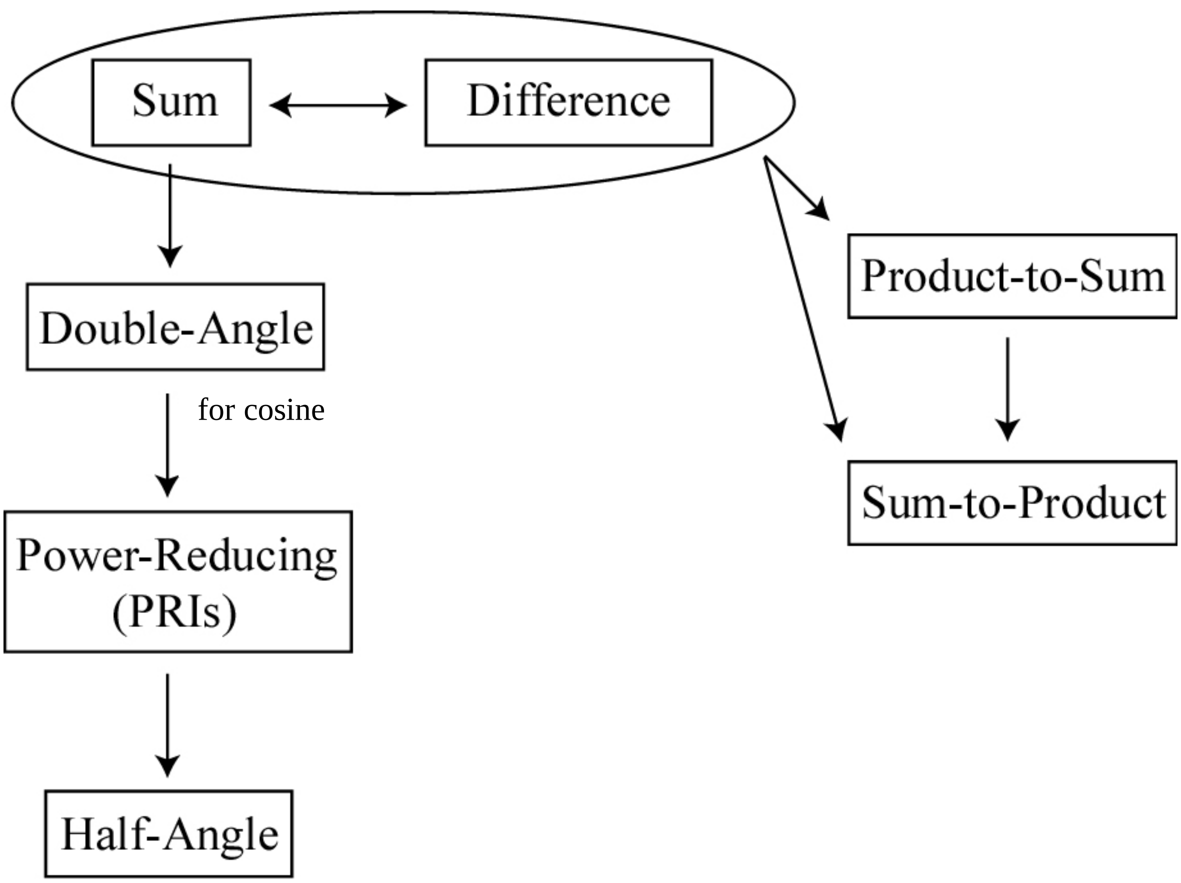
$$x = \frac{1}{4}\left(\frac{4\pi}{3}\right) + \frac{2\pi}{4} n, \quad \text{or} \quad x = \frac{1}{4}\left(\frac{5\pi}{3}\right) + \frac{2\pi}{4} n \quad (n \in \mathbb{Z})$$

$$x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \in \mathbb{Z})$$

Solution set: $\left\{ x \in \mathbb{R} \;\middle|\; x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \in \mathbb{Z}) \right\}$. §

PART E: ADVANCED TRIGONOMETRIC IDENTITIES

These identities may be derived according to the flowchart below.



GROUP 1: SUM IDENTITIES

Memorize:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

Think: "Sum of the mixed-up products"
(Multiplication and addition are commutative, but start with the $\sin u \cos v$ term in anticipation of the Difference Identities.)

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Think: "Cosines [product] – Sines [product]"

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

Think: $"\dfrac{\text{Sum}}{1 - \text{Product}}"$

GROUP 2: DIFFERENCE IDENTITIES

Memorize:

Simply take the Sum Identities above and change every sign in sight!

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

(Make sure that the right side of your identity for $\sin(u+v)$ started with the $\sin u \cos v$ term!)

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Obtaining the Difference Identities from the Sum Identities:

Replace v with $(-v)$ and use the fact that sin and tan are odd, while cos is even.

For example,

$$\begin{aligned} \sin(u-v) &= \sin[u+(-v)] \\ &= \sin u \cos(-v) + \cos u \sin(-v) \\ &= \sin u \cos v - \cos u \sin v \end{aligned}$$

<u>GROUP 3a: DOUBLE-ANGLE (Think: Angle-Reducing, if *u* > 0) IDENTITIES</u>

Memorize:

(Also be prepared to recognize and know these "right-to-left.")

$$\sin(2u) = 2\sin u\cos u$$

<u>Think</u>: "Twice the product"

Reading "right-to-left," we have:

$$2\sin u\cos u = \sin(2u)$$

(This is helpful when simplifying.)

$$\cos(2u) = \cos^2 u - \sin^2 u$$

<u>Think</u>: "Cosines – Sines" (again)

Reading "right-to-left," we have:

$$\cos^2 u - \sin^2 u = \cos(2u)$$

Contrast this with the Pythagorean Identity:

$$\cos^2 u + \sin^2 u = 1$$

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

(Hard to memorize; we'll show how to obtain it.)

Notice that these identities are "angle-reducing" (if $u > 0$) in that they allow you to go from trigonometric functions of $(2u)$ to trigonometric functions of simply u.

Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace *v* with *u*, and simplify.

$$\begin{aligned} \sin(2u) &= \sin(u+u) \\ &= \sin u \cos u + \cos u \sin u \quad \text{(From Sum Identity)} \\ &= \sin u \cos u + \sin u \cos u \quad \text{(Like terms!!)} \\ &= 2 \sin u \cos u \end{aligned}$$

$$\begin{aligned} \cos(2u) &= \cos(u+u) \\ &= \cos u \cos u - \sin u \sin u \quad \text{(From Sum Identity)} \\ &= \cos^2 u - \sin^2 u \end{aligned}$$

$$\begin{aligned} \tan(2u) &= \tan(u+u) \\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad \text{(From Sum Identity)} \\ &= \frac{2 \tan u}{1 - \tan^2 u} \end{aligned}$$

This is a "last resort" if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the $\tan(2u)$ identity is harder to remember, you may prefer to remember the Sum Identity for $\tan(u+v)$ and then derive the $\tan(2u)$ identity this way.

If you're quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.

GROUP 3b: DOUBLE-ANGLE IDENTITIES FOR cos

Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$:

Let's begin with the version we've already seen:

Version 1: $\cos(2u) = \cos^2 u - \sin^2 u$

Also know these two, from "left-to-right," and from "right-to-left":

Version 2: $\cos(2u) = 1 - 2\sin^2 u$

Version 3: $\cos(2u) = 2\cos^2 u - 1$

Obtaining Versions 2 and 3 from Version 1

It's tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ written in different ways.

To obtain Version 2, which contains $\sin^2 u$, we replace $\cos^2 u$ with $\left(1 - \sin^2 u\right)$.

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \underbrace{\left(1 - \sin^2 u\right)}_{\text{from Pythagorean Identity}} - \sin^2 u \\
&= 1 - \sin^2 u - \sin^2 u \\
&= 1 - 2\sin^2 u && (\Rightarrow \text{Version 2})
\end{aligned}$$

To obtain Version 3, which contains $\cos^2 u$, we replace $\sin^2 u$ with $\left(1 - \cos^2 u\right)$.

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \cos^2 u - \underbrace{\left(1 - \cos^2 u\right)}_{\text{from Pythagorean Identity}} \\
&= \cos^2 u - 1 + \cos^2 u \\
&= 2\cos^2 u - 1 && (\Rightarrow \text{Version 3})
\end{aligned}$$

GROUP 4: POWER-REDUCING IDENTITIES ("PRIs")

(These are called the "Half-Angle Formulas" in some books.)

Memorize:

$$\sin^2 u = \frac{1-\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} - \frac{1}{2}\cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} + \frac{1}{2}\cos(2u)$$

Then,

$$\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1-\cos(2u)}{1+\cos(2u)}$$

Actually, you just need to memorize one of the $\sin^2 u$ or $\cos^2 u$ identities and then switch the visible sign to get the other. Think: "sin" is "bad" or "negative"; this is a reminder that the minus sign belongs in the $\sin^2 u$ formula.

Obtaining the Power-Reducing Identities from the Double-Angle Identities for $\cos(2u)$

To obtain the identity for $\sin^2 u$, start with Version 2 of the $\cos(2u)$ identity:

$$\cos(2u) = 1 - 2\sin^2 u$$

Now, solve for $\sin^2 u$.

$$2\sin^2 u = 1 - \cos(2u)$$

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

To obtain the identity for $\cos^2 u$, start with Version 3 of the $\cos(2u)$ identity:

$$\cos(2u) = 2\cos^2 u - 1$$

Now, switch sides and solve for $\cos^2 u$.

$$2\cos^2 u - 1 = \cos(2u)$$

$$2\cos^2 u = 1 + \cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

GROUP 5: HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution $\theta = 2u$. (See **Obtaining …** below.)

The Identities:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

For a given θ, the choices among the $\pm$ signs depend on the Quadrant that $\frac{\theta}{2}$ lies in.

Here, the $\pm$ symbols indicate incomplete knowledge; unlike when we handle the Quadratic Formula, we do **not** take both signs for any of the above formulas for a given θ. There are no $\pm$ symbols in the last two $\tan\left(\frac{\theta}{2}\right)$ formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two $\tan\left(\frac{\theta}{2}\right)$ formulas: Keep either the numerator or the denominator of the radicand of the first formula, place $\sin\theta$ in the other part of the fraction, and remove the radical sign and the $\pm$ symbol.

Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):

For the $\sin\left(\frac{\theta}{2}\right)$ identity, we begin with the PRI:

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

Let $u = \frac{\theta}{2}$, or $\theta = 2u$.

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \text{(by the Square Root Method)}$$

Again, the choice among the $\pm$ signs depends on the Quadrant that $\frac{\theta}{2}$ lies in.

The story is similar for the $\cos\left(\frac{\theta}{2}\right)$ and the $\tan\left(\frac{\theta}{2}\right)$ identities.

What about the last two formulas for $\tan\left(\frac{\theta}{2}\right)$? The key trick is multiplication by trigonometric conjugates. For example:

$$\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}\cdot\frac{(1-\cos\theta)}{(1-\cos\theta)}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\
&= \pm\sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2} \\
&= \pm\left|\frac{1-\cos\theta}{\sin\theta}\right| \qquad \left(\text{because } \sqrt{a^2} = |a|\right)
\end{aligned}$$

Now, $1-\cos\theta \ge 0$ for all real θ, and $\tan\left(\frac{\theta}{2}\right)$ has the same sign as $\sin\theta$ (can you see why?), so …

$$= \frac{1-\cos\theta}{\sin\theta}$$

To get the third formula, use the numerator's (instead of the denominator's) trigonometric conjugate, $1+\cos\theta$, when multiplying into the numerator and the denominator of the radicand in the first few steps.

GROUP 6: PRODUCT-TO-SUM IDENTITIES

These can be verified from right-to-left using the Sum and Difference Identities.

The Identities:

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$
$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$
$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$
$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

GROUP 7: SUM-TO-PRODUCT IDENTITIES

These can be verified from right-to-left using the Product-To-Sum Identities.

The Identities:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$