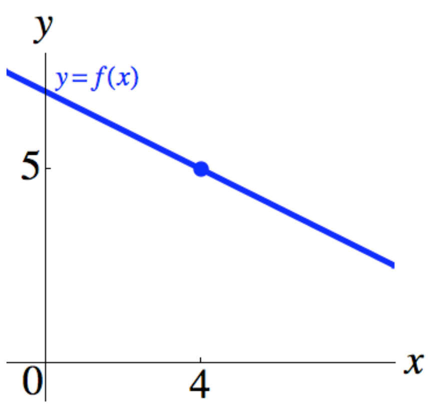
SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

LEARNING OBJECTIVES

• Know rigorous definitions of limits, and use them to rigorously prove limit statements.

PART A: THE "STATIC" APPROACH TO LIMITS

We will use the example $\lim_{x \to 4}\left(7 - \frac{1}{2}x\right) = 5$ in our quest to **rigorously define** what a **limit at a point** is. We consider $\lim_{x \to a} f(x) = L$, where $f(x) = 7 - \frac{1}{2}x$, $a = 4$, and $L = 5$. The graph of $y = f(x)$ is the line below.



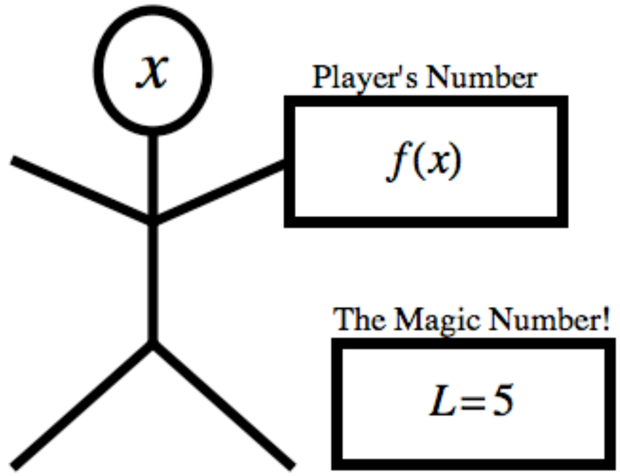
The **"dynamic" view** of limits states that, as *x* "approaches" or "gets closer to" 4, $f(x)$ "approaches" or "gets closer to" 5. (See Section 2.1, Footnote 2.)

The precise approach takes on a more **"static" view**. The idea is that, if *x* is close to 4, then $f(x)$ is close to 5.

The Lottery Analogy

Imagine a lottery in which every $x \in \text{Dom}(f)$ represents a player. However, we disqualify $x = a$ (here, $x = 4$), because that person manages the lottery. (See Section 2.1, Part C.)

Each player is assigned a lottery number by the rule $f(x) = 7 - \frac{1}{2}x$.
The "exact" winning lottery number (the "target") turns out to be $L = 5$.



When Does Player x Win?

In this lottery, more than one player can win, and it is sufficient for a player to be **"close enough"** to the "target" in order to win. In particular, Player x wins $(x \neq a)$ $\Leftrightarrow$ the player's lottery number, $f(x)$, is **strictly within** ε units of L, where $\varepsilon > 0$. The Greek letter ε ("epsilon") often represents a **small positive quantity**. Here, ε is a **tolerance level** that measures how liberal the lottery is in determining winners.

Symbolically:

$$\text{Player } x \text{ wins } (x \neq a) \Leftrightarrow L - \varepsilon < f(x) < L + \varepsilon$$

Subtract L from all three parts.

$$\Leftrightarrow -\varepsilon < f(x) - L < \varepsilon$$

$-1 < r < 1 \Leftrightarrow |r| < 1.$
Similarly:

$$\Leftrightarrow |f(x) - L| < \varepsilon$$

$|f(x) - L|$ is the **distance** (along the y-axis) between Player x's lottery number, $f(x)$, and the "target" L.

Player x **wins** $(x \neq a)$ $\Leftrightarrow$ this distance is less than ε.

Where Do We Look for Winners?

We only care about players that are "close" to $x = a$ (here, $x = 4$), excluding a itself. These players x are strictly between 0 and δ units of a, where $\delta > 0$. Like ε, the Greek letter δ ("delta") often represents a **small positive quantity**. δ is the **half-width** of a **punctured δ-neighborhood** of a.

Symbolically:

$$\text{Player } x \text{ is "close" to } a \iff a - \delta < x < a + \delta \quad (x \neq a)$$

That is, $x \in (a - \delta,\ a + \delta) \setminus \{a\}$.
Subtract a from all three parts.

$$\iff -\delta < x - a < \delta \quad (x \neq a)$$

$$\iff 0 < |x - a| < \delta$$

$|x - a|$ is the **distance** between Player x and a.

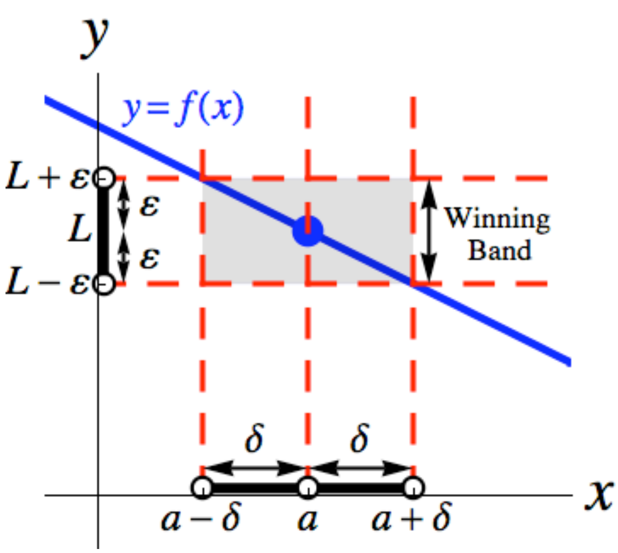
Player x is **"close" to a** $\iff$ this distance is strictly between 0 and δ.

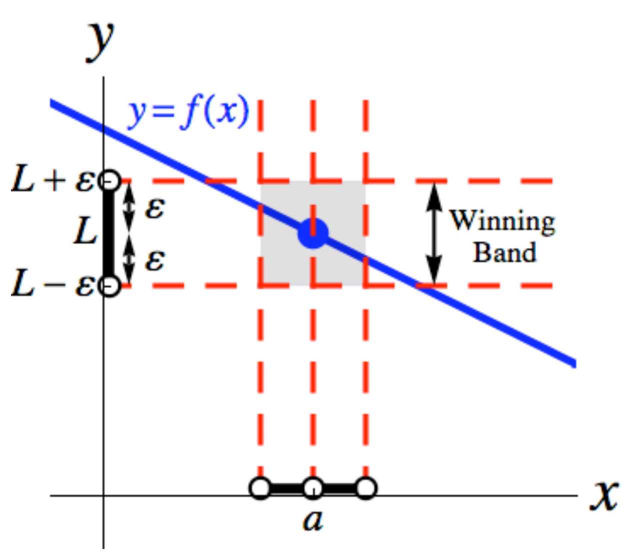
- If the distance is 0, we have $x = a$, which is disqualified.

In the figure on the left, the value for δ is giving us a **punctured δ-neighborhood** of a in which **everyone wins**.

- In this sense, if x is close to a, then $f(x)$ is close to L.

Observe that any **smaller** positive value for δ could also have been chosen. (See the figure on the right. The dashed lines are **not** asymptotes; they indicate the boundaries of the open intervals and the puncture at $x = a$.)





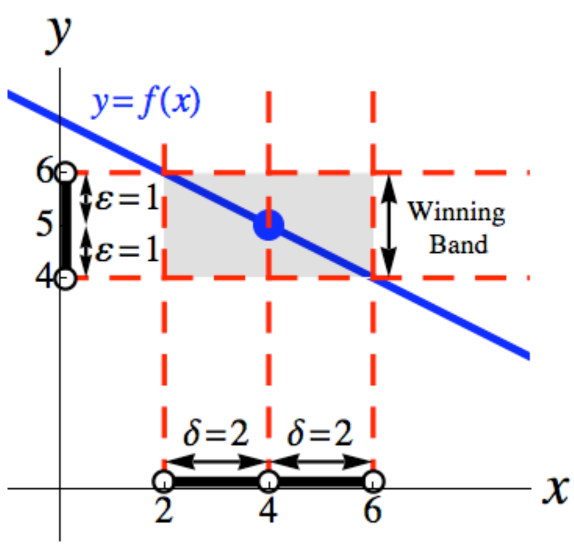
How Does the "Static" Approach to Limits Relate to the "Dynamic" Approach?

Why is $\lim_{x \to 4} \left(7 - \frac{1}{2}x \right) = 5$? Because, **regardless of how small** we make the **tolerance level** ε and how tight we make the lottery for the players, **there is** a value for δ for which the corresponding **punctured** δ**-neighborhood** of $a = 4$ is made up **entirely of winners**. That is, the corresponding "punctured box" (see the shaded boxes in the figures) **traps the graph** of $y = f(x)$ on the punctured δ-neighborhood.

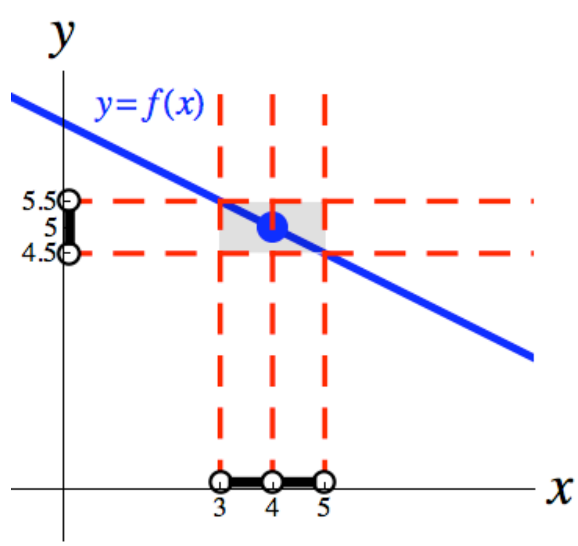
As $\varepsilon \to 0^+$, we can **choose values** for δ in such a way that the corresponding shaded "punctured boxes" **always trap the graph** and zoom in, or collapse in, on the **point** $(4, 5)$. (This would have been the case even if that point had been **deleted** from the graph.) In other words, **there are always winners close to** $a = 4$.

- As *x* gets arbitrarily close to *a*, $f(x)$ gets arbitrarily close to *L*.

If $\varepsilon = 1$, we can choose $\delta = 2$.



If $\varepsilon = 0.5$, we can choose $\delta = 1$.



For this example, if ε is **any positive real number**, we can choose $\delta = 2\varepsilon$. Why is that?

- Graphically, we can exploit the fact that the **slope** of the line $y = 7 - \frac{1}{2}x$ is $-\frac{1}{2}$. Remember, slope $= \frac{\text{rise}}{\text{run}}$. Along the line, an *x*-run of 2 units corresponds to a *y*-drop of 1 unit.

- We will demonstrate this rigorously in Example 1.

PART B: THE PRECISE ε - δ DEFINITION OF A LIMIT AT A POINT

The Precise ε-δ Definition of a Limit at a Point (Version 1)

For $a, L \in \mathbb{R}$, if a function f is defined on a **punctured neighborhood** of a,

$\lim_{x \to a} f(x) = L \Leftrightarrow$ **for every** $\varepsilon > 0$, **there exists** a $\delta > 0$ such that,

if $0 < |x - a| < \delta$ (that is, if x is "close" to a, excluding a itself),

then $|f(x) - L| < \varepsilon$ (that is, $f(x)$ is "close" to L).

Variation Using Interval Form

We can replace $0 < |x - a| < \delta$ with: $x \in (a - \delta,\ a + \delta) \setminus \{a\}$.

We can replace $|f(x) - L| < \varepsilon$ with: $f(x) \in (L - \varepsilon,\ L + \varepsilon)$.

The Precise ε-δ Definition of a Limit at a Point (Version 2: More Symbolic)

For $a, L \in \mathbb{R}$, if a function f is defined on a **punctured neighborhood** of a,

$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni$

$\left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$.

Example 1 (Proving the Limit Statement from Part A)

Prove $\lim_{x \to 4} \left(7 - \frac{1}{2}x \right) = 5$ using a precise ε-δ definition of a limit at a point.

§ Solution

We have: $f(x) = 7 - \frac{1}{2}x$, $a = 4$, and $L = 5$.

We need to show:

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right); \text{ i.e.,}$$

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \left(0 < |x - 4| < \delta \Rightarrow \left| \left(7 - \frac{1}{2}x \right) - (5) \right| < \varepsilon \right).$$

Rewrite $|f(x) - L|$ in terms of $|x - a|$; here, $|x - 4|$:

$$|f(x) - L| = \left| \left(7 - \frac{1}{2}x \right) - (5) \right|$$

$$= \left| -\frac{1}{2}x + 2 \right|$$

Factor out $-\frac{1}{2}$, the coefficient of *x*.

To divide the $+2$ term by $-\frac{1}{2}$, we multiply it by -2 and obtain -4.

$$= \left| -\frac{1}{2}(x - 4) \right|$$

$$= \left| -\frac{1}{2} \right| |x - 4|$$

This is because, if *m* and *n* represent real quantities, then $|mn| = |m||n|$.

$$= \frac{1}{2}|x - 4|$$

We have: $|f(x) - L| = \frac{1}{2}|x - 4|$; call this statement *.

Assuming ε is fixed $(\varepsilon > 0)$, find an appropriate value for δ.

We will find a value for δ that corresponds to a **punctured δ-neighborhood** of $a = 4$ in which **everyone wins**. This means that, for every player *x* in there:

$$\left| f(x) - L \right| < \varepsilon \quad \Leftrightarrow$$
$$\frac{1}{2}\left| x - 4 \right| < \varepsilon \quad (\text{by } *) \quad \Leftrightarrow$$
$$\left| x - 4 \right| < 2\varepsilon$$

We choose $\delta = 2\varepsilon$. We will formally justify this choice in our verification step.

Observe that, since $\varepsilon > 0$, then our $\delta > 0$.

Verify that our choice for δ is appropriate.

We will show that, given ε and our choice for δ $(\delta = 2\varepsilon)$,
$0 < \left| x - a \right| < \delta \Rightarrow \left| f(x) - L \right| < \varepsilon$.

$$0 < \left| x - a \right| < \delta \Rightarrow$$
$$0 < \left| x - 4 \right| < \delta \Rightarrow$$
$$0 < \left| x - 4 \right| < 2\varepsilon \Rightarrow$$
$$0 < \frac{1}{2}\left| x - 4 \right| < \varepsilon \Rightarrow$$
$$\left| f(x) - L \right| < \varepsilon \quad (\text{by } *)$$

Note: It is true that: $0 < \left| f(x) - L \right| < \varepsilon$, but the first inequality $\left(0 < \left| f(x) - L \right|\right)$ does not help us.

Q.E.D.

(“Quod erat demonstrandum” – Latin for “which was to be demonstrated / proven / shown.” This is a formal end to a proof.) §

PART C: DEFINING ONE-SIDED LIMITS AT A POINT

The precise definition of $\lim_{x \to a} f(x) = L$ can be modified for **left-hand** and **right-hand** limits. The **only** changes are the ***x*-intervals** where we look for winners. (See red type.) These *x*-intervals will no longer be symmetric about *a*.

- Therefore, we will use **interval form** instead of absolute value notation when describing these *x*-intervals.
- Also, we will let δ represent the **entire width** of an *x*-interval, not just half the width of a punctured *x*-interval.

The Precise ε-δ Definition of a Left-Hand Limit at a Point

For $a, L \in \mathbb{R}$, if a function f is defined on a **left-neighborhood** of a,

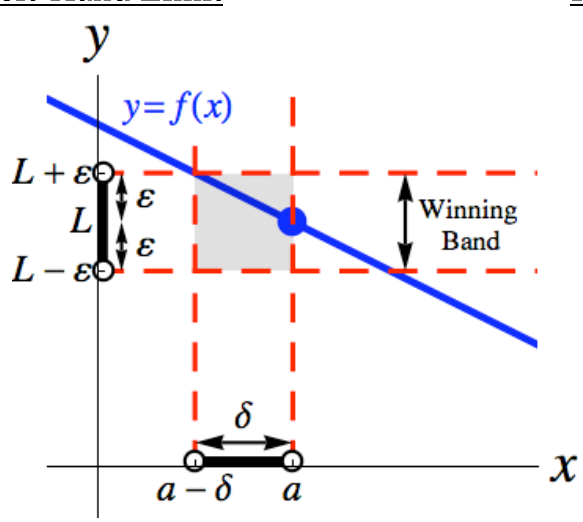
$$\lim_{x \to a^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni \left[x \in (a - \delta, a) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

The Precise ε-δ Definition of a Right-Hand Limit at a Point

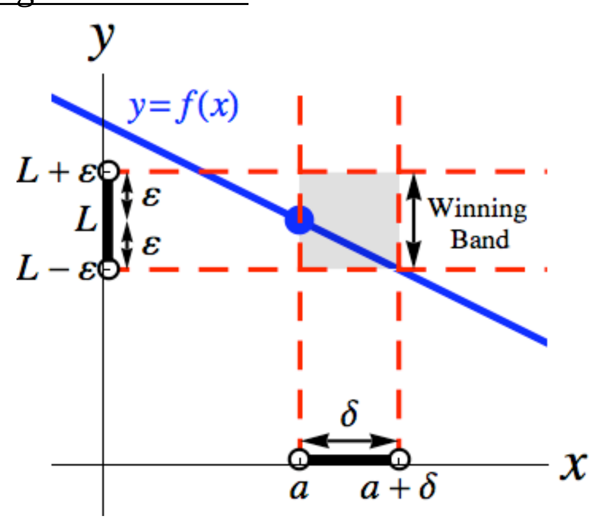
For $a, L \in \mathbb{R}$, if a function f is defined on a **right-neighborhood** of a,

$$\lim_{x \to a^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni \left[x \in (a, a + \delta) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

Left-Hand Limit



Right-Hand Limit



PART D: DEFINING "LONG-RUN" LIMITS

The precise definition of $\lim_{x \to a} f(x) = L$ can also be modified for **"long-run"** limits. Again, the **only** changes are the ***x*-intervals** where we look for winners. (See red type.) These *x*-intervals will be unbounded.

- Therefore, we will use **interval form** instead of absolute value notation when describing these *x*-intervals.
- Also, instead of using δ, we will use *M* (think "Million") and *N* (think "Negative million") to denote "points of no return."

The Precise ε-M Definition of $\lim_{x \to \infty} f(x) = L$

For $L \in \mathbb{R}$, if a function f is defined on some interval (c, ∞), $c \in \mathbb{R}$.

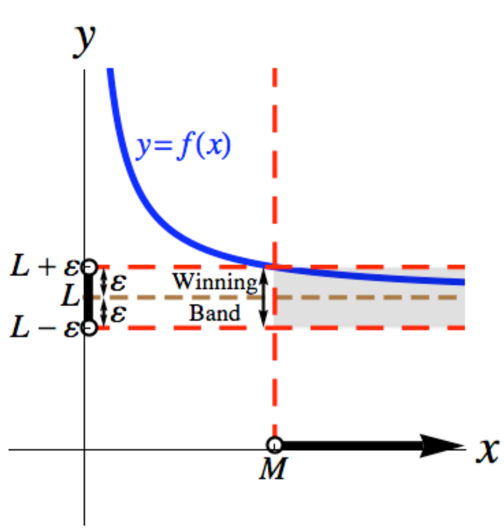
$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists M \in \mathbb{R} \ni$$
$$\left[x > M;\ \text{that is, } x \in (M, \infty) \Rightarrow |f(x) - L| < \varepsilon\right].$$

The Precise ε-N Definition of $\lim_{x \to -\infty} f(x) = L$

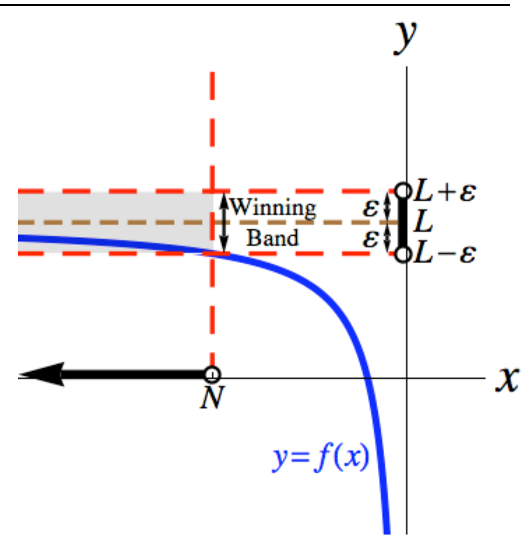
For $L \in \mathbb{R}$, if a function f is defined on some interval $(-\infty, c)$, $c \in \mathbb{R}$.

$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists N \in \mathbb{R} \ni$$
$$\left[x < N;\ \text{that is, } x \in (-\infty, N) \Rightarrow |f(x) - L| < \varepsilon\right].$$

$\lim_{x \to \infty} f(x) = L$; here, $f(x) = \frac{1}{x} + 2$



$\lim_{x \to -\infty} f(x) = L$; here, $f(x) = \frac{1}{x} + 2$

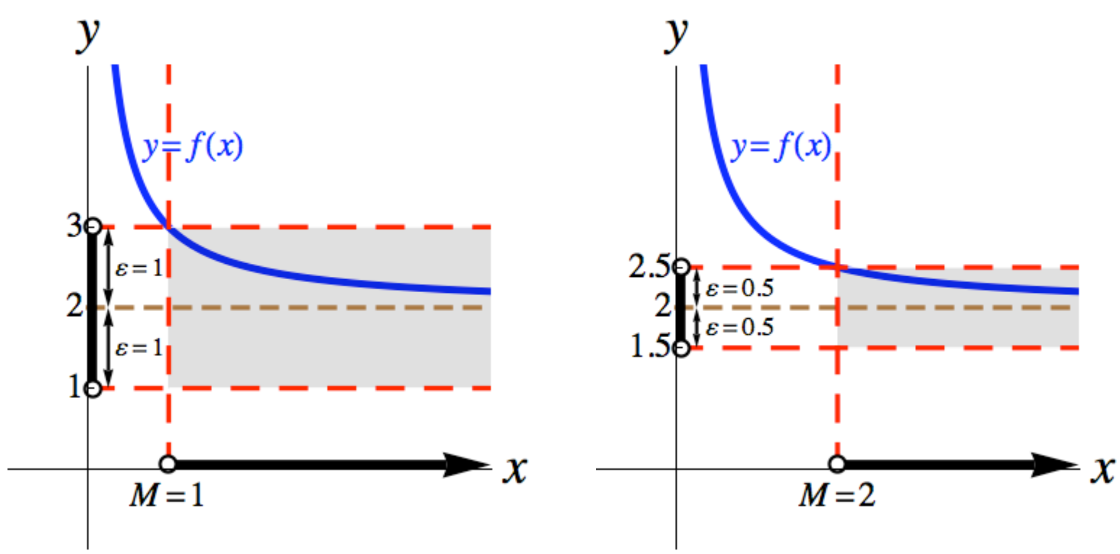


How Does the "Static" Approach to "Long-Run" Limits Relate to the "Dynamic" Approach?

Why is $\lim_{x \to \infty} \left(\frac{1}{x} + 2 \right) = 2$? Because, **regardless of how small** we make the **tolerance level** ε and how tight we make the lottery for the players, **there is** a "point of no return" M **after which all the players win**. That is, the corresponding box (see the shaded boxes in the figures below) **traps the graph** of $y = f(x)$ for all $x > M$.

As $\varepsilon \to 0^+$, we can **choose values** for M in such a way that the corresponding shaded boxes **always trap the graph** and zoom in, or collapse in, on the **HA** $y = 2$. In other words, **there are always winners as** $x \to \infty$.

If $\varepsilon = 1$, we can choose $M = 1$. If $\varepsilon = 0.5$, we can choose $M = 2$.



For this example, if ε is **any positive real number**, we can choose $M = \frac{1}{\varepsilon}$.

PART E: DEFINING INFINITE LIMITS AT A POINT

Challenge to the reader:

Give precise "M-δ" and "N-δ" definitions of $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} f(x) = -\infty$ $(a \in \mathbb{R})$, where the function f is defined on a punctured neighborhood of a.