SECTION 2.4: LIMITS AND INFINITY II

LEARNING OBJECTIVES

• Understand infinite limits at a point and relate them to vertical asymptotes of graphs.
• Be able to evaluate infinite limits at a point, particularly for rational functions expressed in simplified form, and use a short cut to find vertical asymptotes of their graphs.
• Be able to use informal Limit Form notation to analyze infinite limits at a point.

PART A: VERTICAL ASYMPTOTES (“VA”s) and INFINITE LIMITS AT A POINT

In Section 2.1, we discussed finite limits at a point \( a \).
We saw (two-sided) limits where \( \lim_{x \to a} f(x) = L \) \((a, L \in \mathbb{R})\).

In Section 2.3, we discussed finite and infinite limits at \((\pm)\) infinity.
We saw examples where \( \lim_{x \to \pm \infty} f(x) \) or \( \lim_{x \to \pm \infty} f(x) \) is \( L \) \((L \in \mathbb{R}), \infty, \text{ or } -\infty\).

Now, if \( a \in \mathbb{R} \):

\[
f \text{ has an infinite limit at a point } a \iff \lim_{x \to a^+} f(x) \text{ or } \lim_{x \to a^-} f(x) \text{ is } \infty \text{ or } -\infty.
\]

• We read \( \lim_{x \to a} f(x) = \infty \) as “the limit of \( f(x) \) as \( x \) approaches \( a \) is infinity.”

A vertical asymptote, which we will denote by “VA,” is a vertical line that a graph approaches in an “explosive” sense. (See Section 2.1, Example 11.)

Using Infinite Limits at a Point to Find Vertical Asymptotes (VAs)

The graph of \( y = f(x) \) has a vertical asymptote (VA) at \( x = a \) \((a \in \mathbb{R})\)

\[
\iff \lim_{x \to a^+} f(x) \text{ or } \lim_{x \to a^-} f(x) \text{ is } \infty \text{ or } -\infty.
\]

• That is, the graph has a VA at \( x = a \iff \) there is an infinite limit there from one or both sides.
The **number of VAs** the graph has can be a nonnegative integer (0, 1, 2, ...), or it can have infinitely many VAs (consider \( f(x) = \tan x \)).

- If \( f \) is **rational**, then the graph **cannot have infinitely many VAs**.
- If \( f \) is **polynomial**, then the graph has **no VAs**.

**Note:** The graph of \( y = f(x) \) **cannot cross over a VA**, but it can cross over an HA (see Section 2.3, Example 6).

**Example 1 (The Graph of the Reciprocal Function has an “Odd VA”; Revisiting Section 2.3, Example 1)**

Let \( f(x) = \frac{1}{x} \). Evaluate \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \), and show that the graph of \( y = f(x) \) has a vertical asymptote (VA) at \( x = 0 \).

**§ Solution**

Let’s use the **numerical / tabular approach**:

| \( x \) | \(-1\) | \(-\frac{1}{10}\) | \(-\frac{1}{100}\) | \(\to 0^-\) | \(0^+ \leftarrow \frac{1}{100}\) | \(\frac{1}{10}\) | \(1\) |
|---|---|---|---|---|---|---|
| \( f(x) = \frac{1}{x} \) | \(-1\) | \(-10\) | \(-100\) | \(\to -\infty\) | \(\infty \leftarrow 100\) | 10 | 1 |

- Apparently, as \( x \) approaches 0 from the **right**, \( f(x) \) **increases without bound**. That is, \( \lim_{x \to 0^+} f(x) = \infty \).
- Also, as \( x \) approaches 0 from the **left**, \( f(x) \) **decreases without bound**. That is, \( \lim_{x \to 0^-} f(x) = -\infty \).
- Either limit statement implies that the graph of \( y = f(x) \) below has a **vertical asymptote** (“VA”) at \( x = 0 \), the \( y \)-axis.

\[ y = f(x) \]

- \( \lim_{x \to 0} f(x) \) does not exist (DNE). (See Footnote 2.) §
Example 1 gave us the most basic cases of the following Limit Forms.

\[
\begin{align*}
\left( \text{Limit Form } \frac{1}{0^+} \right) & \Rightarrow \infty, \quad \text{and} \quad \left( \text{Limit Form } \frac{1}{0^-} \right) \Rightarrow -\infty
\end{align*}
\]

- These Limit Forms can be rescaled, as described in Section 2.3, Part A.

“Odd and Even VAs”

Assume that the graph of \( y = f(x) \) has a VA at \( x = a \).
(The following terminology is informal and nonstandard.)

- If the two one-sided limits at \( x = a \) are \( \infty \) and \( -\infty \), in either order, then the VA is an “odd VA.”
- If those limits are both \( \infty \) or both \( -\infty \), then the VA is an “even VA.”

- In Example 1, the y-axis was an “odd VA,” partly due to the fact that \( f \) was an odd function. The graph of \( y = f(x) \) “shot off” in different directions around the VA.
- In Example 2 below, the y-axis is an “even VA,” partly due to the fact that \( g \) is an even function, where \( g(x) = \frac{1}{x^2} \). The graph of \( y = g(x) \) “shoots off” in the same direction around the VA.

**Example 2 (A Graph With an “Even VA”)**

Evaluate \( \lim_{x \to 0^+} \frac{1}{x^2}, \lim_{x \to 0^-} \frac{1}{x^2}, \text{ and } \lim_{x \to 0} \frac{1}{x^2} \).

**Solution**

Because \( x^2 > 0 \) for all \( x \neq 0 \), all three give: \( \left( \text{Limit Form } \frac{1}{0^+} \right) \Rightarrow \infty \).

The graph of \( y = \frac{1}{x^2} \) is below.

\[
\begin{align*}
\lim_{x \to 0^+} \frac{1}{x^2} &= \infty, \quad \lim_{x \to 0^-} \frac{1}{x^2} = \infty, \quad \text{and} \quad \lim_{x \to 0} \frac{1}{x^2} = \infty.
\end{align*}
\]
PART B: EVALUATING INFINITE LIMITS FOR RATIONAL FUNCTIONS

Example 3 (Evaluating Infinite Limits at a Point for a Rational Function)

Let \( f(x) = \frac{x + 1}{x^3 + 4x^2} \). Evaluate \( \lim_{x \to -4^+} f(x) \), \( \lim_{x \to -4^-} f(x) \), and \( \lim_{x \to -4} f(x) \).

§ Solution

\[
\lim_{x \to -4} (x + 1) = -4 + 1 = -3, \quad \text{and} \quad \lim_{x \to -4} \left( x^3 + 4x^2 \right) = (-4)^3 + 4(-4)^2 = 0.
\]

All three problems give the Limit Form \( \frac{-3}{0^+} \). For each, we must know how the denominator approaches 0. Since it is easier to analyze signs of products than of sums (for example, do we automatically know the sum of \( a \) and \( b \) if \( a > 0 \) and \( b < 0 \)?), we will factor the denominator.

WARNING 1: Many students improperly use methods such as the “Division Method” and “DTS” from Section 2.3. Those methods are designed to evaluate “long-run” limits, not limits at a point.

\[
\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \frac{x + 1}{x^3 + 4x^2}
\]

\[
= \lim_{x \to -4^+} \frac{x + 1}{x^3 + 4x^2} \quad \left( \text{Limit Form } \frac{-3}{0^+} \right)
\]

In the denominator: Remember that “positive times positive equals positive.”

\[
= -\infty
\]

\[
\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{x + 1}{x^3 + 4x^2}
\]

\[
= \lim_{x \to -4^-} \frac{x + 1}{x^3 + 4x^2} \quad \left( \text{Limit Form } \frac{-3}{0^-} \right)
\]

In the denominator: Remember that “positive times negative equals negative.”

\[
= \infty
\]
\[
\lim_{x \to -4} f(x) \text{ does not exist (DNE). (See Footnote 2.)}
\]

The graph of \( y = f(x) \) is below. Observe the **odd VA** at \( x = -4 \).
(Why is there an HA at the x-axis?)

\[
\begin{array}{c}
\text{Finding VAs for Graphs of “Simplified” Rational Functions}
\\
\text{Let } f(x) = \frac{N(x)}{D(x)}, \text{ where } N(x) \text{ and } D(x) \text{ are nonzero polynomials in } x
\\
\text{with no real zeros in common; this is guaranteed (by the Factor Theorem from Precalculus) if they have no variable factors in common, up to constant multiples. Then,}
\\
The graph of \( y = f(x) \) has a VA at \( x = a \iff a \) is a real zero of \( D(x) \).
\\
\text{Note: The numerator and the denominator of } \frac{x - \frac{1}{3}}{3x - 1} \text{ are common factors}
\\
\text{up to constant multiples (the denominator is 3 times the numerator);}
\\
\text{observe that } \frac{1}{3} \text{ is a real zero of both.}
\end{array}
\]
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Example 4 (Finding VAs for the Graph of a “Simplified” Rational Function; Revisiting Example 3)

Let \( f(x) = \frac{x + 1}{x^3 + 4x^2} \). Find the equations of the vertical asymptotes (VAs) of the graph of \( y = f(x) \). Justify using limits.

§ Solution

\[ f(x) = \frac{x + 1}{x^3 + 4x^2} = \frac{x + 1}{x^2(x + 4)} \], which is simplified. The VAs have equations \( x = 0 \) and \( x = -4 \), corresponding to the real zeros of the denominator.

To justify the VA at \( x = 0 \), show there is an infinite limit there. Either of the following will suffice:

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x + 1}{x^3 + 4x^2} \\
= \lim_{x \to 0^+} \frac{x + 1}{x^2(x + 4)} \\
= \infty
\]

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x + 1}{x^3 + 4x^2} \\
= \lim_{x \to 0^-} \frac{x + 1}{x^2(x + 4)} \\
= \infty
\]

• Since 0 is a real zero of \( D(x) \) with multiplicity 2 (an even number), there is an “even VA” at \( x = 0 \).

To justify the VA at \( x = -4 \), show there is an infinite limit there, as we did in Example 3, by showing either \( \lim_{x \to -4^+} f(x) = -\infty \), or \( \lim_{x \to -4^-} f(x) = \infty \).

• Since \( -4 \) is a real zero of \( D(x) \) with multiplicity 1 (an odd number), there is an “odd VA” at \( x = -4 \).
FOOTNOTES

1. **Alternate definition of an infinite limit at a point.** If we say that $f$ has an infinite limit at $a$
\[ \lim_{x \to a^+} \left| f(x) \right| = \infty \text{ or } \lim_{x \to a^-} \left| f(x) \right| = \infty, \]
we then extend the idea of an “infinite limit” to examples such as the following:

\[ f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \text{ is a rational number } (x \in \mathbb{Q}) \\ -\frac{1}{x}, & \text{if } x \text{ is an irrational number } (x \notin \mathbb{Q}; \text{ really, } x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases} \]

as $x \to 0$. In this work, we will not use this definition.

2. **Infinity and the real projective line.**
   - The affinely extended real number system, denoted by $\overline{\mathbb{R}}$ or $[-\infty, \infty]$, includes two points of infinity, one referred to as $\infty$ (or $+\infty$) and the other referred to as $-\infty$. (We are “adjoining” them to the real number system.) We obtain the two-point compactification of the real numbers. We never refer to $\infty$ and $-\infty$ as real numbers, though.
   - In the projectively extended real number system, denoted by $\mathbb{R}^*$, $\infty$ and $-\infty$ are treated as the same (we collapse them together and identify them with one another as $\infty$), and we then obtain the one-point compactification of the real numbers, also known as the real projective line. Then, $\frac{1}{0} = \infty$, the slope of a vertical line is $\infty$, $\lim_{x \to 0} \frac{1}{x} = \infty$, and $\lim_{x \to -4} \frac{x + 1}{x^2 + 4x} = \infty$.
   - A point at infinity is sometimes added to the complex plane, and it typically corresponds to the “north pole” of a Riemann sphere that the complex plane is wrapped around.
   - See “Projectively Extended Real Numbers” in *MathWorld* (web) and “Real projective line” in *Wikipedia* (web).