SECTION 2.4: LIMITS AND INFINITY II

LEARNING OBJECTIVES

- Understand infinite limits at a point and relate them to vertical asymptotes of graphs.
- Be able to evaluate infinite limits at a point, particularly for rational functions expressed in simplified form, and use a short cut to find vertical asymptotes of their graphs.
- Be able to use informal Limit Form notation to analyze infinite limits at a point.

PART A: VERTICAL ASYMPTOTES ("VA"s) and INFINITE LIMITS AT A POINT

In Section 2.1, we discussed finite limits at a point *a*.
We saw (two-sided) limits where $\lim_{x\to a} f(x) = L \ (a, L \in \mathbb{R})$.

In Section 2.3, we discussed finite and infinite limits at $(\pm)$ infinity.
We saw examples where $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is $L \ (L \in \mathbb{R})$, ∞, or $-\infty$.

Now, if $a \in \mathbb{R}$:

f has an infinite limit at a point *a* $\Leftrightarrow \lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is ∞ or $-\infty$.

- We read $\lim_{x\to a} f(x) = \infty$ as "the limit of $f(x)$ as x approaches a is infinity."
- See Footnote 1 for an alternate definition.

A vertical asymptote, which we will denote by "VA," is a vertical line that a graph **approaches in an "explosive" sense**. (See Section 2.1, Example 11.)

Using Infinite Limits at a Point to Find Vertical Asymptotes (VAs)

The graph of $y = f(x)$ has a **vertical asymptote (VA)** at $x = a \ (a \in \mathbb{R})$

$\Leftrightarrow \lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is ∞ or $-\infty$.

- That is, the graph has a VA at $x = a \Leftrightarrow$ there is an infinite limit there from one or both sides.

The **number of VAs** the graph has can be a nonnegative integer (0, 1, 2, …), or it can have infinitely many VAs (consider $f(x) = \tan x$).

- If f is **rational**, then the graph **cannot have infinitely many VAs**.
- If f is **polynomial**, then the graph has **no VAs**.

<u>Note</u>: The graph of $y = f(x)$ **cannot cross over a VA**, but it can cross over an HA (see Section 2.3, Example 6).

Example 1 (The Graph of the Reciprocal Function has an "Odd VA"; Revisiting Section 2.3, Example 1)

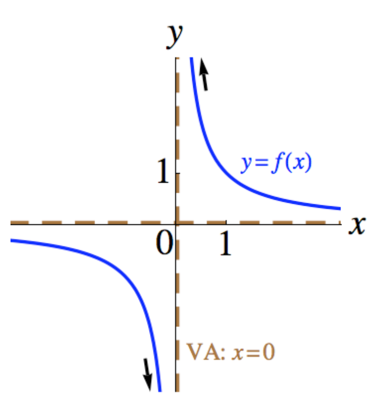
Let $f(x) = \frac{1}{x}$. Evaluate $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$, and show that the graph of $y = f(x)$ has a vertical asymptote (VA) at $x = 0$.

§ Solution

Let's use the **numerical / tabular approach**:

x	-1	$-\frac{1}{10}$	$-\frac{1}{100}$	$\to 0^-$	$0^+ \leftarrow$	$\frac{1}{100}$	$\frac{1}{10}$	1
$f(x) = \frac{1}{x}$	-1	-10	-100	$\to -\infty$	$\infty \leftarrow$	100	10	1

- Apparently, as x approaches 0 from the **right**, $f(x)$ **increases without bound**. That is, $\lim_{x \to 0^+} f(x) = \infty$.
- Also, as x approaches 0 from the **left**, $f(x)$ **decreases without bound**. That is, $\lim_{x \to 0^-} f(x) = -\infty$.
- Either limit statement implies that the graph of $y = f(x)$ below has a **vertical asymptote ("VA")** at $x = 0$, the y-axis.



- $\lim_{x \to 0} f(x)$ does not exist (DNE).

(See Footnote 2.) §

Example 1 gave us the most basic cases of the following Limit Forms.

$$\left(\text{Limit Form } \frac{1}{0^+}\right) \Rightarrow \infty, \text{ and } \left(\text{Limit Form } \frac{1}{0^-}\right) \Rightarrow -\infty$$

• These Limit Forms can be **rescaled**, as described in Section 2.3, Part A.

"Odd and Even VAs"

Assume that the graph of $y = f(x)$ has a VA at $x = a$.
(The following terminology is informal and nonstandard.)

• If the two one-sided limits at $x = a$ are ∞ and $-\infty$, in either order, then the VA is an "odd VA."

• If those limits are **both** ∞ or **both** $-\infty$, then the VA is an "even VA."

• In Example 1, the y-axis was an **"odd VA,"** partly due to the fact that f was an odd function. The graph of $y = f(x)$ "shot off" in **different** directions around the VA.

• In Example 2 below, the y-axis is an **"even VA,"** partly due to the fact that g is an even function, where $g(x) = \frac{1}{x^2}$. The graph of $y = g(x)$ "shoots off" in the **same** direction around the VA.

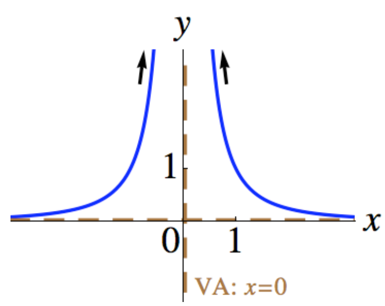
Example 2 (A Graph With an "Even VA")

Evaluate $\lim_{x \to 0^+} \frac{1}{x^2}$, $\lim_{x \to 0^-} \frac{1}{x^2}$, and $\lim_{x \to 0} \frac{1}{x^2}$.

§ Solution

Because $x^2 > 0$ for all $x \neq 0$, **all three** give: $\left(\text{Limit Form } \frac{1}{0^+}\right) \Rightarrow \infty$.

The graph of $y = \frac{1}{x^2}$ is below.



$\lim_{x \to 0^+} \frac{1}{x^2} = \infty$, $\lim_{x \to 0^-} \frac{1}{x^2} = \infty$, and $\lim_{x \to 0} \frac{1}{x^2} = \infty$. §

PART B: EVALUATING INFINITE LIMITS FOR RATIONAL FUNCTIONS

Example 3 (Evaluating Infinite Limits at a Point for a Rational Function)

Let $f(x) = \dfrac{x+1}{x^3+4x^2}$. Evaluate $\lim\limits_{x\to -4^+} f(x)$, $\lim\limits_{x\to -4^-} f(x)$, and $\lim\limits_{x\to -4} f(x)$.

§ Solution

$\lim\limits_{x\to -4} (x+1) = -4+1 = -3$, and $\lim\limits_{x\to -4} \left(x^3+4x^2\right) = (-4)^3 + 4(-4)^2 = 0$.

All three problems give the Limit Form $\dfrac{-3}{0}$. For each, we must know **how** the denominator approaches 0. Since it is easier to analyze **signs of products** than of sums (for example, do we automatically know the sum of *a* and *b* if $a > 0$ and $b < 0$?), we will **factor the denominator**.

WARNING 1: Many students **improperly use methods** such as the "Division Method" and "DTS" from Section 2.3. Those methods are designed to evaluate **"long-run"** limits, **not** limits at a point.

$$\lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} \frac{x+1}{x^3+4x^2}$$

$$= \lim_{x\to -4^+} \frac{\overbrace{x+1}^{\to -3}}{\underbrace{x^2}_{\to 16}\underbrace{(x+4)}_{\to 0^+}} \quad \left(\text{Limit Form } \frac{-3}{0^+}\right)$$

WARNING 2: Write 0^+ and 0^- as necessary.

In the denominator: Remember that "positive times positive equals positive."

$$= -\infty$$

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{x+1}{x^3+4x^2}$$

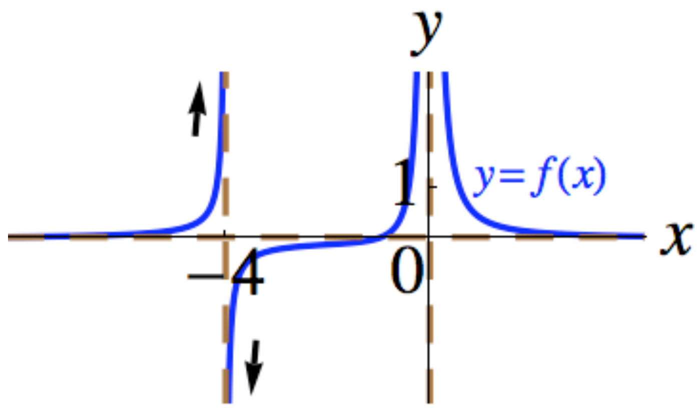
$$= \lim_{x\to -4^-} \frac{\overbrace{x+1}^{\to -3}}{\underbrace{x^2}_{\to 16}\underbrace{(x+4)}_{\to 0^-}} \quad \left(\text{Limit Form } \frac{-3}{0^-}\right)$$

In the denominator: Remember that "positive times negative equals negative."

$$= \infty$$

$\lim_{x \to -4} f(x)$ does not exist (DNE). (See Footnote 2.)

The graph of $y = f(x)$ is below. Observe the **"odd VA"** at $x = -4$.
(Why is there an HA at the *x*-axis?)



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Finding VAs for Graphs of "Simplified" Rational Functions

Let $f(x) = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are nonzero polynomials in x with **no real zeros in common**; this is guaranteed (by the Factor Theorem from Precalculus) if they have **no variable factors in common**, up to constant multiples. Then,

The graph of $y = f(x)$ has a **VA** at $x = a \iff a$ is a **real zero** of $D(x)$.

Note: The numerator and the denominator of $\dfrac{x - \frac{1}{3}}{3x - 1}$ are **common factors up to constant multiples** (the denominator is 3 times the numerator); observe that $\dfrac{1}{3}$ is a **real zero of both**.

Example 4 (Finding VAs for the Graph of a "Simplified" Rational Function; Revisiting Example 3)

Let $f(x) = \dfrac{x+1}{x^3 + 4x^2}$. Find the equations of the vertical asymptotes (VAs) of the graph of $y = f(x)$. **Justify** using limits.

§ Solution

$f(x) = \dfrac{x+1}{x^3 + 4x^2} = \dfrac{x+1}{x^2(x+4)}$, which is **simplified**. The **VAs** have **equations** $x = 0$ and $x = -4$, corresponding to the **real zeros of the denominator**.

To **justify** the VA at $x = 0$, show there is an infinite limit there. **Either** of the following will suffice:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x+1}{x^3 + 4x^2}$$

$$= \lim_{x \to 0^+} \frac{\overbrace{x+1}^{\to 1}}{\underbrace{x^2}_{\to 0^+} \underbrace{(x+4)}_{\to 4}} \quad \left(\text{Limit Form } \frac{1}{0^+} \right)$$

$$= \infty$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x+1}{x^3 + 4x^2}$$

$$= \lim_{x \to 0^-} \frac{\overbrace{x+1}^{\to 1}}{\underbrace{x^2}_{\to 0^+} \underbrace{(x+4)}_{\to 4}} \quad \left(\text{Limit Form } \frac{1}{0^+} \right)$$

$$= \infty$$

- Since 0 is a real zero of $D(x)$ with **multiplicity** 2 (an **even** number), there is an **"even VA"** at $x = 0$.

To **justify** the VA at $x = -4$, show there is an infinite limit there, as we did in Example 3, by showing **either** $\lim_{x \to -4^+} f(x) = -\infty$, or $\lim_{x \to -4^-} f(x) = \infty$.

- Since -4 is a real zero of $D(x)$ with **multiplicity** 1 (an **odd** number), there is an **"odd VA"** at $x = -4$. §

FOOTNOTES

1. **Alternate definition of an infinite limit at a point.** If we say that f has an infinite limit at a $\Leftrightarrow \left(\lim_{x \to a^+} \left| f(x) \right| = \infty \text{ or } \lim_{x \to a^-} \left| f(x) \right| = \infty \right)$, we then extend the idea of an "infinite limit" to examples such as the following:

$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \text{ is a rational number } (x \in \mathbb{Q}) \\ -\dfrac{1}{x}, & \text{if } x \text{ is an irrational number } (x \notin \mathbb{Q}; \text{ really}, x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

as $x \to 0$. In this work, we will not use this definition.

2. **Infinity and the real projective line.**
   • The affinely extended real number system, denoted by $\overline{\mathbb{R}}$ or $[-\infty, \infty]$, includes two points of infinity, one referred to as ∞ (or $+\infty$) and the other referred to as $-\infty$. (We are "adjoining" them to the real number system.) We obtain the two-point compactification of the real numbers. We never refer to ∞ and $-\infty$ as real numbers, though.
   • In the projectively extended real number system, denoted by $\mathbb{R}^*$, ∞ and $-\infty$ are treated as the same (we collapse them together and identify them with one another as ∞), and we then obtain the one-point compactification of the real numbers, also known as the real projective line. Then, $\frac{1}{0} = \infty$, the slope of a vertical line is ∞, $\lim_{x \to 0} \frac{1}{x} = \infty$, and $\lim_{x \to -4} \frac{x+1}{x^2 + 4x} = \infty$.
   • A point at infinity is sometimes added to the complex plane, and it typically corresponds to the "north pole" of a Riemann sphere that the complex plane is wrapped around.
   • See "Projectively Extended Real Numbers" in *MathWorld* (web) and "Real projective line" in *Wikipedia* (web).