TOPIC 3: TRIGONOMETRY II

PART A: FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Memorize these in both “directions” (i.e., left-to-right and right-to-left).

<table>
<thead>
<tr>
<th>Reciprocal Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \csc x = \frac{1}{\sin x} )</td>
</tr>
<tr>
<td>( \sec x = \frac{1}{\cos x} )</td>
</tr>
<tr>
<td>( \cot x = \frac{1}{\tan x} )</td>
</tr>
</tbody>
</table>

**WARNING 1:** Remember that the reciprocal of \( \sin x \) is \( \csc x \), not \( \sec x \).

**TIP 1:** We informally treat “0” and “undefined” as reciprocals when we are dealing with basic trigonometric functions. Your algebra teacher will not want to hear this, though!

<table>
<thead>
<tr>
<th>Quotient Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan x = \frac{\sin x}{\cos x} ) and ( \cot x = \frac{\cos x}{\sin x} )</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Pythagorean Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^2 x + \cos^2 x = 1 )</td>
</tr>
<tr>
<td>( 1 + \cot^2 x = \csc^2 x )</td>
</tr>
<tr>
<td>( \tan^2 x + 1 = \sec^2 x )</td>
</tr>
</tbody>
</table>

**TIP 2:** The second and third Pythagorean Identities can be obtained from the first by dividing both of its sides by \( \sin^2 x \) and \( \cos^2 x \), respectively.

**TIP 3:** The squares of \( \csc x \) and \( \sec x \), which have “Up-U, Down-U” graphs, are all alone on the right sides of the last two identities. They can never be 0 in value. (Why is that? Look at the left sides.)
Cofunction Identities

If \( x \) is measured in radians, then:

\[
\sin x = \cos \left( \frac{\pi}{2} - x \right)
\]

\[
\cos x = \sin \left( \frac{\pi}{2} - x \right)
\]

We have analogous relationships for tangent and cotangent, and for secant and cosecant; remember that they are sometimes undefined.

Think: Cofunctions of complementary angles are equal.

Even / Odd (or Negative Angle) Identities

Among the six basic trigonometric functions, only cosine (and its reciprocal, secant) are even:

\[
\cos(-x) = \cos x \\
\sec(-x) = \sec x
\]

However, the other four are odd:

\[
\sin(-x) = -\sin x \\
\csc(-x) = -\csc x \\
\tan(-x) = -\tan x \\
\cot(-x) = -\cot x
\]

• If \( f \) is an even function, then the graph of \( y = f(x) \) is symmetric about the \( y \)-axis.

• If \( f \) is an odd function, then the graph of \( y = f(x) \) is symmetric about the origin.
PART B: DOMAINS AND RANGES OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x$</td>
<td>$(-\infty, \infty)$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$(-\infty, \infty)$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>Set-builder form: $\left{ x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z}) \right}$</td>
<td>$(-\infty, \infty)$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>Set-builder form: $\left{ x \in \mathbb{R} \mid x \neq \pi n \ (n \in \mathbb{Z}) \right}$</td>
<td>$(-\infty, -1) \cup [1, \infty)$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>Set-builder form: $\left{ x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + \pi n \ (n \in \mathbb{Z}) \right}$</td>
<td>$(-\infty, -1) \cup [1, \infty)$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>Set-builder form: $\left{ x \in \mathbb{R} \mid x \neq \pi n \ (n \in \mathbb{Z}) \right}$</td>
<td>$(-\infty, \infty)$</td>
</tr>
</tbody>
</table>

- The **unit circle** approach explains the domain and range for sine and cosine, as well as the range for tangent (since any real number can be a **slope**).
• Domain for tangent: The “X”s on the unit circle below correspond to an **undefined slope**. Therefore, the corresponding real numbers (the corresponding angle measures in radians) are **excluded** from the domain.

![](image)

• Domain for tangent and secant: The “X”s on the unit circle above also correspond to a cosine value of 0. By the Quotient Identity for tangent \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) and the Reciprocal Identity for secant \( \sec \theta = \frac{1}{\cos \theta} \), we **exclude** the corresponding radian measures from the domains of both functions.

![](image)

• Domain for cotangent and cosecant: The “X”s on the unit circle below correspond to a sine value of 0. By the Quotient Identity for cotangent \( \cot \theta = \frac{\cos \theta}{\sin \theta} \) and the Reciprocal Identity for cosecant \( \csc \theta = \frac{1}{\sin \theta} \), we **exclude** the corresponding radian measures from the domains of both functions.

![](image)

• Range for cosecant and secant: We turn “inside out” the range for both sine and cosine, which is \([-1, 1]\).

• Range for cotangent: This is explained by the fact that the range for tangent is \((-\infty, \infty)\) and the Reciprocal Identity for cotangent: \( \cot \theta = \frac{1}{\tan \theta} \). \( \cot \theta \) is 0 in value \( \Leftrightarrow \) \( \tan \theta \) is undefined.
PART C: GRAPHS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

- The six basic trigonometric functions are periodic, so their graphs can be decomposed into cycles that repeat like wallpaper patterns. The period for tangent and cotangent is $\pi$; it is $2\pi$ for the others.

- A vertical asymptote (“VA”) is a vertical line that a graph approaches in an “explosive” sense. (This idea will be made more precise in Section 2.4.) VAs on the graph of a basic trigonometric function correspond to exclusions from the domain. They are graphed as dashed lines.

- Remember that the domain of a function $f$ corresponds to the $x$-coordinates picked up by the graph of $y = f(x)$, and the range corresponds to the $y$-coordinates.

- Remember that cosine and secant are the only even functions among the six, so their graphs are symmetric about the $y$-axis. The other four are odd, so their graphs are symmetric about the origin.
• We use the graphs of $y = \sin x$ and $y = \cos x$ (in black in the figures below) as guide graphs to help us graph $y = \csc x$ and $y = \sec x$.

![Graphs of $y = \csc x$ and $y = \sec x$](image)

Relationships between the graphs of $y = \csc x$ and $y = \sin x$ (and between the graphs of $y = \sec x$ and $y = \cos x$):

• The VAs on the graph of $y = \csc x$ are drawn through the $x$-intercepts of the graph of $y = \sin x$. This is because $\csc x$ is undefined $\iff \sin x = 0$.

• The reciprocals of 1 and $-1$ are themselves, so $\csc x$ and $\sin x$ take on each of those values simultaneously. This explains how their graphs intersect.

• Because sine and cosecant are reciprocal functions, we know that, between the VAs in the graph of $y = \csc x$, they share the same sign, and one increases $\iff$ the other decreases.
PART D: SOLVING TRIGONOMETRIC EQUATIONS

Example 1 (Solving a Trigonometric Equation)

Solve: \(2 \sin(4x) = -\sqrt{3}\)

\[\text{Solution}\]

\[2 \sin(4x) = -\sqrt{3}\] Isolate the sine expression.

\[\sin(4x) = -\frac{\sqrt{3}}{2}\] Substitution: Let \(\theta = 4x\).

\[\sin \theta = -\frac{\sqrt{3}}{2}\] We will now solve this equation for \(\theta\).

Observe that \(\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}\), so \(\frac{\pi}{3}\) will be the reference angle for our solutions for \(\theta\). Since \(-\frac{\sqrt{3}}{2}\) is a negative sine value, we want “coreference angles” of \(\frac{\pi}{3}\) in Quadrants III and IV.

Our solutions for \(\theta\) are:

\[\theta = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})\]

From this point on, it is a matter of algebra.
To find our solutions for $x$, replace $\theta$ with $4x$, and solve for $x$.

$$4x = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad 4x = \frac{5\pi}{3} + 2\pi n \quad (n \in \mathbb{Z})$$

$$x = \frac{1}{4} \left( \frac{4\pi}{3} \right) + \frac{2\pi}{4} n, \quad \text{or} \quad x = \frac{1}{4} \left( \frac{5\pi}{3} \right) + \frac{2\pi}{4} n \quad (n \in \mathbb{Z})$$

$$x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \in \mathbb{Z})$$

Solution set: \( \left\{ x \in \mathbb{R} \mid x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \in \mathbb{Z}) \right\} \).

**PART E: ADVANCED TRIGONOMETRIC IDENTITIES**

These identities may be derived according to the flowchart below.

![Flowchart](image_url)
GROUP 1: SUM IDENTITIES

Memorize:

\[ \sin(u + v) = \sin u \cos v + \cos u \sin v \]

Think: “Sum of the mixed-up products”
(Multiplication and addition are commutative, but start with the \( \sin u \cos v \) term in anticipation of the Difference Identities.)

\[ \cos(u + v) = \cos u \cos v - \sin u \sin v \]


\[ \tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \]

Think: "Sum
1
Product"
GROUP 3a: DOUBLE-ANGLE (Think: Angle-Reducing, if \( u > 0 \)) IDENTITIES

Memorize:

(Also be prepared to recognize and know these “right-to-left.”)

\[
\sin(2u) = 2 \sin u \cos u
\]

Think: “Twice the product”

Reading “right-to-left,” we have:

\[
2 \sin u \cos u = \sin(2u)
\]

(This is helpful when simplifying.)

\[
\cos(2u) = \cos^2 u - \sin^2 u
\]

Think: “Cosines – Sines” (again)

Reading “right-to-left,” we have:

\[
\cos^2 u - \sin^2 u = \cos(2u)
\]

Contrast this with the Pythagorean Identity:

\[
\cos^2 u + \sin^2 u = 1
\]

\[
\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}
\]

(Hard to memorize; we’ll show how to obtain it.)

Notice that these identities are “angle-reducing” (if \( u > 0 \)) in that they allow you to go from trigonometric functions of \((2u)\) to trigonometric functions of simply \(u\).
Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace $v$ with $u$, and simplify.

\[
\sin(2u) = \sin(u + u)
\]
\[
= \sin u \cos u + \cos u \sin u \quad \text{(From Sum Identity)}
\]
\[
= \sin u \cos u + \sin u \cos u \quad \text{(Like terms!!)}
\]
\[
= 2 \sin u \cos u
\]

\[
\cos(2u) = \cos(u + u)
\]
\[
= \cos u \cos u - \sin u \sin u \quad \text{(From Sum Identity)}
\]
\[
= \cos^2 u - \sin^2 u
\]

\[
\tan(2u) = \tan(u + u)
\]
\[
= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad \text{(From Sum Identity)}
\]
\[
= \frac{2 \tan u}{1 - \tan^2 u}
\]

This is a “last resort” if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the $\tan(2u)$ identity is harder to remember, you may prefer to remember the Sum Identity for $\tan(u + v)$ and then derive the $\tan(2u)$ identity this way.

If you’re quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.
GROUP 3b: DOUBLE-ANGLE IDENTITIES FOR $\cos$

Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$:

Let’s begin with the version we’ve already seen:

Version 1: $\cos(2u) = \cos^2 u - \sin^2 u$

Also know these two, from “left-to-right,” and from “right-to-left”:

Version 2: $\cos(2u) = 1 - 2\sin^2 u$

Version 3: $\cos(2u) = 2\cos^2 u - 1$

Obtaining Versions 2 and 3 from Version 1

It’s tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ written in different ways.

To obtain Version 2, which contains $\sin^2 u$, we replace $\cos^2 u$ with $(1 - \sin^2 u)$.

\[
\cos(2u) = \cos^2 u - \sin^2 u \quad \text{(Version 1)}
\]
\[
= (1 - \sin^2 u) - \sin^2 u
\]
\[
= 1 - \sin^2 u - \sin^2 u
\]
\[
= 1 - \sin^2 u - \sin^2 u
\]
\[
= 1 - 2\sin^2 u \quad (\Rightarrow \text{Version 2})
\]

To obtain Version 3, which contains $\cos^2 u$, we replace $\sin^2 u$ with $(1 - \cos^2 u)$.

\[
\cos(2u) = \cos^2 u - \sin^2 u \quad \text{(Version 1)}
\]
\[
= \cos^2 u - \left(1 - \cos^2 u\right)
\]
\[
= \cos^2 u - 1 + \cos^2 u
\]
\[
= 2\cos^2 u - 1 \quad (\Rightarrow \text{Version 3})
\]
GROUP 4: POWER-REDUCING IDENTITIES ("PRIs")

(These are called the “Half-Angle Formulas” in some books.)

Memorize:  

\[
\begin{align*}
\sin^2 u &= \frac{1 - \cos(2u)}{2} & \text{or} & & \frac{1}{2} - \frac{1}{2}\cos(2u) \\
\cos^2 u &= \frac{1 + \cos(2u)}{2} & \text{or} & & \frac{1}{2} + \frac{1}{2}\cos(2u)
\end{align*}
\]

Then,

\[
\begin{align*}
\tan^2 u &= \frac{\sin^2 u}{\cos^2 u} = \frac{1 - \cos(2u)}{1 + \cos(2u)} \\
\sin^2 u &= \frac{1}{2} - \frac{1}{2}\cos(2u) \\
\cos^2 u &= \frac{1}{2} + \frac{1}{2}\cos(2u)
\end{align*}
\]

Actually, you just need to memorize one of the \(\sin^2 u\) or \(\cos^2 u\) identities and then switch the visible sign to get the other. Think: “\(\sin\)” is “bad” or “negative”; this is a reminder that the minus sign belongs in the \(\sin^2 u\) formula.

Obtaining the Power-Reducing Identities from the Double-Angle Identities for \(\cos(2u)\)

To obtain the identity for \(\sin^2 u\), start with Version 2 of the \(\cos(2u)\) identity:

\[
\cos(2u) = 1 - 2\sin^2 u
\]

Now, solve for \(\sin^2 u\).

\[
\begin{align*}
2\sin^2 u &= 1 - \cos(2u) \\
\sin^2 u &= \frac{1 - \cos(2u)}{2}
\end{align*}
\]

To obtain the identity for \(\cos^2 u\), start with Version 3 of the \(\cos(2u)\) identity:

\[
\cos(2u) = 2\cos^2 u - 1
\]

Now, switch sides and solve for \(\cos^2 u\).

\[
\begin{align*}
2\cos^2 u - 1 &= \cos(2u) \\
2\cos^2 u &= 1 + \cos(2u) \\
\cos^2 u &= \frac{1 + \cos(2u)}{2}
\end{align*}
\]
GROUP 5: HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution $\theta = 2u$. (See Obtaining ... below.)

The Identities:

\[
\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{2}}
\]
\[
\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos\theta}{2}}
\]
\[
\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}
\]

For a given $\theta$, the choices among the $\pm$ signs depend on the Quadrant that $\frac{\theta}{2}$ lies in.

Here, the $\pm$ symbols indicate incomplete knowledge; unlike when we handle the Quadratic Formula, we do not take both signs for any of the above formulas for a given $\theta$. There are no $\pm$ symbols in the last two $\tan\left(\frac{\theta}{2}\right)$ formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two $\tan\left(\frac{\theta}{2}\right)$ formulas: Keep either the numerator or the denominator of the radicand of the first formula, place $\sin\theta$ in the other part of the fraction, and remove the radical sign and the $\pm$ symbol.
Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):

For the \( \sin \left( \frac{\theta}{2} \right) \) identity, we begin with the PRI:

\[
\sin^2 u = \frac{1 - \cos(2u)}{2}
\]

Let \( u = \frac{\theta}{2} \), or \( \theta = 2u \).

\[
\sin^2 \left( \frac{\theta}{2} \right) = \frac{1 - \cos \theta}{2}
\]

\[
\sin \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{(by the Square Root Method)}
\]

Again, the choice among the \( \pm \) signs depends on the Quadrant that \( \frac{\theta}{2} \) lies in.

The story is similar for the \( \cos \left( \frac{\theta}{2} \right) \) and the \( \tan \left( \frac{\theta}{2} \right) \) identities.

What about the last two formulas for \( \tan \left( \frac{\theta}{2} \right) \)? The key trick is multiplication by trigonometric conjugates. For example:

\[
\tan \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}
\]

\[
= \pm \sqrt{\frac{(1 - \cos \theta)(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)}}
\]

\[
= \pm \sqrt{\frac{(1 - \cos \theta)^2}{1 - \cos^2 \theta}}
\]

\[
= \pm \sqrt{\frac{1 - \cos \theta}{\sin^2 \theta}}
\]

\[
= \pm \sqrt{\frac{(1 - \cos \theta)^2}{\sin \theta}}
\]

\[
= \pm \frac{1 - \cos \theta}{\sin \theta} \quad \text{(because } \sqrt{a^2} = |a|)\]
Now, $1 - \cos \theta \geq 0$ for all real $\theta$, and $\tan \left( \frac{\theta}{2} \right)$ has the same sign as $\sin \theta$ (can you see why?), so …

\[ \frac{1 - \cos \theta}{\sin \theta} \]

To get the third formula, use the numerator’s (instead of the denominator’s) trigonometric conjugate, $1 + \cos \theta$, when multiplying into the numerator and the denominator of the radicand in the first few steps.

**GROUP 6: PRODUCT-TO-SUM IDENTITIES**

These can be verified from right-to-left using the Sum and Difference Identities.

**The Identities:**

\[
\begin{align*}
\sin u \sin v &= \frac{1}{2} \left[ \cos(u - v) - \cos(u + v) \right] \\
\cos u \cos v &= \frac{1}{2} \left[ \cos(u - v) + \cos(u + v) \right] \\
\sin u \cos v &= \frac{1}{2} \left[ \sin(u + v) + \sin(u - v) \right] \\
\cos u \sin v &= \frac{1}{2} \left[ \sin(u + v) - \sin(u - v) \right]
\end{align*}
\]

**GROUP 7: SUM-TO-PRODUCT IDENTITIES**

These can be verified from right-to-left using the Product-To-Sum Identities.

**The Identities:**

\[
\begin{align*}
\sin x + \sin y &= 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \\
\sin x - \sin y &= 2 \cos \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right) \\
\cos x + \cos y &= 2 \cos \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \\
\cos x - \cos y &= -2 \sin \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right)
\end{align*}
\]