SECTION 2.5: THE INDETERMINATE FORMS $\frac{0}{0}$ AND $\frac{\infty}{\infty}$

LEARNING OBJECTIVES

- Understand what it means for a Limit Form to be indeterminate.
- Recognize indeterminate forms, and know what other Limit Forms yield.
- Learn techniques for resolving indeterminate forms when evaluating limits, including factoring, rationalizing numerators and denominators, and (in Chapter 10) L'Hôpital's Rule.

PART A: WHAT ARE INDETERMINATE FORMS?

An indeterminate form is a Limit Form that could yield a variety of real values; the limit might not exist. **Further analysis** is required to know what the limit is.

The seven "classic" **indeterminate forms** are:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ \infty^0,\ 0^0,\ \text{and } 1^\infty.$$

- Observe that the first six forms involve 0 and/or $(\pm)\infty$, while the seventh involves 1 and ∞.

In Section 2.3, Parts D and E, we encountered the indeterminate form $\frac{\pm\infty}{\pm\infty}$.

This is simply written as $\frac{\infty}{\infty}$, since **further analysis** is required, anyway.

(Sometimes, signs matter in the forms. For example, $\left(\text{Limit Form } \infty-\infty\right)$ is indeterminate, while $\left(\text{Limit Form } \infty+\infty\right)\Rightarrow\infty$. See Part D.)

Example 1 (0/0 is an Indeterminate Form)

If $c\in\mathbb{R}$,

$$\lim_{x\to 0}\frac{cx}{x}\quad\left(\text{Limit Form } \frac{0}{0}\right)=\lim_{x\to 0} c$$
$$= c$$

- We are taking a **limit** as $x\to 0$, so the fact that $\frac{cx}{x}$ is undefined at $x=0$ is irrelevant. (See Section 2.1, Part C.)

c could be 2, $-\pi$, etc. $\left(\text{Limit Form } \frac{0}{0}\right)$ can yield **any real number**.

We already know that $\left(\text{Limit Form } \frac{0}{0}\right)$ is **indeterminate**, but we can further show that it can yield **nonexistent limits**:

$$\lim_{x\to 0^+} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0^+} \frac{1}{x} \quad \left(\text{Limit Form } \frac{1}{0^+}\right) = \infty.$$

$$\lim_{x\to 0^-} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0^-} \frac{1}{x} \quad \left(\text{Limit Form } \frac{1}{0^-}\right) = -\infty.$$

$$\lim_{x\to 0} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0} \frac{1}{x}, \text{ which does not exist (DNE).}$$

• We will use $\left(\text{Limit Form } \frac{0}{0}\right)$ when we define derivatives in Chapter 3.

• In turn, L'Hôpital's Rule will use **derivatives** to resolve indeterminate forms, particularly $\frac{0}{0}$ and $\frac{\infty}{\infty}$. (See Chapter 10.) §

Example 2 (∞/∞ is an Indeterminate Form)

If $c \neq 0$,

$$\lim_{x\to \infty} \frac{cx}{x} \quad \left(\text{Limit Form } \frac{\infty}{\infty}\right) = \lim_{x\to \infty} c$$
$$= c$$

c could be 2, $-\pi$, etc. $\left(\text{Limit Form } \frac{\infty}{\infty}\right)$ can yield **any real number**.

(In the Exercises, you will demonstrate how it can yield 0 and ∞.) §

Example 3 (1/0 is Not an Indeterminate Form)

$\left(\text{Limit Form } \frac{1}{0}\right) \Rightarrow \infty$, $-\infty$, or "DNE." We know a lot! The form is **not indeterminate**, although we must know **how** the denominator approaches 0.

$\left(\text{Limit Form } \frac{1}{0^+}\right) \Rightarrow \infty$. $\left(\text{Limit Form } \frac{1}{0^-}\right) \Rightarrow -\infty$. $\lim_{x\to\infty} \frac{1}{\frac{\sin x}{x}}$ "DNE";

see Section 2.3, Example 6. §

PART B: RESOLVING THE $\frac{0}{0}$ FORM BY FACTORING AND CANCELING; GRAPHS OF RATIONAL FUNCTIONS

Let $f(x)=\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are nonzero polynomials in *x*.

We do **not** require simplified form, as we did in Section 2.4. If *a* is a **real zero** of **both** $N(x)$ and $D(x)$, then we can use the Factor Theorem from Precalculus to help us **factor** $N(x)$ and $D(x)$ and **simplify** $f(x)$.

> Factor Theorem
>
> a is a **real zero** of $D(x) \Leftrightarrow (x-a)$ is a **factor** of $D(x)$.
>
> • This also applied to Section 2.4, but it now helps that the same goes for $N(x)$.

Example 4 (Factoring and Canceling/Dividing to Resolve a 0/0 Form)

Let $f(x)=\frac{x^2-1}{x^2-x}$. Evaluate: a) $\lim\limits_{x\to 1} f(x)$ and b) $\lim\limits_{x\to 0^+} f(x)$.

§ Solution to a)

The Limit Form is $\frac{0}{0}$:

$$\lim_{x\to 1} N(x)=\lim_{x\to 1}\left(x^2-1\right)=(1)^2-1=0\text{, and}$$

$$\lim_{x\to 1} D(x)=\lim_{x\to 1}\left(x^2-x\right)=(1)^2-(1)=0.$$

1 is a **real zero** of **both** $N(x)$ and $D(x)$, so $(x-1)$ is a **common factor**.

We will **cancel** $(x-1)$ factors and **simplify** $f(x)$ to resolve the $\frac{0}{0}$ form.

WARNING 1: Some instructors prefer "divide out" to "cancel."

TIP 1: It often saves time to **begin by factoring** and worry about Limit Forms later.

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1}\frac{x^2-1}{x^2-x} \quad \left(\text{Limit Form } \frac{0}{0}\right)\\ &= \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}\\ &= \lim_{x\to 1}\frac{x+1}{x}\\ &= \frac{(1)+1}{(1)}\\ &= 2\end{aligned}$$

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§ Solution to b)

The Limit Form is $\dfrac{-1}{0}$:

$$\lim_{x\to 0^+} N(x) = \lim_{x\to 0^+}\left(x^2-1\right) = (0)^2 - 1 = -1, \text{ and}$$

$$\lim_{x\to 0^+} D(x) = \lim_{x\to 0^+}\left(x^2-x\right) = (0)^2 - (0) = 0.$$

Here, when we **cancel** $(x-1)$ factors and **simplify** $f(x)$, it is a matter of **convenience**. It takes work to see that $\lim_{x\to 0^+} D(x) = \lim_{x\to 0^+}\left(x^2-x\right) = 0^-$,

and then $\left(\text{Limit Form } \dfrac{-1}{0^-}\right) \Rightarrow \infty$.

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\frac{x^2-1}{x^2-x} \quad \left(\text{Limit Form } \frac{-1}{0}\right)\\ &= \lim_{x\to 0^+}\frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}\\ &= \lim_{x\to 0^+}\frac{x+1}{x} \quad \left(\text{Limit Form } \frac{1}{0^+}\right)\\ &= \infty\end{aligned}$$

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<u>The Graph of a Rational Function f at a Point a</u>

The graph of $y = f(x)$ has **one** of the following at $x = a$ $(a \in \mathbb{R})$:

1) The **point** $\left(a, f(a)\right)$, if $f(a)$ is real $\left(a \in \text{Dom}(f)\right)$.

In 2) and 3) below,

- $(x-a)$ is a **factor** of the denominator, $D(x)$.
- That is, a is a **real zero** of $D(x)$, and $a \notin \text{Dom}(f)$.

2) A **VA**, if simplifying $f(x)$ yields the Limit Form $\frac{c}{0}$ as $x \to a$ $(c \neq 0)$.

   - That is, there is at least one $(x-a)$ **factor** of $D(x)$ that **cannot be canceled/divided out**. It will still **force** the denominator towards 0 as $x \to a$.

3) A **hole** at the point (a, L), if $f(a)$ is undefined $\left(a \notin \text{Dom}(f)\right)$, **but** $\lim_{x \to a} f(x) = L$ $(L \in \mathbb{R})$.

   - That is, $(x-a)$ is a **factor** of $D(x)$, **but all such factors** can be **canceled/divided out** by $(x-a)$ factor(s) in the numerator. Then, the denominator is **no longer forced** towards 0.
   - A hole can only occur if we start with the Limit Form $\frac{0}{0}$, because a **denominator** approaching 0 can only be **prevented** from "exploding" $f(x)$ if the **numerator** approaches 0, as well. (If the numerator **fails** to prevent this, we get a **VA**.)

Example 5 (VAs and Holes on the Graph of a Rational Function; Revisiting Example 4)

Let $f(x) = \frac{x^2 - 1}{x^2 - x}$. Identify any **vertical asymptotes (VAs)** and **holes** on the graph of $y = f(x)$.

§ Solution

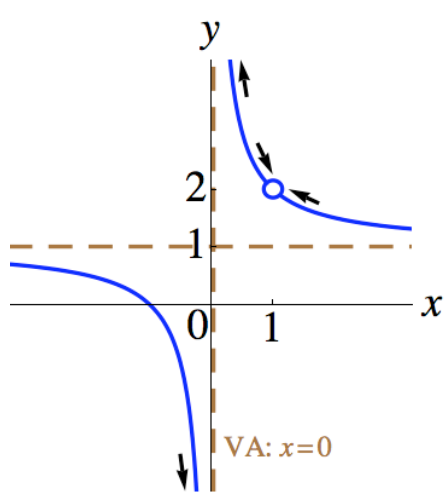
In Example 4, we saw that: $f(x) = \frac{x^2 - 1}{x^2 - x} = \frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}} = \frac{x+1}{x} \quad (x \neq 1)$.

The **real zeros** of $x^2 - x$ are 0 and 1, so they correspond to **VAs or holes**.

• In 4a, we found that: $\lim_{x \to 1} f(x) = 2$, even though $1 \notin \text{Dom}(f)$, so the graph has a **hole** at the point $(1, 2)$. As $x \to 1$, the factor $(x-1) \to 0$. When we simplify $f(x)$, we **cancel (divide out) all** of the $(x-1)$ factors in the **denominator**. The new denominator, x, **no longer approaches 0**, and the overall limit **exists**.

• In 4b, we found that: $\lim_{x \to 0^+} f(x) = \infty$, so the graph has a **VA** at $x = 0$ (the y-axis). When we simplify $f(x)$, we **cannot cancel** (divide out) the x factor in the **denominator**. As $x \to 0$, the new denominator, x, **still approaches 0**.

The graph of $y = \frac{x^2 - 1}{x^2 - x}$ $\left(\text{or } y = \frac{x+1}{x} \ (x \neq 1), \text{ or } y = 1 + \frac{1}{x} \ (x \neq 1) \right)$ is below.



Since 0 is a zero of the new denominator, x, with multiplicity 1, the VA at $x = 0$ is an **"odd VA."**

Why is there an **HA** at $y = 1$? §

PART C: RESOLVING THE $\dfrac{0}{0}$ FORM BY RATIONALIZING; GRAPHS OF ALGEBRAIC FUNCTIONS

Graphs of algebraic functions can also have points, VAs, and holes. Unlike graphs of rational functions, they can also have **"blank spaces"** where there are no points for infinitely many real values of *x*.

Example 6 (Rationalizing a Numerator to Resolve a 0/0 Form)

Evaluate $\displaystyle\lim_{x\to 0}\frac{\sqrt{9-x}-3}{x}$.

§ Solution

Observe that $\sqrt{9-x}$ is **real** on a punctured neighborhood of 0.
We assume $x \approx 0$.

$$\lim_{x\to 0}\frac{\sqrt{9-x}-3}{x} \qquad \left(\text{Limit Form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left[\frac{\left(\sqrt{9-x}-3\right)}{x}\cdot\frac{\left(\sqrt{9-x}+3\right)}{\left(\sqrt{9-x}+3\right)}\right] \qquad \left(\text{Rationalizing the numerator}\right)$$

$$= \lim_{x\to 0}\frac{\left(\sqrt{9-x}\right)^2-(3)^2}{x\left(\sqrt{9-x}+3\right)} \qquad \left(\textbf{WARNING 2: } \text{Write the entire denominator! It's not just } x.\right)$$

$$= \lim_{x\to 0}\frac{(9-x)-9}{x\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x\to 0}\frac{\overset{(-1)}{\cancel{-x}}}{\cancel{x}\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x\to 0}\frac{-1}{\sqrt{9-x}+3}$$

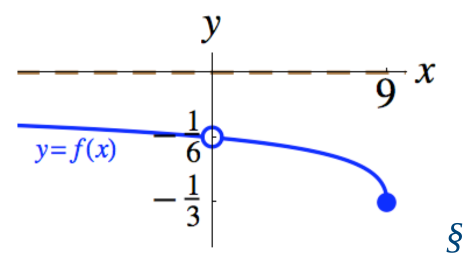
$$= \frac{-1}{\sqrt{9-(0)}+3}$$

$$= -\frac{1}{6}$$

$\lim_{x \to 0} f(x) = -\frac{1}{6}$, even though $0 \notin \text{Dom}(f)$, where $f(x) = \frac{\sqrt{9-x}-3}{x}$.

Therefore, the graph of $y = f(x)$ has a **hole** at the point $\left(0, -\frac{1}{6}\right)$.

The graph of $y = f(x)$ is below. What is $\text{Dom}(f)$?



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PART D: LIMIT FORMS THAT ARE NOT INDETERMINATE

Cover up the "Yields" columns below and guess at the results of the Limit Forms $(c \in \mathbb{R})$. Experiment with sequences of numbers and with extreme numbers.

For example, for $\infty^{-\infty}$, or $\frac{1}{\infty^{\infty}}$, look at $(1000)^{-10,000} = \frac{1}{(1000)^{10,000}}$.

Fractions

Limit Form	Yields
$\frac{1}{\infty}$	0^+
$\frac{1}{0^+}$	∞
$\frac{\infty}{1}$	∞
$\frac{\infty}{0^+}$	∞
$\frac{0^+}{\infty}$	0^+

Sums, Differences, Products

Limit Form	Yields
$\infty + c$	∞
$-\infty + c$	$-\infty$
$\infty + \infty$	∞
$-\infty - \infty$	$-\infty$
$\infty \cdot 1$	∞
$\infty \cdot \infty$	∞

With Exponents

Limit Form	Yields
∞^{∞}	∞
$\infty^{-\infty}$	0^+
0^{∞}	0
2^{∞}	∞
$\left(\frac{1}{2}\right)^{\infty}$	0