SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

LEARNING OBJECTIVES

• Understand and be able to rigorously apply the Squeeze (Sandwich) Theorem when evaluating limits at a point and “long-run” limits at \((\pm)\) infinity.

PART A: APPLYING THE SQUEEZE (SANDWICH) THEOREM TO LIMITS AT A POINT

We will formally state the Squeeze (Sandwich) Theorem in Part B.

Example 1 below is one of many basic examples where we use the Squeeze (Sandwich) Theorem to show that \(\lim_{x \to 0} f(x) = 0\), where \(f(x)\) is the product of a sine or cosine expression and a monomial of even degree.

• The idea is that “something approaching 0” times “something bounded” (that is, trapped between two real numbers) will approach 0. Informally,

\[
\text{(Limit Form } 0 \cdot \text{bounded)} \Rightarrow 0.
\]

Example 1 (Applying the Squeeze (Sandwich) Theorem to a Limit at a Point)

Let \(f(x) = x^2 \cos\left(\frac{1}{x}\right)\). Prove that \(\lim_{x \to 0} f(x) = 0\).

§ Solution

• We first bound \(\cos\left(\frac{1}{x}\right)\),

\[-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad (\forall x \neq 0) \Rightarrow
\]

which is real for all \(x \neq 0\).

• Multiply all three parts by \(x^2\) so that the middle part becomes \(f(x)\).

\[-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2 \quad (\forall x \neq 0) \Rightarrow
\]

WARNING 1: We must observe that \(x^2 > 0\) for all \(x \neq 0\), or at least on a punctured neighborhood of \(x = 0\), so that we can multiply by \(x^2\) without reversing inequality symbols.
As \( x \to 0 \), the **left and right parts approach 0**. Therefore, by the Squeeze (Sandwich) Theorem, the **middle part**, \( f(x) \), is **forced to approach 0**, also. The middle part is “squeezed” or “sandwiched” between the left and right parts, so it **must approach the same limit** as the other two do.

\[
\lim_{x \to 0} (-x^2) = 0, \quad \text{and} \quad \lim_{x \to 0} x^2 = 0, \quad \text{so} \quad \lim_{x \to 0} x^2 \cos \left( \frac{1}{x} \right) = 0 \quad \text{by the Squeeze Theorem.}
\]

Shorthand: As \( x \to 0 \),

\[
-x^2 \to 0 \leq x^2 \cos \left( \frac{1}{x} \right) \leq x^2 \to 0 \quad (\forall x \neq 0).
\]

Therefore, \( \to 0 \) by the Squeeze (Sandwich) Theorem

The graph of \( y = x^2 \cos \left( \frac{1}{x} \right) \), together with the squeezing graphs of \( y = -x^2 \) and \( y = x^2 \), is below.

(The axes are scaled differently.)
In Example 2 below, $f(x)$ is the **product** of a **sine or cosine expression** and a **monomial of odd degree**.

**Example 2 (Handling Complications with Signs)**

Let $f(x) = x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right)$. Use the Squeeze Theorem to find $\lim_{x \to 0} f(x)$.

**§ Solution 1 (Using Absolute Value)**

- We first **bound** $\sin \left( \frac{1}{\sqrt[3]{x}} \right)$, 
  
  $-1 \leq \sin \left( \frac{1}{\sqrt[3]{x}} \right) \leq 1$ \hspace{1em} (\forall x \neq 0) \hspace{1em} \Rightarrow$

  which is **real** for all $x \neq 0$.

- **WARNING 2**: The problem with multiplying all three parts by $x^3$ is that $x^3 < 0$ when $x < 0$. The $\leq$ inequality symbols would have to be **reversed** for $x < 0$.

  Instead, we use **absolute value** here. We could write

  $0 \leq \left| \sin \left( \frac{1}{\sqrt[3]{x}} \right) \right| \leq 1$ \hspace{1em} (\forall x \neq 0),

  but we assume that absolute values are **nonnegative**.

- **Multiply** both sides of the inequality by $|x^3|$. We know

  $|x^3| \left| \sin \left( \frac{1}{\sqrt[3]{x}} \right) \right| \leq |x^3|$ \hspace{1em} (\forall x \neq 0) \hspace{1em} \Rightarrow$

  $|x^3| > 0$ \hspace{1em} (\forall x \neq 0).

- “The **product** of absolute values equals the absolute value of the product.”

  $|x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right)| \leq |x^3|$ \hspace{1em} (\forall x \neq 0) \hspace{1em} \Rightarrow$

- If, say, $|a| \leq 4$, then

  $-4 \leq a \leq 4$. Similarly:

  $-|x^3| \leq x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right) \leq |x^3|$ \hspace{1em} (\forall x \neq 0) \hspace{1em} \Rightarrow$
• Now, apply the Squeeze (Sandwich) Theorem.

\[ \lim_{x \to 0} \left(-x^3 \right) = 0, \text{ and } \lim_{x \to 0} x^3 = 0, \text{ so} \]
\[ \lim_{x \to 0} x^3 \sin \left(\frac{1}{\sqrt[3]{x}}\right) = 0 \text{ by the Squeeze Theorem.} \]

Shorthand: As \( x \to 0 \),
\[ -\lim_{x \to 0} x^3 \leq \lim_{x \to 0} x^3 \sin \left(\frac{1}{\sqrt[3]{x}}\right) \leq \lim_{x \to 0} x^3 \quad (\forall x \neq 0). \]

§ Solution 2 (Split Into Cases: Analyze Right-Hand and Left-Hand Limits Separately)

First, we analyze: \( \lim_{x \to 0^+} x^3 \sin \left(\frac{1}{\sqrt[3]{x}}\right) \).

Assume \( x > 0 \), since we are taking a limit as \( x \to 0^+ \).

• We first bound \( \sin \left(\frac{1}{\sqrt[3]{x}}\right) \), \[-1 \leq \sin \left(\frac{1}{\sqrt[3]{x}}\right) \leq 1 \quad (\forall x > 0) \quad \Rightarrow\]
which is real for all \( x \neq 0 \).

• Multiply all three parts by \( x^3 \) so that the middle part becomes \( f(x) \). We know \( x^3 > 0 \) for all \( x > 0 \).

• Now, apply the Squeeze (Sandwich) Theorem.

\[ \lim_{x \to 0^+} \left(-x^3 \right) = 0, \text{ and } \lim_{x \to 0^+} x^3 = 0, \text{ so} \]
\[ \lim_{x \to 0^+} x^3 \sin \left(\frac{1}{\sqrt[3]{x}}\right) = 0 \text{ by the Squeeze Theorem.} \]

Shorthand: As \( x \to 0^+ \),
\[ -\lim_{x \to 0^+} x^3 \leq \lim_{x \to 0^+} x^3 \sin \left(\frac{1}{\sqrt[3]{x}}\right) \leq \lim_{x \to 0^+} x^3 \quad (\forall x > 0). \]
Second, we analyze: $\lim_{x \to 0^-} x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right)$.

Assume $x < 0$, since we are taking a limit as $x \to 0^-$.

- We first **bound** $\sin \left( \frac{1}{\sqrt[3]{x}} \right)$, $-1 \leq \sin \left( \frac{1}{\sqrt[3]{x}} \right) \leq 1 \quad (\forall x < 0) \implies$

  which is **real** for all $x \neq 0$.

- **Multiply** all three parts by $x^3$ so that the middle part becomes $f(x)$. We know $x^3 < 0$ for all $x < 0$, so we **reverse** the $\leq$ inequality symbols.

- **Reversing the compound inequality** will make it easier to read.

- Now, apply the **Squeeze (Sandwich) Theorem**. $\lim_{x \to 0^-} x^3 = 0$, and $\lim_{x \to 0^-} (-x^3) = 0$, so

  $\lim_{x \to 0^-} x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right) = 0$ by the Squeeze (Sandwich) Theorem.

  **Shorthand:** As $x \to 0^-$, $x^3 \leq x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right) \leq -x^3 \quad (\forall x < 0)$.

  Therefore, $\to 0$ by the Squeeze (Sandwich) Theorem.

Now, $\lim_{x \to 0^+} x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right) = 0$, and $\lim_{x \to 0^-} x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right) = 0$, so

$\lim_{x \to 0} x^3 \sin \left( \frac{1}{\sqrt[3]{x}} \right) = 0$. §
Example 3 (Limits are Local)

Use \( \lim_{x \to 0} x^2 = 0 \) and \( \lim_{x \to 0} x^6 = 0 \) to show that \( \lim_{x \to 0} x^4 = 0 \).

§ Solution

Let \( I = (-1, 1) \setminus \{0\} \). \( I \) is a punctured neighborhood of 0.

Shorthand: As \( x \to 0 \),

\[
\frac{x^6}{x^2} \leq x^4 \leq \frac{x^2}{x^2} \quad \text{Therefore, } x 
\]

\[
\leq \frac{x^2}{x^2} \quad \text{(\( \forall x \in I \))}
\]

Therefore, by the Squeeze (Sandwich) Theorem

\[
\text{WARNING 3: The direction of the } \leq \text{ inequality symbols may confuse students. Observe that } 
\]

\[
\left( \frac{1}{2} \right)^4 = \frac{1}{16}, \quad \left( \frac{1}{2} \right)^2 = \frac{1}{4}, \quad \text{and } \frac{1}{16} < \frac{1}{4}. \]

We conclude: \( \lim_{x \to 0} x^4 = 0 \).

We do not need the compound inequality to hold true for all nonzero values of \( x \). We only need it to hold true on some punctured neighborhood of 0 so that we may apply the Squeeze (Sandwich) Theorem to the two-sided limit \( \lim_{x \to 0} x^4 \). This is because “Limits are Local.”

As seen below, the graphs of \( y = x^6 \) and \( y = x^2 \) squeeze (from below and above, respectively) the graph of \( y = x^4 \) on \( I \). In Chapter 6, we will be able to find the areas of the bounded regions.
PART B: THE SQUEEZE (SANDWICH) THEOREM

We will call $B$ the “bottom” function and $T$ the “top” function.

The Squeeze (Sandwich) Theorem

Let $B$ and $T$ be functions such that $B(x) \leq f(x) \leq T(x)$ on a punctured neighborhood of $a$.

If $\lim_{x\to a} B(x) = L$ and $\lim_{x\to a} T(x) = L$ ($L \in \mathbb{R}$), then $\lim_{x\to a} f(x) = L$.

Variation for Right-Hand Limits at a Point

Let $B(x) \leq f(x) \leq T(x)$ on some right-neighborhood of $a$.

If $\lim_{x\to a^+} B(x) = L$ and $\lim_{x\to a^+} T(x) = L$ ($L \in \mathbb{R}$), then $\lim_{x\to a^+} f(x) = L$.

Variation for Left-Hand Limits at a Point

Let $B(x) \leq f(x) \leq T(x)$ on some left-neighborhood of $a$.

If $\lim_{x\to a^-} B(x) = L$ and $\lim_{x\to a^-} T(x) = L$ ($L \in \mathbb{R}$), then $\lim_{x\to a^-} f(x) = L$.

PART C: VARIATIONS FOR “LONG-RUN” LIMITS

In the upcoming Example 4, $f(x)$ is the quotient of a sine or cosine expression and a polynomial.

• The idea is that “something bounded” divided by “something approaching $(\pm)\,\text{infinity}$” will approach 0. Informally,

\[
\left( \lim_{x \to \pm \infty} \text{bounded} \right) \Rightarrow 0.
\]
Example 4 (Applying the Squeeze (Sandwich) Theorem to a “Long-Run” Limit; Revisiting Section 2.3, Example 6)

Evaluate: a) \( \lim_{x \to \infty} f(x) \) and b) \( \lim_{x \to -\infty} f(x) \), where \( f(x) = \frac{\sin x}{x} \).

§ Solution to a)

Assume \( x > 0 \), since we are taking a limit as \( x \to \infty \).

- We first bound \( \sin x \).
  \[-1 \leq \sin x \leq 1 \quad (\forall x > 0) \quad \Rightarrow \]

- Divide all three parts by \( x \) \((x > 0)\) so that the middle part becomes \( f(x) \).
  \[- \frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \quad (\forall x > 0) \quad \Rightarrow \]

- Now, apply the Squeeze (Sandwich) Theorem.
  \[ \lim_{x \to \infty} \left( - \frac{1}{x} \right) = 0, \quad \text{and} \quad \lim_{x \to \infty} \frac{1}{x} = 0, \quad \text{so} \]
  \[ \lim_{x \to \infty} \frac{\sin x}{x} = 0 \quad \text{by the Squeeze Theorem.} \]

Shorthand: As \( x \to \infty \),

\[ - \frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \quad \text{for} \quad x \to 0 \quad \text{by the Squeeze (Sandwich) Theorem} \]

§ Solution to b)

Assume \( x < 0 \), since we are taking a limit as \( x \to -\infty \).

- We first bound \( \sin x \).
  \[-1 \leq \sin x \leq 1 \quad (\forall x < 0) \quad \Rightarrow \]

- Divide all three parts by \( x \) so that the middle part becomes \( f(x) \). But \( x < 0 \), so we must reverse the \( \leq \) inequality symbols.
  \[- \frac{1}{x} \geq \frac{\sin x}{x} \geq \frac{1}{x} \quad (\forall x < 0) \quad \Rightarrow \]

- Reversing the compound inequality will make it easier to read.
  \[ \frac{1}{x} \leq \frac{\sin x}{x} \leq - \frac{1}{x} \quad (\forall x < 0) \quad \Rightarrow \]
• Now, apply the **Squeeze (Sandwich) Theorem**.

\[
\lim_{{x \to -\infty}} \frac{1}{x} = 0, \text{ and } \lim_{{x \to -\infty}} \left( -\frac{1}{x} \right) = 0, \text{ so } \\
\lim_{{x \to -\infty}} \frac{\sin x}{x} = 0 \text{ by the Squeeze Theorem.}
\]

Shorthand: As \( x \to -\infty \),

\[
\frac{1}{\lim_{{x \to 0}}} \leq \frac{\sin x}{x} \leq -\frac{1}{\lim_{{x \to 0}}} \quad (\forall x < 0)
\]

Therefore, \( \to 0 \) by the Squeeze (Sandwich) Theorem.

The graph of \( y = \frac{\sin x}{x} \), together with the squeezing graphs of \( y = -\frac{1}{x} \) and \( y = \frac{1}{x} \), is below. We can now justify the **HA** at \( y = 0 \) (the x-axis).

**Variation for “Long-Run” Limits to the Right**

Let \( B(x) \leq f(x) \leq T(x) \) on some \( x \)-interval of the form \((c, \infty), \ c \in \mathbb{R}\).

If \( \lim_{{x \to \infty}} B(x) = L \) and \( \lim_{{x \to \infty}} T(x) = L \) \( (L \in \mathbb{R}) \), then \( \lim_{{x \to \infty}} f(x) = L \).

• In Example 4a, we used \( c = 0 \). We need the compound inequality to hold “forever” after some point \( c \).

**Variation for “Long-Run” Limits to the Left**

Let \( B(x) \leq f(x) \leq T(x) \) on some \( x \)-interval of the form \((-\infty, c), \ c \in \mathbb{R}\).

If \( \lim_{{x \to -\infty}} B(x) = L \) and \( \lim_{{x \to -\infty}} T(x) = L \) \( (L \in \mathbb{R}) \), then \( \lim_{{x \to -\infty}} f(x) = L \).