SECTION 2.2:
PROPERTIES OF LIMITS and ALGEBRAIC FUNCTIONS

LEARNING OBJECTIVES

- Know properties of limits, and use them to evaluate limits of functions, particularly algebraic functions.
- Understand how the properties of limits justify the limit theorems in Section 2.1.
- Be able to use informal Limit Form notation to analyze limits.
- Learn to exercise caution when handling $\left(\text{Limit Form } \sqrt[\text{even}]{0}\right)$.

PART A: PROPERTIES OF LIMITS / THE ALGEBRA OF LIMITS; LIMIT FORMS

Assume that: $\lim_{x\to a} f(x) = L_1$, and $\lim_{x\to a} g(x) = L_2$, where $a, L_1, L_2 \in \mathbb{R}$.

1) The limit of a **sum** equals the **sum** of the limits.

$$\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
$$= L_1 + L_2$$

- We may refer to this as the Sum Rule of Limits.

For example, as $x \to a$, if $f(x) \to 2$ and $g(x) \to 3$, then $\left[f(x)+g(x)\right] \to 5$.
We can represent this **informally** using a Limit Form: $(\text{Limit Form } 2+3) \Rightarrow 5$.

WARNING 1: Limit Forms. There is no standard notation for Limit Forms, and they represent footnotes to the rigorous evaluation of limits. Different instructors may have different rules on when Limit Forms need to be written.

2) The limit of a **difference** equals the **difference** of the limits.

$$\lim_{x\to a}\left[f(x)-g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$
$$= L_1 - L_2$$

For example, $(\text{Limit Form } 5-3) \Rightarrow 2$.

3) <u>The limit of a **product** equals the **product** of the limits.</u>

$$\lim_{x \to a} \left[f(x) g(x) \right], \text{ or } \lim_{x \to a} f(x) g(x) = \left[\lim_{x \to a} f(x) \right] \left[\lim_{x \to a} g(x) \right]$$
$$= L_1 L_2$$

For example, $\left(\text{Limit Form } 2 \cdot 3 \right) \Rightarrow 6$.

4) <u>The limit of a **quotient** equals the **quotient** of the limits, if the limit of the divisor (or denominator) is **not zero**.</u>

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right], \text{ or } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$
$$= \frac{L_1}{L_2}, \text{ if } L_2 \neq 0$$

For example, $\left(\text{Limit Form } \frac{6}{2} \right) \Rightarrow 3$.

5) <u>The limit of a (positive integer) **power** equals the **power** of the limit.</u>

If n is a positive integer $\left(n \in \mathbb{Z}^+ \right)$, then:

$$\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n$$
$$= \left(L_1 \right)^n$$

• This is a direct consequence of Property 3. For instance,

$$\lim_{x \to a} x^2 = \lim_{x \to a} xx = \left(\lim_{x \to a} x \right) \left(\lim_{x \to a} x \right) = \left(\lim_{x \to a} x \right)^2.$$

For example, $\left(\text{Limit Form } 2^{(\text{constant } 3)} \right) \Rightarrow 8$.

• The seemingly simpler statement $\left(\text{Limit Form } 2^3 \right) \Rightarrow 8$ is also true, but it actually says something more powerful. It says that "something approaching 2" raised to an "exponent **approaching** 3" will approach 8. However, this idea **falls apart** when the base $f(x)$ approaches a **negative** number. It is true that $\left(\text{Limit Form } (-2)^{(\text{constant } 3)} \right) \Rightarrow -8$, for example, but it is **not** true that $\left(\text{Limit Form } (-2)^3 \right) \Rightarrow -8$. Think about why $(-2)^{3.5}$, or $(-2)^{7/2}$, is **not** a real number; we will address this issue in Part B.

6) <u>The limit of a **constant multiple** equals the **constant multiple** of the limit. ("Constant Factors Pop Out.")</u>

If $c \in \mathbb{R}$, then:

$$\lim_{x \to a} \left[c \cdot f(x) \right], \text{ or } \lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x)$$
$$= cL_1$$

For example, **twice** "something that approaches 3" will approach 6.

• In multivariable calculus, if *y* is **independent** of *x*, then we can pop out *y*.

<u>Note</u>: Properties 5, 6, and 7 (upcoming) are generalized in Section 2.8, Footnote 6.

<u>Limit Operators are Linear</u>

Properties 1), 2), and 6) imply that limit operators are <u>linear operators</u>. This means that we can take limits **term-by-term**, and then **constant factors "pop out,"** assuming the limits exist. (See Footnote 1.)

• This is a key property that is shared by differentiation and integration operators in later chapters.

Properties 1-6, building on the **elementary rules** $\lim_{x \to a} c = c$ and $\lim_{x \to a} x = a$ $(a, c \in \mathbb{R})$, justify the **Basic Limit Theorem for Rational Functions** in Section 2.1. A demonstration follows.

Example 1 (Demonstrating How the Properties of Limits Justify the Basic Limit Theorem for Rational Functions)

Evaluate $\lim_{x\to 4} \frac{3x^2-1}{x+5}$ using the properties of limits in this section.

§ Solution

$$\lim_{x\to 4} \frac{3x^2-1}{x+5} = \frac{\lim_{x\to 4}\left(3x^2-1\right)}{\lim_{x\to 4}\left(x+5\right)} \quad \left(\text{by Property 4 on } \textbf{quotients}\right)$$

$$= \frac{\lim_{x\to 4} 3x^2 - \lim_{x\to 4} 1}{\lim_{x\to 4} x + \lim_{x\to 4} 5} \quad \left(\text{by Properties 1, 2 on } \textbf{sums}, \textbf{differences}\right)$$

$$= \frac{\lim_{x\to 4} 3x^2 - 1}{4+5} \quad \left(\text{by } \textbf{elementary rules}\right)$$

$$= \frac{3\left(\lim_{x\to 4} x^2\right) - 1}{4+5} \quad \left(\text{by Property 6 on } \textbf{constant multiples}\right)$$

$$= \frac{3\left(\lim_{x\to 4} x\right)^2 - 1}{4+5} \quad \left(\begin{array}{l}\text{by Property 5 on } \textbf{powers}\text{, or} \\ \text{by Property 3 on } \textbf{products}\text{: } x^2 = xx\end{array}\right)$$

$$= \frac{3(4)^2 - 1}{4+5} \quad \left(\text{by } \textbf{elementary rules}\text{; see } \underline{\text{Note 1}} \text{ below}\right)$$

$$= \frac{47}{9}$$

Note 1: Observe that the limit can be evaluated by simply substituting $x = 4$ into $\frac{3x^2-1}{x+5}$, as the **Basic Limit Theorem for Rational Functions** suggests.

Note 2: Observe that all indicated limits **exist** and there are **no zero denominator** issues, so we could apply Properties 1-6. Our use of the "$=$" sign is appropriate here, though we often use it informally even when the limit turns out not to exist. §

Properties of One-Sided Limits

Properties 1-6 extend naturally to one-sided limits. For example,

$$\lim_{x\to a^-}\left[f(x)+g(x)\right] = \lim_{x\to a^-} f(x) + \lim_{x\to a^-} g(x), \text{ and}$$

$$\lim_{x\to a^+}\left[f(x)+g(x)\right] = \lim_{x\to a^+} f(x) + \lim_{x\to a^+} g(x),$$

provided the indicated limits exist.

PART B: PROPERTIES OF LIMITS OF ROOTS

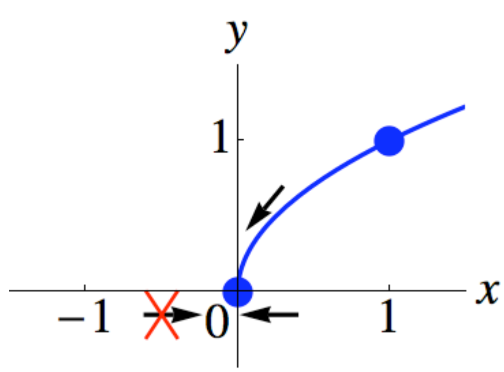
We now motivate Property 7, a much more complicated property on **roots**.

Example 2 (Evaluating the Limit of a Square Root)

Evaluate $\lim_{x\to 1}\sqrt{x}$, $\lim_{x\to -1}\sqrt{x}$, $\lim_{x\to 0^+}\sqrt{x}$, $\lim_{x\to 0^-}\sqrt{x}$, and $\lim_{x\to 0}\sqrt{x}$.

§ Solution

The graph of $y=\sqrt{x}$ is below. We emphasize the interesting cases where $a=0$.



$\lim_{x\to 1}\sqrt{x} = \sqrt{1} = 1$, evidently.

$\lim_{x\to -1}\sqrt{x}$ does not exist (DNE).

- Actually, this is **not** because $\sqrt{-1}$ is imaginary. It is because there is no **punctured neighborhood** of $x=-1$ on which $\sqrt{x}$ is real. There is **no way** to approach $x=-1$ through the **domain** of f, where f is the (principal) square root function.

Review Section 2.1, Example 6. $\text{Dom}(f) = [0, \infty)$ here, as well.

$$\lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0.$$

$$\lim_{x \to 0^-} \sqrt{x} \text{ does not exist (DNE).}$$

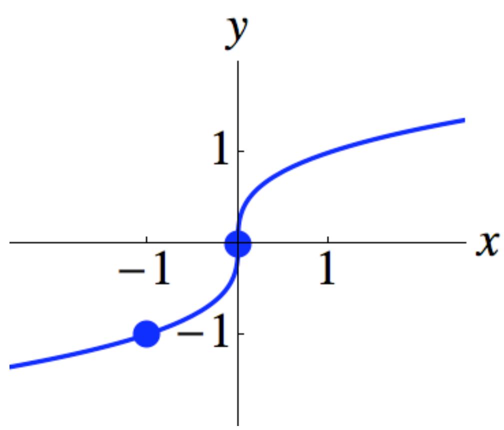
Therefore, $\lim_{x \to 0} \sqrt{x}$ does not exist (DNE). §

Example 3 (Evaluating the Limit of a Cube Root)

Evaluate $\lim_{x \to -1} \sqrt[3]{x}$ and $\lim_{x \to 0} \sqrt[3]{x}$.

§ Solution

The graph of $y = \sqrt[3]{x}$ is below.



The **domain** of the cube root function is $\mathbb{R}$. The (principal) cube roots of **negative** real numbers are (negative) **real numbers**; this is a key difference from square roots. It turns out that substituting $x = a$ works here for both limits.

$$\lim_{x \to -1} \sqrt[3]{x} = \sqrt[3]{-1} = -1.$$

$$\lim_{x \to 0} \sqrt[3]{x} = \sqrt[3]{0} = 0.$$

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Property 7 now extends our observations from Examples 2 and 3 to more **general radicands**, not just *x*, and also to **general types of roots**.

WARNING 2: In theory, **even roots** tend to require more thought than **odd roots**.

As before, assume $\lim_{x\to a} f(x) = L_1$.

7) The limit of a **root** equals the **root** of the limit … sometimes.

If n is a positive integer $\left(n \in \mathbb{Z}^+\right)$, and either

- (n is odd), or
- (n is even, and $L_1 > 0$), then:

$$\begin{aligned}\lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{\lim_{x\to a} f(x)} \\ &= \sqrt[n]{L_1}\end{aligned}$$

For example, $\left(\text{Limit Form } \sqrt{4}\right) \Rightarrow 2$, and $\left(\text{Limit Form } \sqrt[3]{-8}\right) \Rightarrow -2$.

(The index of a radical, such as the "3" in $\sqrt[3]{-8}$, is assumed to be a constant.)

WARNING 3: The Limit Form $\sqrt[\text{even}]{0}$, corresponding to $L_1 = 0$, could either yield a **limit value of 0** or a limit that **does not exist (DNE)**. Informally, $\left(\text{Limit Form } \sqrt[\text{even}]{0}\right) \Rightarrow 0$ or "DNE," but further analysis is required to determine which is the case.

Limit Forms such as $\sqrt{-1}$ and $\sqrt[4]{-5}$ imply that the limits **do not exist (DNE)**.

Property 7* below elaborates on limits of **even roots**.

7*) Properties of Limits of Even Roots

Let n be a positive **even** integer.

- If $L_1 > 0$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$ by Property 7.
- If $L_1 < 0$, then $\lim_{x\to a} \sqrt[n]{f(x)}$ does not exist (DNE). The one-sided limits $\lim_{x\to a^+} \sqrt[n]{f(x)}$ and $\lim_{x\to a^-} \sqrt[n]{f(x)}$ also do not exist (DNE).
- If $L_1 = 0$, then $\lim_{x\to a} \sqrt[n]{f(x)} = 0$ or "DNE." In particular,
  - •• $\lim_{x\to a} \sqrt[n]{f(x)} = 0 \Leftrightarrow f(x) \geq 0$ on some **punctured neighborhood** of a; change this to a **right-neighborhood** for a **right-hand limit** and a **left-neighborhood** for a **left-hand limit**.
  - •• Otherwise, the limit does not exist (DNE).

PART C: LIMITS OF ALGEBRAIC FUNCTIONS

Our understanding of Property 7 will now allow us to extend our **Basic Limit Theorem for Rational Functions** to more general **algebraic functions**.

Remember that:

- all **constant** functions are also **polynomial** functions,
- all **polynomial** functions are also **rational** functions, and
- all **rational** functions are also **algebraic** functions.

> Basic Limit Theorem for Algebraic Functions
>
> If f is an algebraic function, $a \in \text{Dom}(f)$, and
> **no radicand of any even root approaches 0 in the limit (informally, the Limit Form $\sqrt[\text{even}]{0}$ does not appear),**
>
> then $\lim_{x \to a} f(x) = f(a)$.
>
> • To evaluate the limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

If the Limit Form $\sqrt[\text{even}]{0}$ does appear, this substitution method **might** still work, but further analysis is required. How is the radicand approaching 0?

Example 4 (Evaluating the Limit of an Algebraic Function)

Let $f(x) = \dfrac{\sqrt[3]{x-4}}{(3x-9)^2} + \sqrt{x+3}$. Evaluate $\lim_{x \to 2} f(x)$.

§ Solution

f is an algebraic function. Observe that:

$f(x)$ is real $\Leftrightarrow \left[x+3 \ge 0 \text{ and } (3x-9)^2 \ne 0 \right]$. As a result,

$\text{Dom}(f) = \left\{ x \in \mathbb{R} \,\middle|\, x \ge -3 \text{ and } x \ne 3 \right\} = [-3, \infty) \setminus \{3\} = [-3, 3) \cup (3, \infty)$.

We observe that $2 \in \text{Dom}(f)$, and the Limit Form $\sqrt[\text{even}]{0}$ will **not** appear, so we **substitute ("plug in")** $x = 2$ and evaluate $f(2)$.

TIP 1: As a practical matter, when we evaluate the limit of an algebraic function, we often **substitute immediately and see what happens**. **(**We might not have time to find the domain.) If we end up with a **real number**, and if any $\sqrt[\text{even}]{0}$ Limit Forms encountered **only yield 0** (not "DNE"), then that number will be the **limit value**.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left[\frac{\sqrt[3]{x-4}}{(3x-9)^2} + \sqrt{x+3}\right]$$
$$= \frac{\sqrt[3]{(2)-4}}{\left[3(2)-9\right]^2} + \sqrt{(2)+3}$$
$$= \frac{\sqrt[3]{-2}}{9} + \sqrt{5}$$
$$= -\frac{\sqrt[3]{2}}{9} + \sqrt{5}, \text{ or } \sqrt{5} - \frac{\sqrt[3]{2}}{9}, \text{ or } \frac{9\sqrt{5} - \sqrt[3]{2}}{9}$$

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We confront the Limit Form $\sqrt[\text{even}]{0}$ in the following Examples.

Example 5 (Resolving the Limit Form $\sqrt[\text{even}]{0}$ *)*

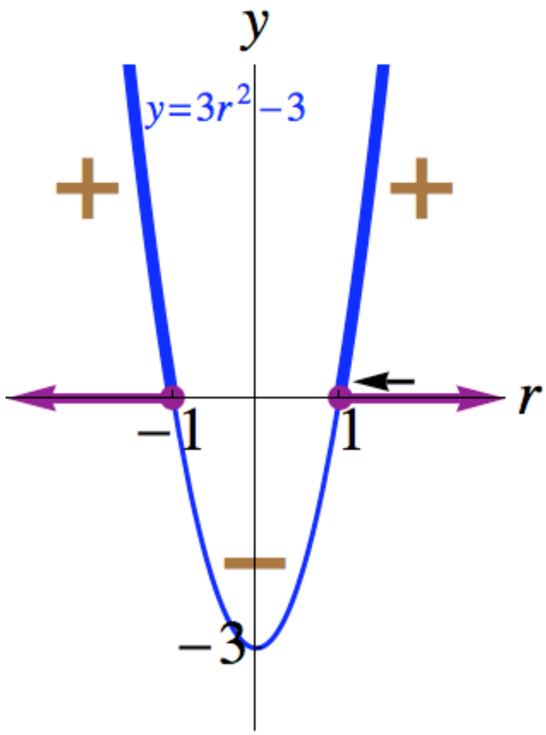
Evaluate $\lim_{r\to 1^+} \sqrt{3r^2 - 3}$.

§ Solution

• The radicand $3r^2 - 3$ is rational. By the **Extended Limit Theorem for Rational Functions** in Section 2.1, we find that $\lim_{r\to 1^+} \left(3r^2 - 3\right) = 0$, so we are facing the Limit Form $\sqrt[\text{even}]{0}$.

• We use Property 7*. We will show that $3r^2 - 3 \geq 0$ on a **right-neighborhood** of $r = 1$, and then $\lim_{r\to 1^+} \sqrt{3r^2 - 3} = 0$. Otherwise, the limit would not exist (DNE).

• The graph of $y = 3r^2 - 3$ follows. It is an upward-opening parabola in the *ry*-plane. The zeros of $3r^2 - 3$, -1 and 1, correspond to the *r*-intercepts. The **domain** of $\sqrt{3r^2 - 3}$ consists of the *r*-values that make $y = 3r^2 - 3 \geq 0$. It corresponds to the parts of the parabola that lie **above or on** the *r*-axis. This is important, because we are only allowed to approach $r = 1$ through this **domain** (in purple). In fact, here, we can approach $r = 1$ **from the right**.



Therefore, $\lim_{r \to 1^+} \sqrt{3r^2 - 3} = 0$.

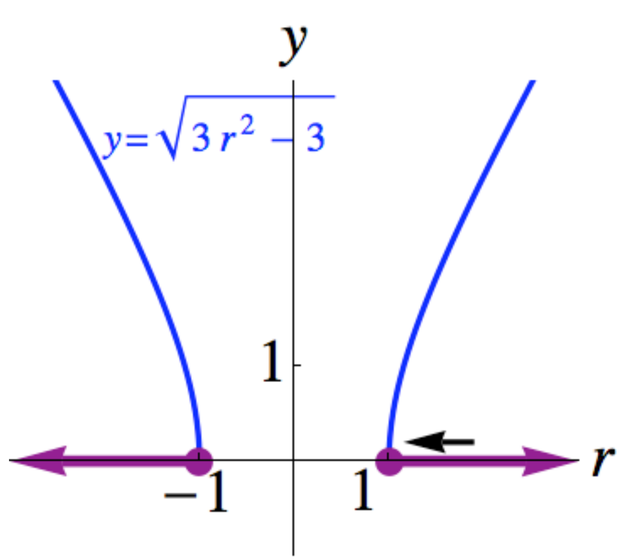
(For more, see Section 2.7: Nonlinear Inequalities in the Precalculus notes.)

Here's a **non-graphical** approach. As $r \to 1^+$, $r > 1$. Now,

$$r > 1 \Rightarrow$$
$$r^2 > 1 \Rightarrow$$
$$3r^2 > 3 \Rightarrow$$
$$3r^2 - 3 > 0$$

Therefore, $\lim_{r \to 1^+} \sqrt{3r^2 - 3} = 0$.

The graph of $y = \sqrt{3r^2 - 3}$ is below. Observe that the graph **disappears** where $3r^2 - 3 < 0$; this is where we fall **outside the domain** (in purple).



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Example 6 (Evaluating a Limit Using Example 5 and Properties of Limits)

Evaluate $\lim_{r \to 1^+} \left(7\sqrt{3r^2 - 3} + 5 \right)$.

§ Solution

$$\begin{aligned} \lim_{r \to 1^+} \left(7\sqrt{3r^2 - 3} + 5 \right) &= \lim_{r \to 1^+} 7\sqrt{3r^2 - 3} + \lim_{r \to 1^+} 5 \quad (\text{by Prop. 1 on } \textbf{sums}) \\ &= 7\left(\lim_{r \to 1^+} \sqrt{3r^2 - 3} \right) + 5 \quad \left(\begin{array}{l} \text{by Prop. 6 on } \textbf{constant} \\ \textbf{multiples, elem. rules} \end{array} \right) \\ &= 7(0) + 5 \quad (\text{by Example 5}) \\ &= 5 \end{aligned}$$

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Example 7 (Resolving the Limit Form $\sqrt[\text{even}]{0}$ *)*

Evaluate $\lim_{x \to -7} \sqrt{(x+7)^2}$.

§ Solution 1

As $x \to -7$, $(x+7)^2 \to 0$.

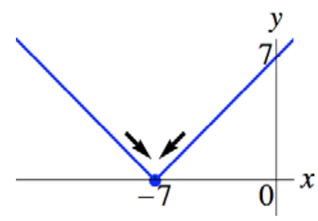
$(x+7)^2 \geq 0$ for all real x $(\forall x \in \mathbb{R})$. Therefore,

$\lim_{x \to -7} \sqrt{(x+7)^2} = 0$. §

§ Solution 2

$$\begin{aligned} \lim_{x \to -7} \sqrt{(x+7)^2} &= \lim_{x \to -7} |x+7| \\ &= |-7+7| \\ &= 0 \end{aligned}$$

Below is the graph of $y = \sqrt{(x+7)^2}$, or $y = |x+7|$.



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FOOTNOTES

1. **Limits of linear combinations.** The fact that limit operators are linear implies that the limit of a linear combination of $f(x)$ and $g(x)$ equals the linear combination of the limits:

$$\begin{aligned} \lim_{x \to a} \left[c \cdot f(x) + d \cdot g(x) \right] &= c \cdot \lim_{x \to a} f(x) + d \cdot \lim_{x \to a} g(x) \\ &= cL_1 + dL_2 \quad (c, d \in \mathbb{R}) \end{aligned}$$