CHAPTER 2:

Limits and Continuity

- The conventional approach to calculus is founded on limits.
- In this chapter, we will develop the concept of a limit by example.
- Properties of limits will be established along the way.
- We will use limits to analyze asymptotic behaviors of functions and their graphs.
- Limits will be formally defined near the end of the chapter.
- Continuity of a function (at a point and on an interval) will be defined using limits.

SECTION 2.1: AN INTRODUCTION TO LIMITS

LEARNING OBJECTIVES

- Understand the concept of (and notation for) a limit of a rational function at a point in its domain, and understand that "limits are local."
- Evaluate such limits.
- Distinguish between one-sided (left-hand and right-hand) limits and two-sided limits – and what it means for such limits to exist.
- Use numerical / tabular methods to guess at limit values.
- Distinguish between limit values and function values at a point.
- Understand the use of neighborhoods and punctured neighborhoods in the evaluation of one-sided and two-sided limits.
- Evaluate some limits involving piecewise-defined functions.

PART A: THE LIMIT OF A FUNCTION AT A POINT

Our study of calculus begins with an understanding of the expression $\lim_{x \to a} f(x)$, where a is a real number (in short, $a \in \mathbb{R}$) and f is a function. This is read as:

"the limit of $f(x)$ as x approaches a."

- **WARNING 1:** $\to$ means "approaches." Avoid using this symbol outside the context of limits.

- $\lim_{x \to a}$ is called a limit operator. Here, it is applied to the function f.

$\lim_{x \to a} f(x)$ is the real number that $f(x)$ approaches as x approaches a, **if such a number exists**. If $f(x)$ does, indeed, approach a real number, we denote that number by L (for limit value). We say the limit **exists**, and we write:

$$\lim_{x \to a} f(x) = L, \quad \text{or} \quad f(x) \to L \text{ as } x \to a.$$

These statements will be **rigorously defined** in Section 2.7.

When we **evaluate** $\lim_{x \to a} f(x)$, we do one of the following:

- We find the limit value L (in simplified form).

  We write: $\lim_{x \to a} f(x) = L$.

- We say the limit is ∞ (infinity) or $-\infty$ (negative infinity).

  We write: $\lim_{x \to a} f(x) = \infty$, or $\lim_{x \to a} f(x) = -\infty$.

- We say the limit **does not exist ("DNE")** in some other way.

  We write: $\lim_{x \to a} f(x)$ DNE.

  (The "DNE" notation is used by Swokowski but few other authors.)

If we say the limit is ∞ or $-\infty$, the limit is still **nonexistent**. Think of ∞ and $-\infty$ as "special cases of DNE" that we do write when appropriate; they indicate **why** the limit does not exist.

$\lim_{x \to a} f(x)$

exists	does not exist
The limit is a real number, L.	"DNE" ($\boxed{\infty}$ $\boxed{-\infty}$)

$\lim_{x \to a} f(x)$ is called a limit at a point, because $x = a$ corresponds to a **point** on the real number line. Sometimes, this is related to a point on the graph of f.

Example 1 (Evaluating the Limit of a Polynomial Function at a Point)

Let $f(x) = 3x^2 + x - 1$. Evaluate $\lim_{x \to 1} f(x)$.

§ Solution

f is a **polynomial** function with implied domain $\text{Dom}(f) = \mathbb{R}$.

We **substitute** ("plug in") $x = 1$ and evaluate $f(1)$.

WARNING 2: Sometimes, the **limit value** $\lim_{x \to a} f(x)$ does not equal the **function value** $f(a)$. (See Part C.)

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(3x^2 + x - 1\right)$$

WARNING 3: Use **grouping symbols** when taking the limit of an expression consisting of **more than one term**.

$$= 3(1)^2 + (1) - 1$$

WARNING 4: Do not omit the limit operator $\lim_{x\to 1}$ until this substitution phase.

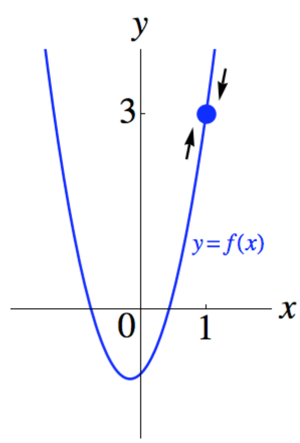
WARNING 5: When performing **substitutions**, be prepared to use **grouping symbols**. Omit them only if you are sure they are unnecessary.

$$= 3$$

We can write: $\lim_{x\to 1} f(x) = 3$, or $f(x) \to 3$ as $x \to 1$.

• Be prepared to work with function and variable names other than f and x. For example, if $g(t) = 3t^2 + t - 1$, then $\lim_{t\to 1} g(t) = 3$, also.

The graph of $y = f(x)$ is below.



Imagine that the arrows in the figure represent two lovers running towards each other along the parabola. What is the y-coordinate of the point they are approaching as they approach $x = 1$?
It is 3, the limit value.

TIP 1: Remember that ***y*-coordinates** of points along the graph correspond to **function values**. §

Example 2 (Evaluating the Limit of a Rational Function at a Point)

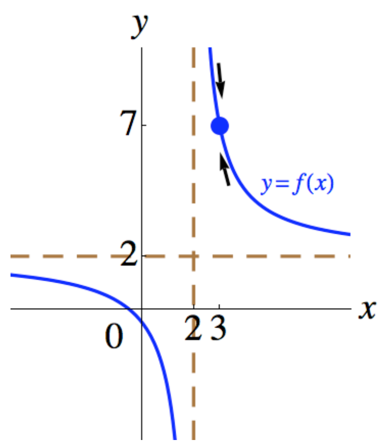
Let $f(x) = \dfrac{2x+1}{x-2}$. Evaluate $\lim\limits_{x \to 3} f(x)$.

§ Solution

f is a **rational** function with implied domain $\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid x \neq 2\}$.
We observe that 3 is in the **domain** of f (in short, $3 \in \mathrm{Dom}(f)$), so we **substitute** ("plug in") $x = 3$ and evaluate $f(3)$.

$$\begin{aligned}\lim_{x \to 3} f(x) &= \lim_{x \to 3} \frac{2x+1}{x-2} \\ &= \frac{2(3)+1}{(3)-2} \\ &= 7\end{aligned}$$

The graph of $y = f(x)$ is below.



Note: As is often the case, you might not know how to draw the graph until later.

• **Asymptotes.** The dashed lines are asymptotes, which are lines that a graph approaches

- in a "long-run" sense
  (see the horizontal asymptote, or "HA," at $y = 2$), or
- in an "explosive" sense
  (see the vertical asymptote, or "VA," at $x = 2$).

"HA"s and "VA"s will be defined using limits in Sections 2.3 and 2.4, respectively.

• **"Limits are Local."** What if the lover on the left is running along the left branch of the graph? In fact, we ignore the left branch, because of the following key principle of limits.

"Limits [at a Point] are Local"

When analyzing $\lim_{x \to a} f(x)$, we only consider the behavior of f in the **"immediate vicinity"** of $x = a$.

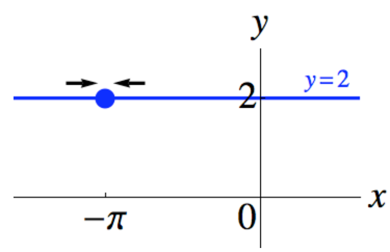
In fact, we may exclude consideration of $x = a$ itself, as we will see in Part C.

In the graph, we only care what happens **"immediately around"** $x = 3$. Section 2.7 will feature a rigorous approach. §

Example 3 (Evaluating the Limit of a Constant Function at a Point)

$$\lim_{x \to -\pi} 2 = 2.$$

(Observe that substituting $x = -\pi$ technically works here, since there is no "*x*" in "2," anyway.)



• **A constant approaches itself.** We can write $2 \to 2$ ("2 approaches 2") as $x \to -\pi$. When we think of a sequence of numbers approaching 2, we may think of distinct numbers such as 2.1, 2.01, 2.001, …. However, the **constant sequence** 2, 2, 2, … is also said to approach 2. §

All **constant** functions are also **polynomial** functions, and all **polynomial** functions are also **rational** functions. The following theorem applies to all three Examples thus far.

Basic Limit Theorem for Rational Functions

If f is a rational function, and $a \in \text{Dom}(f)$,

then $\lim_{x \to a} f(x) = f(a)$.

• To evaluate the limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

We will justify this theorem in Section 2.2.

PART B: ONE- AND TWO-SIDED LIMITS; EXISTENCE OF LIMITS

$\lim_{x \to a}$ is a **two-sided** limit operator in $\lim_{x \to a} f(x)$, because we must consider the behavior of f as x approaches a from **both** the left **and** the right.

$\lim_{x \to a^-}$ is a **one-sided** left-hand limit operator. $\lim_{x \to a^-} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the left**."

$\lim_{x \to a^+}$ is a **one-sided** right-hand limit operator. $\lim_{x \to a^+} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the right**."

Example 4 (Using a Numerical / Tabular Approach to Guess a Left-Hand Limit Value)

Guess the value of $\lim_{x \to 3^-} (x+3)$ using a **table** of function values.

§ Solution

Let $f(x) = x + 3$. $\lim_{x \to 3^-} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **lesser (or lower) numbers**. That is, we approach $x = 3$ from the **left** along the real number line.

We select an **increasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but less than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

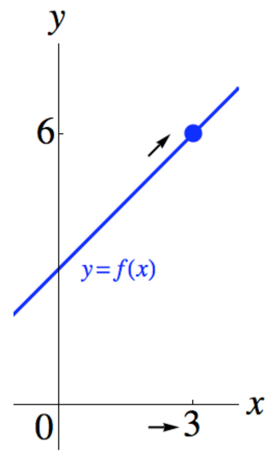
x	2.9	2.99	2.999	$\to 3^-$
$f(x) = x + 3$	5.9	5.99	5.999	$\to 6$ (?)

We guess: $\lim_{x \to 3^-} (x+3) = 6$.

WARNING 6: Do not confuse superscripts with signs of numbers. Be careful about associating the "$-$" superscript with negative numbers. Here, we consider **positive** numbers that are close to 3.

- If we were taking a limit as ***x* approached 0**, then we would associate the "$-$" superscript with **negative** numbers and the "+" superscript with **positive** numbers.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the left of** $x = 3$.



WARNING 7: The numerical / tabular approach is **unreliable**, and it is typically **unacceptable** as a method for evaluating limits on exams. (See Part D, Example 11 to witness a failure of this method.) However, it may help us guess at limit values, and it strengthens our understanding of limits. §

Example 5 (Using a Numerical / Tabular Approach to Guess a Right-Hand Limit Value)

Guess the value of $\lim_{x \to 3^+} (x+3)$ using a **table** of function values.

§ Solution

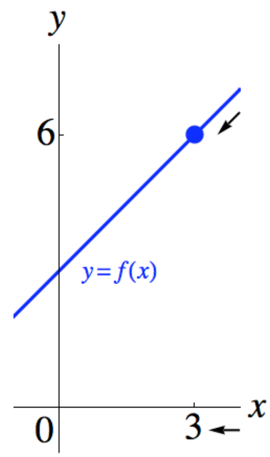
Let $f(x) = x + 3$. $\lim_{x \to 3^+} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **greater (or higher) numbers**. That is, we approach $x = 3$ from the **right** along the real number line.

We select a **decreasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but greater than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

x	$3^+ \leftarrow$	3.001	3.01	3.1
$f(x) = x + 3$	6 (?) $\leftarrow$	6.001	6.01	6.1

We guess: $\lim_{x \to 3^+} (x+3) = 6$.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the right of** $x = 3$.



§

Existence of a Two-Sided Limit at a Point

$$\lim_{x\to a} f(x) = L \Leftrightarrow \left[\lim_{x\to a^-} f(x) = L, \text{ and } \lim_{x\to a^+} f(x) = L \right], \quad (a, L \in \mathbb{R}).$$

• A two-sided limit **exists** $\Leftrightarrow$ the corresponding left-hand and right-hand limits **exist**, and they are **equal**.

• If either one-sided limit **does not exist (DNE)**, or if the two one-sided limits are **unequal**, then the two-sided limit **does not exist (DNE)**.

Our guesses, $\lim_{x\to 3^-} (x+3) = 6$ and $\lim_{x\to 3^+} (x+3) = 6$, imply $\lim_{x\to 3} (x+3) = 6$.

In fact, all three limits can be evaluated by **substituting** $x = 3$ into $(x+3)$:

$$\lim_{x\to 3^-} (x+3) = 3+3 = 6; \ \lim_{x\to 3^+} (x+3) = 3+3 = 6; \ \lim_{x\to 3} (x+3) = 3+3 = 6.$$

This procedure is generalized in the following theorem.

Extended Limit Theorem for Rational Functions

If f is a rational function, and $a \in \text{Dom}(f)$,

then $\lim_{x\to a^-} f(x) = f(a)$, $\lim_{x\to a^+} f(x) = f(a)$, and $\lim_{x\to a} f(x) = f(a)$.

• To evaluate each limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

WARNING 8: Substitution might not work if f is not a rational function.

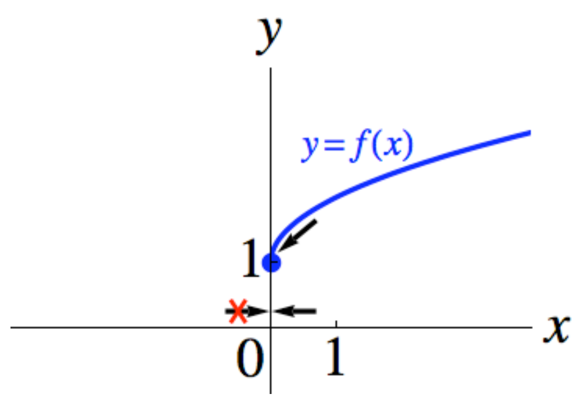
Example 6 (Pitfalls of Substituting into a Function that is Not Rational)

Let $f(x) = \sqrt{x} + 1$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

Observe that $\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\} = [0, \infty)$, because $\sqrt{x}$ is **real** when $x \geq 0$, but it is **not real** when $x < 0$.

This is important, because x is only allowed to approach 0 (or whatever a is) **through** $\mathrm{Dom}(f)$. Here, x is allowed to approach 0 from the right but **not** from the left.



Right-Hand Limit: $\lim_{x \to 0^+} f(x) = 1$.

Substituting $x = 0$ works: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\sqrt{x} + 1\right) = \sqrt{0} + 1 = 1$.

Left-Hand Limit: $\lim_{x \to 0^-} f(x)$ does not exist (DNE).

Substituting $x = 0$ **does not work** here.

Two-Sided Limit: $\lim_{x \to 0} f(x)$ does not exist (DNE).

This is because the corresponding left-hand limit does not exist (DNE).

Observe that f is **not** a rational function, so the aforementioned theorem does **not** apply, even though $0 \in \mathrm{Dom}(f)$. f is, however, an **algebraic** function, and we will discuss algebraic functions in Section 2.2. §

PART C: IGNORING THE FUNCTION AT *a*

Example 7 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Examples 4 and 5)

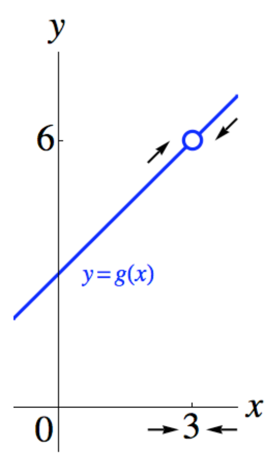
Let $g(x) = x + 3,\ (x \neq 3)$.

(We are deleting 3 from the domain of the function in Examples 4 and 5; this changes the function.)

Evaluate $\lim\limits_{x \to 3^-} g(x)$, $\lim\limits_{x \to 3^+} g(x)$, and $\lim\limits_{x \to 3} g(x)$.

§ Solution

Since $3 \notin \text{Dom}(g)$, we must delete the point $(3, 6)$ from the graph of $y = x + 3$ to obtain the graph of g below.



We say that g has a removable discontinuity at $x = 3$ (see Section 2.8), and the graph of g has a hole at the point $(3, 6)$.

Observe that, as x approaches 3 from the left **and** from the right, $g(x)$ **approaches** 6, even though $g(x)$ never equals 6.

$g(3)$ is undefined, yet the following statements are true:

$$\lim_{x \to 3^-} g(x) = 6,$$
$$\lim_{x \to 3^+} g(x) = 6, \text{ and}$$
$$\lim_{x \to 3} g(x) = 6.$$

There literally **does not have to be a point** at $x = 3$ (in general, $x = a$) for these limits to exist! Observe that substituting $x = 3$ into $(x + 3)$ works. §

Example 8 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Example 7)

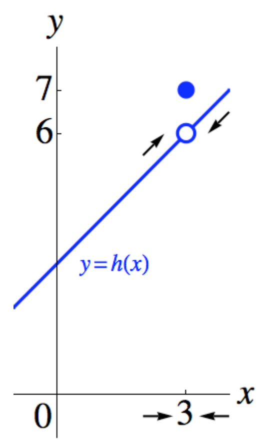
Let the function h be defined **piecewise** as follows: $h(x) = \begin{cases} x+3, & x \neq 3 \\ 7, & x = 3 \end{cases}$.

(A piecewise-defined function applies different evaluation rules to different subsets of (groups of numbers in) its domain. This type of function can lead to interesting limit problems.)

Evaluate $\lim_{x \to 3} h(x)$.

§ Solution

h is identical to the function g from Example 7, except that $3 \in \mathrm{Dom}(h)$, and $h(3) = 7$. As a result, we must add the point $(3, 7)$ to the graph of g to obtain the graph of h below.



As with g, h also has a **removable discontinuity** at $x = 3$, and its graph also has a **hole** at the point $(3, 6)$.

Observe that, as x approaches 3 from the left **and** from the right, $h(x)$ also **approaches** 6.

$\lim_{x \to 3} h(x) = 6$ once again, even though $h(3) = 7$.

WARNING 2 repeat (applied to f): Sometimes, the **limit value** $\lim_{x \to a} f(x)$ does not equal the **function value** $f(a)$. §

As in Example 7, observe that substituting $x = 3$ into $(x+3)$ works. §

The existence (or value) of $\lim_{x \to a} f(x)$ **need not** depend on the existence (or value) of $f(a)$.

• Sometimes, it **does help** to know what $f(a)$ is when evaluating $\lim_{x \to a} f(x)$.
In Section 2.8, we will say that f is <u>continuous</u> at a $\Leftrightarrow$ $\lim_{x \to a} f(x) = f(a)$, provided that $\lim_{x \to a} f(x)$ and $f(a)$ exist. We appreciate **continuity**, because we can then simply **substitute** $x = a$ to evaluate a limit, which was what we did when we applied the **Basic Limit Theorem for Rational Functions** in Part A.

• In Examples 7 and 8, we dealt with functions that were **not** continuous at $x = 3$, yet **substituting** $x = 3$ into $(x + 3)$ allowed us to evaluate the one- and two-sided limits at $a = 3$. We will develop theorems that cover these Examples. We first need the following definitions.

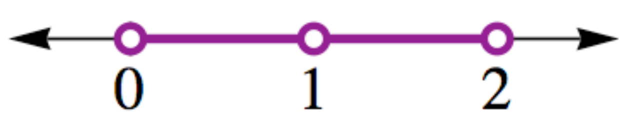
A <u>neighborhood of *a*</u> is an **open interval** along the real number line that is **symmetric** about *a*.

For example, the interval $(0, 2)$ is a **neighborhood** of 1. Since 1 is the **midpoint** of $(0, 2)$, the neighborhood is **symmetric** about 1.

A <u>punctured (or deleted) neighborhood of *a*</u> is constructed by taking a neighborhood of *a* and **deleting** *a* itself.

For example, the set $(0, 2) \setminus \{1\}$, which can be written as $(0, 1) \cup (1, 2)$, is a **punctured neighborhood** of 1. It is a set of numbers that are **"immediately around"** 1 on the real number line.

• The notation $(0, 2) \setminus \{1\}$ indicates that we can construct it by taking the **neighborhood** $(0, 2)$ and **deleting** 1.



<u>"Puncture Theorem" for Limits of Locally Rational Functions</u>

Let r be a rational function, and let $a \in \mathrm{Dom}(r)$.
Let $f(x) = r(x)$ on a punctured neighborhood of $x = a$.
Then, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} r(x) = r(a)$.

• To evaluate the limits, substitute ("plug in") $x = a$ into $r(x)$, and evaluate $r(a)$.

• That is, if a function rule is given by a **rational** expression $r(x)$ **locally (immediately) around** $x = a$, where $a \in \mathrm{Dom}(r)$, then **evaluate** the rational expression **at** a to obtain the **limit** of the function at a.

Refer to Examples 7 and 8. Let $r(x) = x + 3$. Observe that r is a rational function, and $3 \in \mathrm{Dom}(r)$. Both the g and h functions were defined by $x + 3$ **locally (immediately) around** $x = 3$. More precisely, they were defined by $x + 3$ on some **punctured neighborhood** of $x = 3$, say $(2.9, 3.1) \setminus \{3\}$. Therefore,

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6, \text{ and}$$

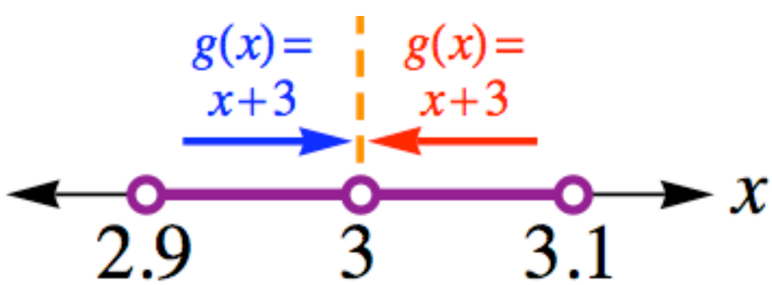
$$\lim_{x \to 3} h(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6.$$

It is easier to write:

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6, \text{ and}$$

$$\lim_{x \to 3} h(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6.$$

The figure below refers to g, but it also applies to h.
The dashed line segment at $x = 3$ reiterates the **puncture** there.



Why does the theorem only require that a function be **locally** rational about *a*? Consider the following Example.

Example 9 (Limits are Local)

Let $f(t) = \begin{cases} t+2, & t<0 \\ \sqrt{t}, & t \geq 0 \end{cases}$. Evaluate $\lim\limits_{t \to -1} f(t)$.

§ Solution

Observe that $f(t) = t+2$ is the **only** rule that is relevant as *t* approaches -1 **locally** from the left **and** from the right. We only consider values of *t* that are **"immediately around"** $a = -1$. **"Limits are Local!"**

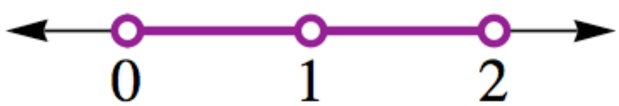
It is **irrelevant** that the rule $f(t) = \sqrt{t}$ is different, or that it is not rational. §

The following definitions will prove helpful in our study of **one-sided limits**.

A left-neighborhood of *a* is an **open interval** of the form (c, a), where $c < a$.
A right-neighborhood of *a* is an **open interval** of the form (a, c), where $c > a$.
A **punctured neighborhood** of *a* consists of **both** a left-neighborhood of *a* **and** a right-neighborhood of *a*.

For example, the interval $(0,1)$ is a **left-neighborhood** of 1. It is a set of numbers that are **"immediately to the left"** of 1 on the real number line.

The interval $(1,2)$ is a **right-neighborhood** of 1. It is a set of numbers that are **"immediately to the right"** of 1 on the real number line.



We now modify the "Puncture Theorem" for **one-sided limits**.

• Basically, when evaluating a **left-hand limit** such as $\lim\limits_{x \to a^-} f(x)$, we use the function rule that governs the *x*-values **"immediately to the left"** of *a* on the real number line.

• Likewise, when evaluating a **right-hand limit** such as $\lim\limits_{x \to a^+} f(x)$, we use the rule that governs the *x*-values **"immediately to the right"** of *a*.

Variation of the "Puncture Theorem" for Left-Hand Limits

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a left-neighborhood of $x = a$.
Then, $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} r(x) = r(a)$.

Variation of the "Puncture Theorem" for Right-Hand Limits

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a right-neighborhood of $x = a$.
Then, $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} r(x) = r(a)$.

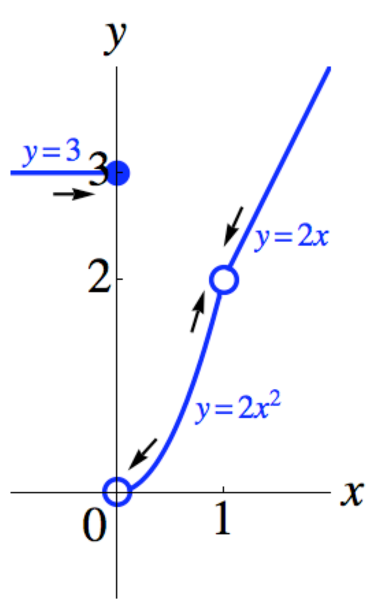
Example 10 (Evaluating One-Sided and Two-Sided Limits of a Piecewise-Defined Function)

Let $f(x) = \begin{cases} 3, & \text{if } x \le 0 \\ 2x^2, & \text{if } 0 < x < 1 \\ 2x, & \text{if } x > 1 \end{cases}$.

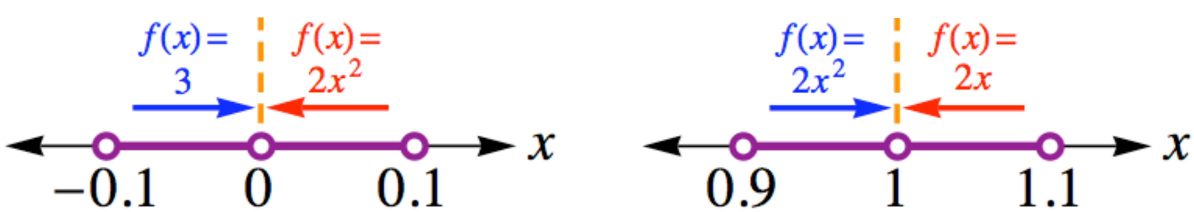
Evaluate the one-sided and two-sided limits of f at 1 and at 0.

§ Solution

The graph of $y = f(x)$ is below. It helps, but it is **not** required to evaluate limits. Instead, we can evaluate limits of **relevant** function rules.



$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^2$ $= 2(1)^2$ $= 2$	The left-hand limit as $x \to 1^-$: We use the rule $f(x) = 2x^2$, because it applies to a **left-neighborhood** of 1, say $(0.9, 1)$.
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x$ $= 2(1)$ $= 2$	The right-hand limit as $x \to 1^+$: We use the rule $f(x) = 2x$, because it applies to a **right-neighborhood** of 1, say $(1, 1.1)$.
$\lim_{x \to 1} f(x) = 2$	The two-sided limit as $x \to 1$: The left-hand and right-hand limits at 1 **exist**, and they are **equal**, so the two-sided limit exists and equals their common value.
$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3$ $= 3$	The left-hand limit as $x \to 0^-$: We use the rule $f(x) = 3$, because it applies to a **left-neighborhood** of 0, say $(-0.1, 0)$.
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^2$ $= 2(0)^2$ $= 0$	The right-hand limit as $x \to 0^+$: We use the rule $f(x) = 2x^2$, because it applies to a **right-neighborhood** of 0, say $(0, 0.1)$.
$\lim_{x \to 0} f(x)$ does not exist (DNE)	The two-sided limit as $x \to 0$: The left-hand and right-hand limits at 0 **exist**, but they are **unequal**, so the two-sided limit does not exist (DNE).



§

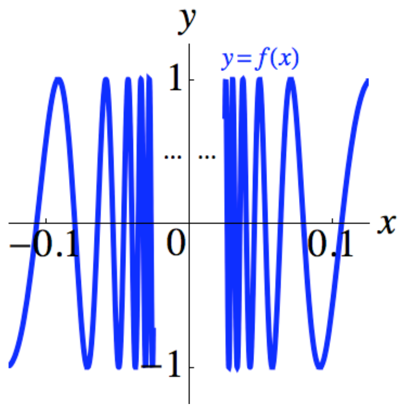
PART D: NONEXISTENT LIMITS

Example 11 (Nonexistent Limits)

Let $f(x) = \sin\left(\frac{1}{x}\right)$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. Ask your instructor if s/he might have you even attempt to draw this. In a sense, the classic sine wave is being turned "inside out" relative to the *y*-axis.



As *x* approaches 0 from the right (or from the left), the function values **oscillate** between -1 and 1.

They do **not** approach a **single real number**. Therefore,

$\lim_{x \to 0^+} f(x)$ does not exist (DNE),

$\lim_{x \to 0^-} f(x)$ does not exist (DNE), and

$\lim_{x \to 0} f(x)$ does not exist (DNE).

Note 1: The *y*-axis is **not a vertical asymptote (VA)** here, because the graph and the function values are **not "exploding" without bound** around the *y*-axis.

Note 2: Here is an example of how the **numerical / tabular approach** introduced in Part B **might lead us astray**:

x	$0^+ \leftarrow$	$\frac{1}{3\pi}$	$\frac{1}{2\pi}$	$\frac{1}{\pi}$
$f(x) = \sin\left(\frac{1}{x}\right)$	0 (?) $\leftarrow$ NO!	0	0	0

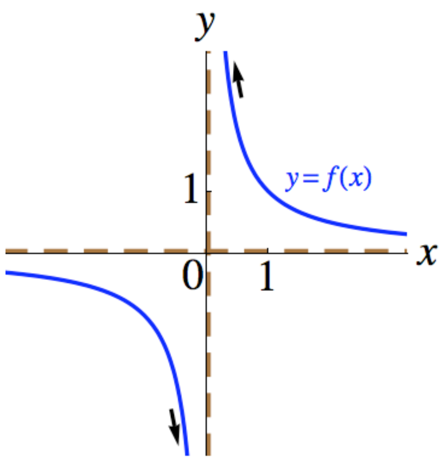
§

Example 12 (Infinite and/or Nonexistent Limits)

Let $f(x)=\dfrac{1}{x}$. Evaluate $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0} f(x)$.

§ Solution

The graph of $y=f(x)$ is below. We will discuss this graph in later sections.



As x approaches 0 from the **right**, the function values **increase without bound**.

Therefore, $\lim\limits_{x\to 0^+} f(x)=\infty$.

As x approaches 0 from the **left**, the function values **decrease without bound**.

Therefore, $\lim\limits_{x\to 0^-} f(x)=-\infty$.

∞ and $-\infty$ are **mismatched**.

Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist (DNE).

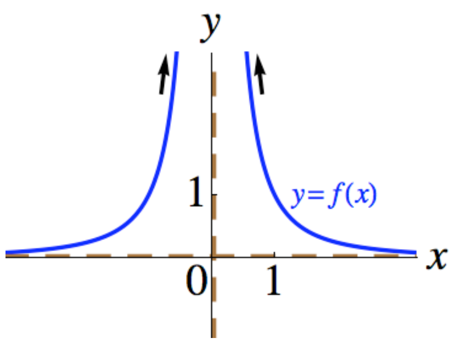
In fact, all three limits **do not exist**. For example, $\lim\limits_{x\to 0^+} f(x)$, **does not exist**, because the function values **do not approach a single real number** as x approaches 0 from the right. The expressions ∞ and $-\infty$ indicate **why** the one-sided limits do not exist, and we write ∞ and $-\infty$ where appropriate. §

Example 13 (Infinite and Nonexistent Limits)

Let $f(x)=\dfrac{1}{x^2}$. Evaluate $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0} f(x)$.

§ Solution

The graph of $y=f(x)$ is below. Observe that f is an even function.



$\lim\limits_{x\to 0^+} f(x)=\infty$,

$\lim\limits_{x\to 0^-} f(x)=\infty$, and

$\lim\limits_{x\to 0} f(x)=\infty$. §

Example 14 (A Nonexistent Limit)

Let $f(x) = \dfrac{|x|}{x}$. Evaluate $\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0^-} f(x)$, and $\lim\limits_{x \to 0} f(x)$.

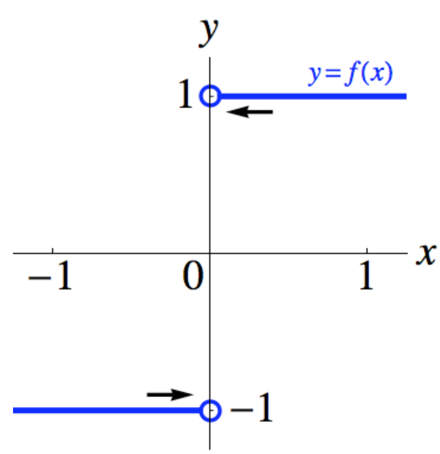
§ Solution

Note: f is **not** a rational function, but it is an **algebraic function**, since $f(x) = \dfrac{|x|}{x} = \dfrac{\sqrt{x^2}}{x}$.

Remember that: $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

Then, $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & \text{if } x > 0 \\ \dfrac{-x}{x} = -1, & \text{if } x < 0 \end{cases}$, and $f(0)$ is undefined.

The graph of $y = f(x)$ is below.



$\lim\limits_{x \to 0^+} f(x) = 1$,

$\lim\limits_{x \to 0^-} f(x) = -1$, and

$\lim\limits_{x \to 0} f(x)$ does not exist (DNE),

due to the fact that the right-hand and left-hand limits are **unequal**. §

FOOTNOTES

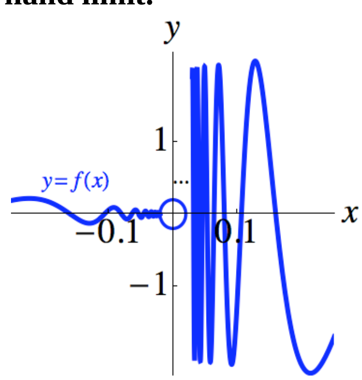
1. **Limits do not require continuity.** In Section 2.8, we will discuss continuity, a property of functions that helps our lovers run along the graph of a function without having to jump or hop. In Exercises 1-3, we could imagine the lovers running towards each other (one from the left, one from the right) while staying on the graph of f and without having to jump or hop, provided they were placed on appropriate parts of the graph. Sometimes, the "run" requires jumping or hopping. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is a rational number } (x \in \mathbb{Q}) \\ x, & \text{if } x \text{ is an irrational number } (x \notin \mathbb{Q};\ \text{really, } x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$.
   It turns out that $\lim\limits_{x \to 0} f(x) = 0$.

2. **Misconceptions about limits.**
   See "Why Is the Limit Concept So Difficult for Students?" by Sally Jacobs in the Fall 2002 edition (vol.24, No.1) of *The AMATYC Review*, pp.25-34.
   • Students can be misled by the use of the word "limit" in real-world contexts. For example, a speed limit is a bound that is not supposed to be exceeded; there is no such restriction on limits in calculus.
   • Limit values can sometimes be attained. For example, if a function f is continuous at $x = a$ (see Examples 1-3), then the function value takes on the limit value at $x = a$.
   • Limit values do not have to be attained. See Examples 7 and 8.

   Observations:
   • The dynamic view of limits, which involves ideas of motion and "approaching" (for example, our lovers), may be more accessible to students than the static view preferred by many textbook authors. The static view is exemplified by the formal definitions of limits we will see in Section 2.7. The dynamic view greatly assists students in transitioning to the static view and the formal definitions.
   • Leading mathematicians in 18^{th}- and 19^{th}-century Europe heatedly debated ideas of limits.

3. **Multivariable calculus.** When we go to higher dimensions, there may be more than two possible approaches (not just left-hand and right-hand) when analyzing limits at a point! Neighborhoods can take the form of disks or balls.

4. **An example where a left-hand limit exists but not the right-hand limit.**
   Let $f(x) = \dfrac{x + |x|(1+x)}{x} \sin\left(\dfrac{1}{x}\right) = \begin{cases} -x\sin\left(\dfrac{1}{x}\right), & \text{if } x < 0 \\ (2+x)\sin\left(\dfrac{1}{x}\right), & \text{if } x > 0 \end{cases}$.



   Then, $\lim\limits_{x \to 0^-} f(x) = 0$, which can be proven by the Squeeze (Sandwich) Theorem in Section 2.6. However, $\lim\limits_{x \to 0^+} f(x)$ does not exist (DNE).
   See William F. Trench, *Introduction to Real Analysis* (free online at: http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF), p.39.

SECTION 2.2: PROPERTIES OF LIMITS and ALGEBRAIC FUNCTIONS

LEARNING OBJECTIVES

- Know properties of limits, and use them to evaluate limits of functions, particularly algebraic functions.
- Understand how the properties of limits justify the limit theorems in Section 2.1.
- Be able to use informal Limit Form notation to analyze limits.
- Learn to exercise caution when handling $\left(\text{Limit Form } \sqrt[\text{even}]{0}\right)$.

PART A: PROPERTIES OF LIMITS / THE ALGEBRA OF LIMITS; LIMIT FORMS

Assume that: $\lim_{x \to a} f(x) = L_1$, and $\lim_{x \to a} g(x) = L_2$, where $a, L_1, L_2 \in \mathbb{R}$.

1) The limit of a **sum** equals the **sum** of the limits.

$$\lim_{x \to a} \left[f(x) + g(x) \right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
$$= L_1 + L_2$$

- We may refer to this as the Sum Rule of Limits.

For example, as $x \to a$, if $f(x) \to 2$ and $g(x) \to 3$, then $\left[f(x) + g(x) \right] \to 5$. We can represent this **informally** using a Limit Form: $(\text{Limit Form } 2+3) \Rightarrow 5$.

WARNING 1: Limit Forms. There is no standard notation for Limit Forms, and they represent footnotes to the rigorous evaluation of limits. Different instructors may have different rules on when Limit Forms need to be written.

2) The limit of a **difference** equals the **difference** of the limits.

$$\lim_{x \to a} \left[f(x) - g(x) \right] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$
$$= L_1 - L_2$$

For example, $(\text{Limit Form } 5-3) \Rightarrow 2$.

3) <u>The limit of a **product** equals the **product** of the limits.</u>

$$\lim_{x \to a} \left[f(x) g(x) \right], \text{ or } \lim_{x \to a} f(x) g(x) = \left[\lim_{x \to a} f(x) \right] \left[\lim_{x \to a} g(x) \right]$$
$$= L_1 L_2$$

For example, $\left(\text{Limit Form } 2 \cdot 3 \right) \Rightarrow 6$.

4) <u>The limit of a **quotient** equals the **quotient** of the limits, if the limit of the divisor (or denominator) is **not zero**.</u>

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right], \text{ or } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$
$$= \frac{L_1}{L_2}, \text{ if } L_2 \neq 0$$

For example, $\left(\text{Limit Form } \frac{6}{2} \right) \Rightarrow 3$.

5) <u>The limit of a (positive integer) **power** equals the **power** of the limit.</u>

If n is a positive integer $\left(n \in \mathbb{Z}^+ \right)$, then:

$$\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n$$
$$= \left(L_1 \right)^n$$

• This is a direct consequence of Property 3. For instance,

$$\lim_{x \to a} x^2 = \lim_{x \to a} xx = \left(\lim_{x \to a} x \right) \left(\lim_{x \to a} x \right) = \left(\lim_{x \to a} x \right)^2.$$

For example, $\left(\text{Limit Form } 2^{(\text{constant } 3)} \right) \Rightarrow 8$.

• The seemingly simpler statement $\left(\text{Limit Form } 2^3 \right) \Rightarrow 8$ is also true, but it actually says something more powerful. It says that "something approaching 2" raised to an "exponent **approaching** 3" will approach 8. However, this idea **falls apart** when the base $f(x)$ approaches a **negative** number. It is true that $\left(\text{Limit Form } (-2)^{(\text{constant } 3)} \right) \Rightarrow -8$, for example, but it is **not** true that $\left(\text{Limit Form } (-2)^3 \right) \Rightarrow -8$. Think about why $(-2)^{3.5}$, or $(-2)^{7/2}$, is **not** a real number; we will address this issue in Part B.

6) The limit of a **constant multiple** equals the **constant multiple** of the limit. ("Constant Factors Pop Out.")

If $c \in \mathbb{R}$, then:

$$\lim_{x \to a} \left[c \cdot f(x) \right], \text{ or } \lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x)$$
$$= cL_1$$

For example, **twice** "something that approaches 3" will approach 6.

• In multivariable calculus, if y is **independent** of x, then we can pop out y.

Note: Properties 5, 6, and 7 (upcoming) are generalized in Section 2.8, Footnote 6.

Limit Operators are Linear

Properties 1), 2), and 6) imply that limit operators are linear operators. This means that we can take limits **term-by-term**, and then **constant factors "pop out,"** assuming the limits exist. (See Footnote 1.)

• This is a key property that is shared by differentiation and integration operators in later chapters.

Properties 1-6, building on the **elementary rules** $\lim_{x \to a} c = c$ and $\lim_{x \to a} x = a$ $(a, c \in \mathbb{R})$, justify the **Basic Limit Theorem for Rational Functions** in Section 2.1. A demonstration follows.

Example 1 (Demonstrating How the Properties of Limits Justify the Basic Limit Theorem for Rational Functions)

Evaluate $\lim_{x\to 4}\frac{3x^2-1}{x+5}$ using the properties of limits in this section.

§ Solution

$$\lim_{x\to 4}\frac{3x^2-1}{x+5}=\frac{\lim_{x\to 4}\left(3x^2-1\right)}{\lim_{x\to 4}\left(x+5\right)} \quad \left(\text{by Property 4 on } \textbf{quotients}\right)$$

$$=\frac{\lim_{x\to 4}3x^2-\lim_{x\to 4}1}{\lim_{x\to 4}x+\lim_{x\to 4}5} \quad \left(\text{by Properties 1, 2 on } \textbf{sums}, \textbf{differences}\right)$$

$$=\frac{\lim_{x\to 4}3x^2-1}{4+5} \quad \left(\text{by } \textbf{elementary rules}\right)$$

$$=\frac{3\left(\lim_{x\to 4}x^2\right)-1}{4+5} \quad \left(\text{by Property 6 on } \textbf{constant multiples}\right)$$

$$=\frac{3\left(\lim_{x\to 4}x\right)^2-1}{4+5} \quad \left(\begin{array}{l}\text{by Property 5 on } \textbf{powers}\text{, or}\\ \text{by Property 3 on } \textbf{products}\text{: } x^2=xx\end{array}\right)$$

$$=\frac{3(4)^2-1}{4+5} \quad \left(\text{by } \textbf{elementary rules}\text{; see } \underline{\text{Note 1}} \text{ below}\right)$$

$$=\frac{47}{9}$$

Note 1: Observe that the limit can be evaluated by simply substituting $x=4$ into $\frac{3x^2-1}{x+5}$, as the **Basic Limit Theorem for Rational Functions** suggests.

Note 2: Observe that all indicated limits **exist** and there are **no zero denominator** issues, so we could apply Properties 1-6. Our use of the "$=$" sign is appropriate here, though we often use it informally even when the limit turns out not to exist. §

Properties of One-Sided Limits

Properties 1-6 extend naturally to one-sided limits. For example,

$$\lim_{x \to a^-} \left[f(x) + g(x) \right] = \lim_{x \to a^-} f(x) + \lim_{x \to a^-} g(x), \text{ and}$$

$$\lim_{x \to a^+} \left[f(x) + g(x) \right] = \lim_{x \to a^+} f(x) + \lim_{x \to a^+} g(x),$$

provided the indicated limits exist.

PART B: PROPERTIES OF LIMITS OF ROOTS

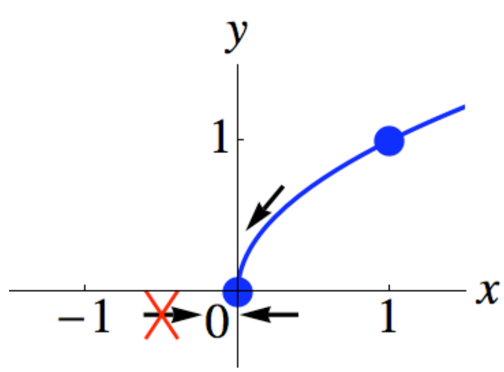
We now motivate Property 7, a much more complicated property on **roots**.

Example 2 (Evaluating the Limit of a Square Root)

Evaluate $\lim_{x \to 1} \sqrt{x}$, $\lim_{x \to -1} \sqrt{x}$, $\lim_{x \to 0^+} \sqrt{x}$, $\lim_{x \to 0^-} \sqrt{x}$, and $\lim_{x \to 0} \sqrt{x}$.

§ Solution

The graph of $y = \sqrt{x}$ is below. We emphasize the interesting cases where $a = 0$.



$\lim_{x \to 1} \sqrt{x} = \sqrt{1} = 1$, evidently.

$\lim_{x \to -1} \sqrt{x}$ does not exist (DNE).

- Actually, this is **not** because $\sqrt{-1}$ is imaginary. It is because there is no **punctured neighborhood** of $x = -1$ on which $\sqrt{x}$ is real. There is **no way** to approach $x = -1$ through the **domain** of f, where f is the (principal) square root function.

Review Section 2.1, Example 6. $\mathrm{Dom}(f)=[0,\infty)$ here, as well.

$$\lim_{x\to 0^+}\sqrt{x}=\sqrt{0}=0.$$

$\lim_{x\to 0^-}\sqrt{x}$ does not exist (DNE).

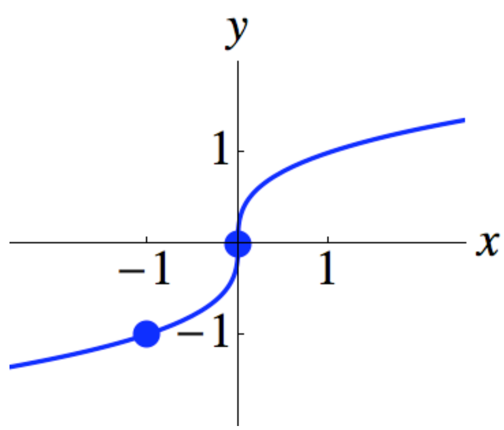
Therefore, $\lim_{x\to 0}\sqrt{x}$ does not exist (DNE). §

Example 3 (Evaluating the Limit of a Cube Root)

Evaluate $\lim_{x\to -1}\sqrt[3]{x}$ and $\lim_{x\to 0}\sqrt[3]{x}$.

§ Solution

The graph of $y=\sqrt[3]{x}$ is below.



The **domain** of the cube root function is $\mathbb{R}$. The (principal) cube roots of **negative** real numbers are (negative) **real numbers**; this is a key difference from square roots. It turns out that substituting $x=a$ works here for both limits.

$$\lim_{x\to -1}\sqrt[3]{x}=\sqrt[3]{-1}=-1.$$

$$\lim_{x\to 0}\sqrt[3]{x}=\sqrt[3]{0}=0.$$

§

Property 7 now extends our observations from Examples 2 and 3 to more **general radicands**, not just *x*, and also to **general types of roots**.

WARNING 2: In theory, **even roots** tend to require more thought than **odd roots**.

As before, assume $\lim_{x \to a} f(x) = L_1$.

7) The limit of a **root** equals the **root** of the limit … sometimes.

If n is a positive integer $\left(n \in \mathbb{Z}^+\right)$, and either

- (n is odd), or
- (n is even, and $L_1 > 0$), then:

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$$

For example, $\left(\text{Limit Form } \sqrt{4}\right) \Rightarrow 2$, and $\left(\text{Limit Form } \sqrt[3]{-8}\right) \Rightarrow -2$.

(The index of a radical, such as the "3" in $\sqrt[3]{-8}$, is assumed to be a constant.)

WARNING 3: The Limit Form $\sqrt[\text{even}]{0}$, corresponding to $L_1 = 0$, could either yield a **limit value of 0** or a limit that **does not exist (DNE)**. Informally, $\left(\text{Limit Form } \sqrt[\text{even}]{0}\right) \Rightarrow 0$ or "DNE," but further analysis is required to determine which is the case.

Limit Forms such as $\sqrt{-1}$ and $\sqrt[4]{-5}$ imply that the limits **do not exist (DNE)**.

Property 7* below elaborates on limits of **even roots**.

7*) Properties of Limits of Even Roots

Let n be a positive **even** integer.

- If $L_1 > 0$, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$ by Property 7.
- If $L_1 < 0$, then $\lim_{x \to a} \sqrt[n]{f(x)}$ does not exist (DNE). The one-sided limits $\lim_{x \to a^+} \sqrt[n]{f(x)}$ and $\lim_{x \to a^-} \sqrt[n]{f(x)}$ also do not exist (DNE).
- If $L_1 = 0$, then $\lim_{x \to a} \sqrt[n]{f(x)} = 0$ or "DNE." In particular,
  - •• $\lim_{x \to a} \sqrt[n]{f(x)} = 0 \Leftrightarrow f(x) \geq 0$ on some **punctured neighborhood** of a; change this to a **right-neighborhood** for a **right-hand limit** and a **left-neighborhood** for a **left-hand limit**.
  - •• Otherwise, the limit does not exist (DNE).

PART C: LIMITS OF ALGEBRAIC FUNCTIONS

Our understanding of Property 7 will now allow us to extend our **Basic Limit Theorem for Rational Functions** to more general **algebraic functions**.

Remember that:
- all **constant** functions are also **polynomial** functions,
- all **polynomial** functions are also **rational** functions, and
- all **rational** functions are also **algebraic** functions.

Basic Limit Theorem for Algebraic Functions

If f is an algebraic function, $a \in \mathrm{Dom}(f)$, and **no radicand of any even root approaches 0 in the limit (informally, the Limit Form $\sqrt[\text{even}]{0}$ does not appear),**

then $\lim_{x \to a} f(x) = f(a)$.

- To evaluate the limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

If the Limit Form $\sqrt[\text{even}]{0}$ does appear, this substitution method **might** still work, but further analysis is required. How is the radicand approaching 0?

Example 4 (Evaluating the Limit of an Algebraic Function)

Let $f(x) = \dfrac{\sqrt[3]{x-4}}{(3x-9)^2} + \sqrt{x+3}$. Evaluate $\lim_{x \to 2} f(x)$.

§ Solution

f is an algebraic function. Observe that:

$f(x)$ is real $\Leftrightarrow \left[x+3 \geq 0 \text{ and } (3x-9)^2 \neq 0 \right]$. As a result,

$\mathrm{Dom}(f) = \left\{ x \in \mathbb{R} \mid x \geq -3 \text{ and } x \neq 3 \right\} = [-3, \infty) \setminus \{3\} = [-3, 3) \cup (3, \infty)$.

We observe that $2 \in \mathrm{Dom}(f)$, and the Limit Form $\sqrt[\text{even}]{0}$ will **not** appear, so we **substitute ("plug in")** $x = 2$ and evaluate $f(2)$.

TIP 1: As a practical matter, when we evaluate the limit of an algebraic function, we often **substitute immediately and see what happens**. **(**We might not have time to find the domain.) If we end up with a **real number**, and if any $\sqrt[\text{even}]{0}$ Limit Forms encountered **only yield 0** (not "DNE"), then that number will be the **limit value**.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left[\frac{\sqrt[3]{x-4}}{(3x-9)^2} + \sqrt{x+3}\right]$$
$$= \frac{\sqrt[3]{(2)-4}}{\left[3(2)-9\right]^2} + \sqrt{(2)+3}$$
$$= \frac{\sqrt[3]{-2}}{9} + \sqrt{5}$$
$$= -\frac{\sqrt[3]{2}}{9} + \sqrt{5}, \text{ or } \sqrt{5} - \frac{\sqrt[3]{2}}{9}, \text{ or } \frac{9\sqrt{5} - \sqrt[3]{2}}{9}$$

§

We confront the Limit Form $\sqrt[\text{even}]{0}$ in the following Examples.

Example 5 (Resolving the Limit Form $\sqrt[\text{even}]{0}$)

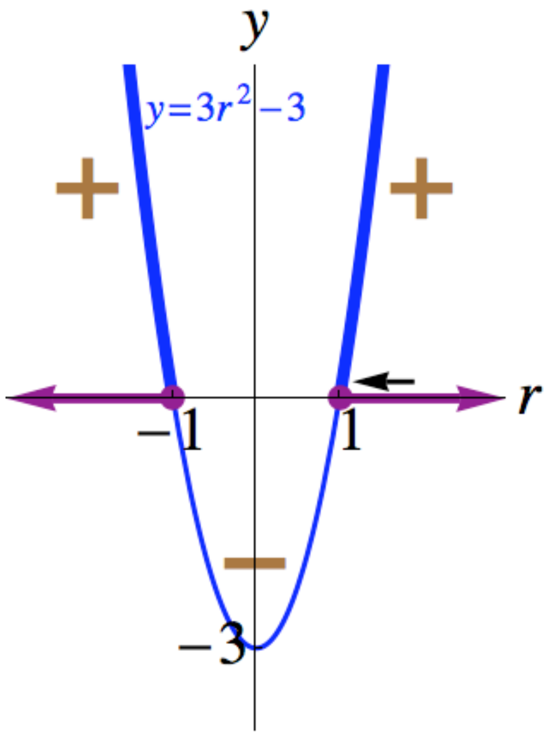
Evaluate $\displaystyle\lim_{r\to 1^+} \sqrt{3r^2 - 3}$.

§ Solution

• The radicand $3r^2 - 3$ is rational. By the **Extended Limit Theorem for Rational Functions** in Section 2.1, we find that $\displaystyle\lim_{r\to 1^+} \left(3r^2 - 3\right) = 0$, so we are facing the Limit Form $\sqrt[\text{even}]{0}$.

• We use Property 7*. We will show that $3r^2 - 3 \ge 0$ on a **right-neighborhood** of $r = 1$, and then $\displaystyle\lim_{r\to 1^+} \sqrt{3r^2 - 3} = 0$. Otherwise, the limit would not exist (DNE).

• The graph of $y = 3r^2 - 3$ follows. It is an upward-opening parabola in the *ry*-plane. The zeros of $3r^2 - 3$, -1 and 1, correspond to the *r*-intercepts. The **domain** of $\sqrt{3r^2 - 3}$ consists of the *r*-values that make $y = 3r^2 - 3 \ge 0$. It corresponds to the parts of the parabola that lie **above or on** the *r*-axis. This is important, because we are only allowed to approach $r = 1$ through this **domain** (in purple). In fact, here, we can approach $r = 1$ **from the right**.



Therefore, $\lim_{r \to 1^+} \sqrt{3r^2 - 3} = 0$.

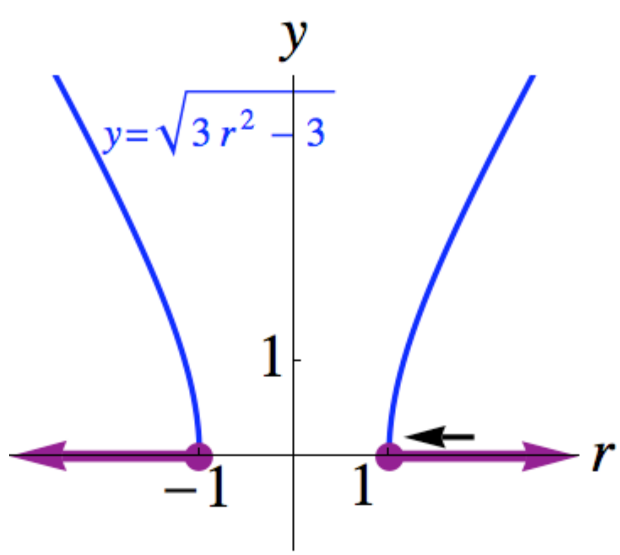
(For more, see Section 2.7: Nonlinear Inequalities in the Precalculus notes.)

Here's a **non-graphical** approach. As $r \to 1^+$, $r > 1$. Now,

$$r > 1 \Rightarrow$$
$$r^2 > 1 \Rightarrow$$
$$3r^2 > 3 \Rightarrow$$
$$3r^2 - 3 > 0$$

Therefore, $\lim_{r \to 1^+} \sqrt{3r^2 - 3} = 0$.

The graph of $y = \sqrt{3r^2 - 3}$ is below. Observe that the graph **disappears** where $3r^2 - 3 < 0$; this is where we fall **outside the domain** (in purple).



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Example 6 (Evaluating a Limit Using Example 5 and Properties of Limits)

Evaluate $\lim_{r\to 1^+}\left(7\sqrt{3r^2-3}+5\right)$.

§ Solution

$$\begin{aligned}\lim_{r\to 1^+}\left(7\sqrt{3r^2-3}+5\right) &= \lim_{r\to 1^+} 7\sqrt{3r^2-3} + \lim_{r\to 1^+} 5 \quad (\text{by Prop. 1 on } \textbf{sums})\\ &= 7\left(\lim_{r\to 1^+}\sqrt{3r^2-3}\right)+5 \quad \left(\begin{array}{l}\text{by Prop. 6 on } \textbf{constant}\\ \textbf{multiples}, \textbf{ elem. rules}\end{array}\right)\\ &= 7(0)+5 \quad (\text{by Example 5})\\ &= 5\end{aligned}$$

§

Example 7 (Resolving the Limit Form $\sqrt[\text{even}]{0}$)

Evaluate $\lim_{x\to -7}\sqrt{(x+7)^2}$.

§ Solution 1

As $x\to -7$, $(x+7)^2 \to 0$.

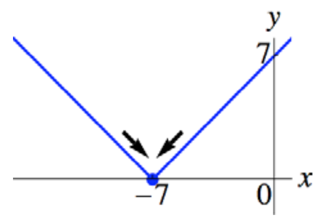
$(x+7)^2 \geq 0$ for all real x $(\forall x\in\mathbb{R})$. Therefore,

$\lim_{x\to -7}\sqrt{(x+7)^2} = 0$. §

§ Solution 2

$$\begin{aligned}\lim_{x\to -7}\sqrt{(x+7)^2} &= \lim_{x\to -7}|x+7|\\ &= |-7+7|\\ &= 0\end{aligned}$$

Below is the graph of $y=\sqrt{(x+7)^2}$, or $y=|x+7|$.



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FOOTNOTES

1. **Limits of linear combinations.** The fact that limit operators are linear implies that the limit of a linear combination of $f(x)$ and $g(x)$ equals the linear combination of the limits:

$$\begin{aligned}\lim_{x\to a}\left[c\cdot f(x) + d\cdot g(x)\right] &= c\cdot\lim_{x\to a} f(x) + d\cdot \lim_{x\to a} g(x)\\ &= cL_1 + dL_2 \quad (c, d\in\mathbb{R})\end{aligned}$$

SECTION 2.3: LIMITS AND INFINITY I

LEARNING OBJECTIVES

- Understand "long-run" limits and relate them to horizontal asymptotes of graphs.
- Be able to evaluate "long-run" limits, possibly by using short cuts for polynomial, rational, and/or algebraic functions.
- Be able to use informal Limit Form notation to analyze "long-run" limits.
- Know how to use "long-run" limits in real-world modeling.

PART A: HORIZONTAL ASYMPTOTES ("HA"s) and "LONG-RUN" LIMITS

A horizontal asymptote, which we will denote by "HA," is a horizontal line that a graph **approaches in a "long-run" sense**. We graph asymptotes as **dashed lines**.

"Long-Run" Limits

We will informally call $\lim_{x \to \infty} f(x)$ the "long-run" limit to the right and $\lim_{x \to -\infty} f(x)$ the "long-run" limit to the left.

- We read $\lim_{x \to \infty} f(x)$ as "the limit of $f(x)$ as x approaches infinity."

Using "Long-Run" Limits to Find Horizontal Asymptotes (HAs)

The graph of $y = f(x)$ has a **horizontal asymptote (HA)** at $y = L$ $(L \in \mathbb{R})$

$$\Leftrightarrow \left(\lim_{x \to \infty} f(x) = L, \text{ or } \lim_{x \to -\infty} f(x) = L \right).$$

- That is, the graph has an HA at $y = L$ $\Leftrightarrow$ one (or both) of the "long-run" limits is L.

The graph can have 0, 1, or 2 HAs. The following property implies that, if f is **rational**, then its graph **cannot have two HAs**.

"Twin (Long-Run) Limits" Property of Rational Functions

If f is a **rational** function, then $\lim_{x \to \infty} f(x) = L \Leftrightarrow \lim_{x \to -\infty} f(x) = L$ $(L \in \mathbb{R})$.

- That is, if $f(x)$ has a "long-run" limit value L as x "explodes" in **one** direction along the x-axis, then L must **also** be the "long-run" limit value as x "explodes" in the **other** direction.

Example 1 (The Graph of the Reciprocal Function has One HA.)

Let $f(x) = \frac{1}{x}$. Evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$, and identify any horizontal asymptotes (HAs) of the graph of $y = f(x)$.

§ Solution

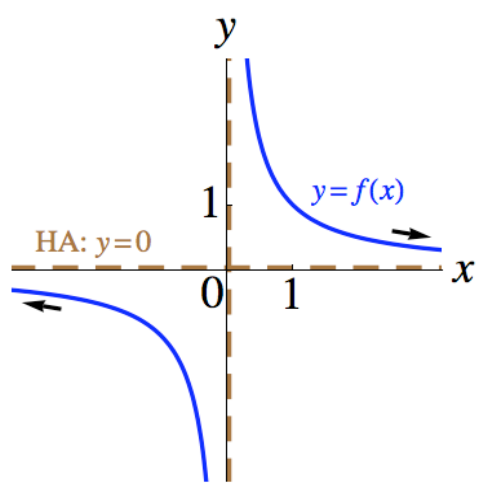
Let's use the **numerical / tabular approach**:

x	$-\infty \leftarrow$	-100	-10	-1	1	10	100	$\to \infty$
$f(x) = \frac{1}{x}$	$0 \leftarrow$	$-\frac{1}{100}$	$-\frac{1}{10}$	-1	1	$\frac{1}{10}$	$\frac{1}{100}$	$\to 0$

• Apparently, as x **increases without bound**, $f(x)$ approaches 0.
That is, $\lim_{x \to \infty} f(x) = 0$.

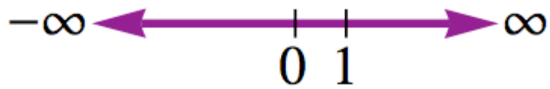
• Also, as x **decreases without bound**, $f(x)$ approaches 0.
That is, $\lim_{x \to -\infty} f(x) = 0$.

• Either limit statement implies that the graph of $y = f(x)$ below has a **horizontal asymptote (HA)** at $y = 0$, the x-axis. We will discuss the **vertical asymptote ("VA")** at the y-axis in Section 2.4.



Note: The graph of $y = \frac{1}{x}$ is a "rotated" **hyperbola**, a type of conic section with two branches. Its **asymptotes** are the coordinate axes (the x- and y-axes). §

x can only approach ∞ **from the left** and $-\infty$ **from the right**.



(It is now harder to apply our motto, "Limits are Local." Abstractly, we could consider the behavior of f on a sort of left-neighborhood of ∞, or on a sort of right-neighborhood of $-\infty$.)

• In Example 1, as $x \to \infty$, y or $f(x)$ approaches 0 **from above** (that is, **from greater values**). This is denoted by $f(x) \to 0^+$. In Section 2.4, we will see the need for this notation, as opposed to just $f(x) \to 0$, particularly when a limit analysis is a **piece** of a larger limit problem.

• Likewise, as $x \to -\infty$, y or $f(x)$ approaches 0 **from below** (that is, **from lesser values**). This is denoted by $f(x) \to 0^-$.

Example 1 gave us the most basic cases of the following Limit Forms.

$$\left(\text{Limit Form } \frac{1}{\infty}\right) \Rightarrow 0^+, \text{ and } \left(\text{Limit Form } \frac{1}{-\infty}\right) \Rightarrow 0^-$$

• It is often sufficient to simply write "0" as opposed to "0^+" or "0^-," especially if it is your "final answer" to a given limit problem. In Example 6, we will **have** to write "0," as neither 0^+ nor 0^- would be appropriate.

The following property covers variations on such Limit Forms.

<u>Rescaling Property of Limit Forms</u>

The following rules apply to Limit Forms that do **not** yield a nonzero real number. They must yield 0 (perhaps as 0^+ or 0^-), ∞, $-\infty$, or "DNE."

• If the Limit Form is **multiplied or divided** by a **positive** real number, then the resulting Limit Form yields the **same** result as the first.

• If the Limit Form is **multiplied or divided** by a **negative** real number, then the resulting Limit Form yields the **opposite** result.

(If the first Limit Form yields "DNE," then so does the second. Also, 0^+ and 0^- are opposites.)

In Section 2.2, Limit Property 6 on **constant multiples** told us how to rescale Limit Forms that **do** yield a nonzero real number. For example, twice a Limit Form that yields 3 will yield 6.

Example Set 2 (Rescaling Limit Forms)

$$\left(\text{Limit Form } \frac{2}{\infty}\right) \Rightarrow 0^+ \qquad \left(\text{Limit Form } \frac{3}{-\infty}\right) \Rightarrow 0^-$$

$$\left(\text{Limit Form } \frac{-\pi}{\infty}\right) \Rightarrow 0^- \qquad \left(\text{Limit Form } \frac{-4.1}{-\infty}\right) \Rightarrow 0^+$$

In fact, $\left(\text{Limit Form } \frac{c}{\infty}\right) \Rightarrow 0$ for all real c $(\forall c \in \mathbb{R})$.

§

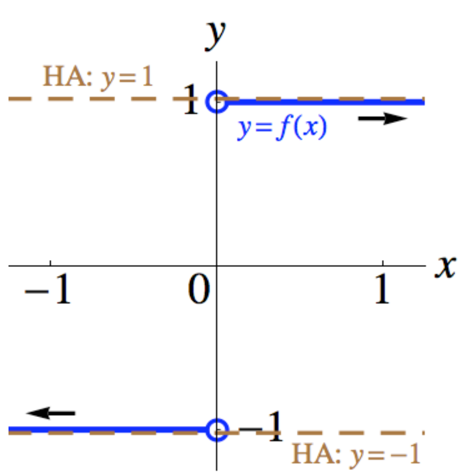
Example 3 (A Graph with Two HAs; Revisiting Example 14 in Section 2.1)

Let $f(x) = \frac{|x|}{x}$.

Identify any horizontal asymptotes (HAs) of the graph of $y = f(x)$.

§ Solution

We obtained the graph of $y = f(x)$ below in Section 2.1, Example 14.



Observe that $\lim_{x \to \infty} f(x) = 1$, and $\lim_{x \to -\infty} f(x) = -1$.

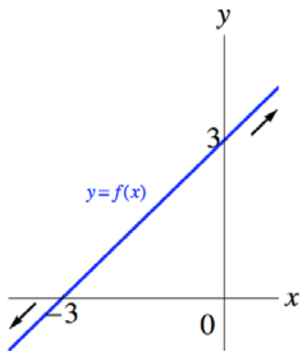
Therefore, the graph has two HAs, at $y = 1$ (a "<u>right-hand HA</u>") and at $y = -1$ (a "<u>left-hand HA</u>").

- Usually, when a graph exhibits this kind of flatness and coincides with the HAs, we don't even bother drawing the dashed lines.

Although f is piecewise rational, it is **not** a rational function overall, so the "Twin (Long-Run) Limits Property" does **not** apply. §

Example 4 (A Graph with No HAs)

Let $f(x) = x + 3$.



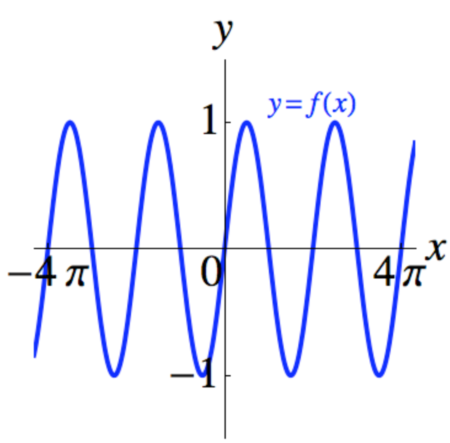
$\lim_{x \to \infty} f(x) = \infty$, and $\lim_{x \to -\infty} f(x) = -\infty$.

Neither long-run limit exists, so the graph has **no HAs**.

Because of these **nonexistent** limits, the "Twin (Long-Run) Limits Property" does **not** apply. §

Example 5 (A Graph with No HAs)

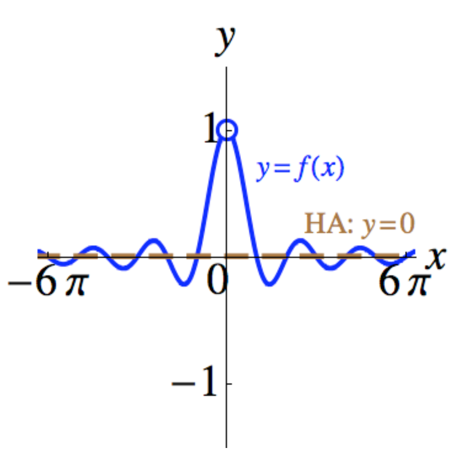
Let $f(x) = \sin x$.



The graph has **no HAs**, because $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ do not exist (DNE). This is because the function values **oscillate** between -1 and 1 and **do not** approach a single real number as $x \to \infty$, nor as $x \to -\infty$. We cannot even say that the limit is ∞ or $-\infty$. §

Example 6 (A Graph That Crosses Over Its HA)

Let $f(x) = \dfrac{\sin x}{x}$.



The graph has one HA, at $y = 0$, since $\lim_{x \to \infty} f(x) = 0$, and $\lim_{x \to -\infty} f(x) = 0$.
These are proven using the Squeeze (Sandwich) Theorem from Section 2.6.
A graph can cross over its HA; here, it happens infinitely many times!

• HAs relate to **long-run** behaviors of $f(x)$, not local behaviors.

• Note: In Section 3.4, we will show why the **hole** at $(0, 1)$ is important! §

PART B : "LONG - RUN" LIMIT RULES FOR $\frac{c}{x^k}$

The following rules will help us evaluate "long-run" limits of algebraic functions. Observe that $\frac{1}{x}$ is a basic example of $\frac{c}{x^k}$.

$$\boxed{\lim_{x\to\infty} x^k = \infty \quad \left(k \in \mathbb{R}^+\right)}$$

"Long-Run" Limit Rules for $\frac{c}{x^k}$

If c is a real number and k is a positive rational number $\left(c \in \mathbb{R},\ k \in \mathbb{Q}^+\right)$, then:

- $\lim_{x\to\infty} \frac{c}{x^k} = 0$, because $\left(\text{Limit Form } \frac{c}{\infty}\right) \Rightarrow 0$.
- $\lim_{x\to-\infty} \frac{c}{x^k} = 0$, **if** x^k is **real** for $x<0$, because $\left(\text{Limit Form } \frac{c}{\pm\infty}\right) \Rightarrow 0$;

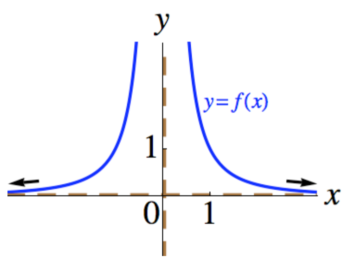
otherwise, $\lim_{x\to-\infty} \frac{c}{x^k}$ does not exist (DNE).

WARNING 1: The "DNE" case arises for a "long-run" limit as $x \to -\infty$ when the **denominator** of $\frac{c}{x^k}$ involves an **even root**.

• What about other values of k? See the Exercises for a case where $k<0$.
See Footnote 1 on positive **irrational** values of k.

Example 7 (Applying the "Long - Run" Limit Rules for $\frac{c}{x^k}$)

Let $f(x) = \frac{1}{x^2}$.



$$\lim_{x\to\infty} \frac{1}{x^2} = 0, \text{ since } \left(\text{Limit Form } \frac{1}{\infty}\right) \Rightarrow 0^+.$$

$$\lim_{x\to-\infty} \frac{1}{x^2} = 0, \text{ since } \left(\text{Limit Form } \frac{1}{\infty}\right) \Rightarrow 0^+.$$

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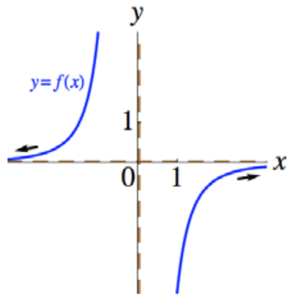
Example 8 (Applying the "Long - Run" Limit Rules for $\frac{c}{x^k}$)

Let $f(x) = \dfrac{-\pi}{x^3}$.

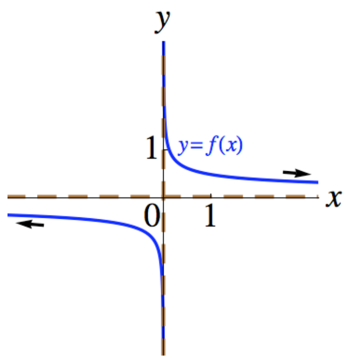


$$\lim_{x\to\infty} \frac{-\pi}{x^3} = 0, \text{ since } \left(\text{Limit Form } \frac{-\pi}{\infty}\right) \Rightarrow 0^-.$$

$$\lim_{x\to-\infty} \frac{-\pi}{x^3} = 0, \text{ since } \left(\text{Limit Form } \frac{-\pi}{-\infty}\right) \Rightarrow 0^+. \ \S$$

Example 9 (Applying the "Long - Run" Limit Rules for $\frac{c}{x^k}$)

Let $f(x) = \dfrac{1}{2\left(\sqrt[3]{x}\right)} = \dfrac{(1/2)}{x^{1/3}}$.



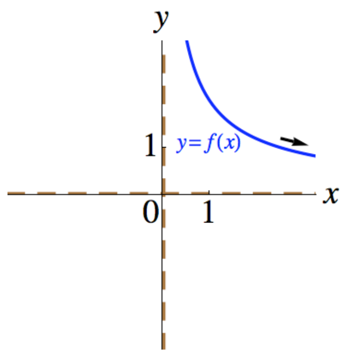
$$\lim_{x\to\infty} \frac{(1/2)}{x^{1/3}} = 0, \text{ since } \left(\text{Limit Form } \frac{1/2}{\infty}\right) \Rightarrow 0^+.$$

$$\lim_{x\to-\infty} \frac{(1/2)}{x^{1/3}} = 0, \text{ since } \left(\text{Limit Form } \frac{1/2}{-\infty}\right) \Rightarrow 0^-.$$

Observe that $x^{1/3}$, or $\sqrt[3]{x}$, is **real** (and negative) for all $x < 0$, so the desired limit is 0. Furthermore, we can say it is 0^-, since $x^{1/3} \to -\infty$ as $x \to -\infty$. §

Example 10 (Applying the "Long - Run" Limit Rules for $\frac{c}{x^k}$)

Let $f(x) = \dfrac{2}{\sqrt[4]{x^3}} = \dfrac{2}{x^{3/4}}$.



$$\lim_{x\to\infty} \frac{2}{x^{3/4}} = 0, \text{ since } \left(\text{Limit Form } \frac{2}{\infty}\right) \Rightarrow 0^+.$$

$\displaystyle\lim_{x\to-\infty} \frac{2}{x^{3/4}}$ does not exist (DNE); see Footnote 2.

Observe that $x^{3/4}$, or $\sqrt[4]{x^3}$, involves an **even root**, so it is **not real** for all $x < 0$. §

PART C: "LONG-RUN" LIMITS OF POLYNOMIAL FUNCTIONS

Constant functions are the **only** polynomial functions whose graphs have an HA.

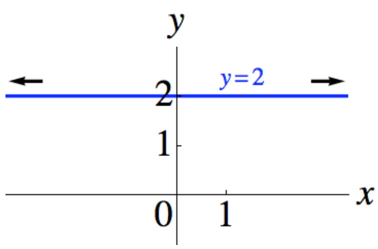
"Long-Run" Limits of Constant Functions

If $c \in \mathbb{R}$, then: $\lim_{x \to \infty} c = c$, and $\lim_{x \to -\infty} c = c$.

The graph of $y = c$ has itself as its **sole HA**.

Example 11 (The Graph of a Constant Function Has One HA)

$\lim_{x \to \infty} 2 = 2$, and $\lim_{x \to -\infty} 2 = 2$.



The graph has an **HA** at $y = 2$, but we omit the dashed line here. §

On the other hand, a **nonconstant polynomial** function either increases or decreases without bound (it "explodes") in the "long run" to the right. It also "explodes" to the left. Its graph has **no HAs**.

"Long-Run" Limits of Nonconstant Polynomial Functions

If f is a **nonconstant polynomial** function, then:

$\lim_{x \to \infty} f(x) = \infty$ or $-\infty$, and

$\lim_{x \to -\infty} f(x) = \infty$ or $-\infty$.

The graph of $y = f(x)$ has **no HAs**.

• In Example 4, we looked at $f(x) = x + 3$. (Granted, "explosive" may be too strong a term for the "long-run" behavior of that linear function f.)

• The following short cut will help us determine whether a "long-run" limit is ∞ or $-\infty$.

"Dominant Term Substitution (DTS)" Short Cut for Polynomial Functions

Let f be a **polynomial** function. The **"long-run"** limits of $f(x)$ are the **same** as those of its dominant term, which is the **leading term**. We **substitute** by replacing $f(x)$ with its dominant term.

• For more on **dominant terms**, see Footnote 4.

• This technique can be extended **carefully** to other functions, as we will see. (See Part E and Footnotes 5 and 6 for **pitfalls**.)

This short cut is justified by factoring and the following:

$$\boxed{(\text{Limit Form } \infty \cdot 1) \Rightarrow \infty.}$$

WARNING 2: "DTS" is used to evaluate **"long-run"** limits, **not** limits at a point.

Example 12 (Evaluating a "Long-Run" Limit of a Polynomial Function)

Evaluate $\lim_{x\to\infty}\left(x^8 - x^6\right)$.

§ Solution 1 (Using the "DTS" Short Cut)

There is a tension between the two terms, x^8 and $-x^6$, because $x^8 \to \infty$ as $x \to \infty$, while $-x^6 \to -\infty$. (Review the **"long-run"** behavior of **monomials** such as these and their graphs in Section 2.2 of the Precalculus notes.)

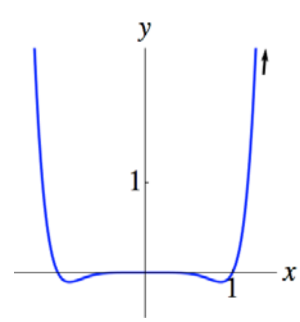
• In Section 2.5, we will see that $(\text{Limit Form } \infty - \infty)$ is **indeterminate**; further analysis is required.

The **leading term**, x^8, dictates the **"long-run" behavior** of their sum, because its magnitude "overwhelms" the magnitude of $-x^6$ in the "long run." (See Footnotes 3 and 4.) The graph of $y = x^8 - x^6$ below shares the **"long-run" upward-opening bowl** shape that the graph of $y = x^8$ does.

WARNING 3: x^8 is the **leading term** because it is the term of **highest degree**, **not** because it is written first.

Applying the **"DTS" short cut**:

$$\lim_{x\to\infty}\left(x^8 - x^6\right) = \lim_{x\to\infty} x^8$$
$$= \infty$$



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§ Solution 2 (Using a Factoring Method to Rigorously Justify "DTS")

We begin by **factoring** out the **leading term** $\left(x^8\right)$, **not** the GCF $\left(\pm x^6\right)$.

$$\begin{aligned}\lim_{x\to\infty}\left(x^8 - x^6\right) &= \lim_{x\to\infty} x^8\left(1-\frac{x^6}{x^8}\right)\\ &= \lim_{x\to\infty} \underbrace{x^8}_{\to\infty}\underbrace{\left(1-\frac{1}{x^2}\right)}_{\to(1)}\\ &= \infty\end{aligned}$$

As $x\to\infty$, $x^8\to\infty$. By Part B, $\dfrac{1}{x^2}\to 0$, and thus $\left(1-\dfrac{1}{x^2}\right)\to 1$. Then, Limit Form $(\infty\cdot 1)\Rightarrow\infty$.

§

PART D: "LONG-RUN" LIMITS OF RATIONAL FUNCTIONS

Let f be a rational function. The "Twin (Long-Run) Limits" Property from Part A implies that the graph of $y = f(x)$ can have **no HAs or exactly one HA**.

"Long-run" limits of $f(x)$ can be found **rigorously** by using the "Division Method" below. It is related to, but easier to apply than, our Factoring Method from Example 12.

> <u>"Division Method" for Evaluating "Long-Run" Limits of Rational Functions</u>
>
> Let $f(x) = \dfrac{N(x)}{D(x)}$, where the numerator $N(x)$ and the denominator $D(x)$ are nonzero polynomials in x. **Divide** (each term of) $N(x)$ and $D(x)$ by the **highest power of x** (the power of x in the leading term) **in the denominator** $D(x)$. The **"long-run"** limits of the resulting expression will be the **same** as those of $f(x)$.

• This procedure ensures that the new denominator will approach a **nonzero real number**, namely the leading coefficient of $D(x)$.
The overall "long-run" limits will then be easy to find.

<u>WARNING 4</u>: The "Division Method" is used to evaluate **"long-run"** limits, **not** limits at a point. Students often forget this.

By **comparing the degrees** of $N(x)$ and $D(x)$, denoted by $\deg(N(x))$ and $\deg(D(x))$, we will categorize rational functions into **three cases**, each with its own short cut for **identifying HAs** and/or **evaluating "long-run" limits**.

Case 1: Equal Degrees

If $\deg(N(x)) = \deg(D(x))$, then the **sole HA** of the graph of $y = f(x)$ is at $y = L$ $(L \neq 0)$, where $L = \dfrac{\text{the leading coefficient of } N(x)}{\text{the leading coefficient of } D(x)}$, the **ratio of the leading coefficients**. Also,

$$\lim_{x \to \infty} f(x) = L, \text{ and } \lim_{x \to -\infty} f(x) = L.$$

Example 13 (Evaluating "Long-Run" Limits of a Rational Function; Case 1: Equal Degrees)

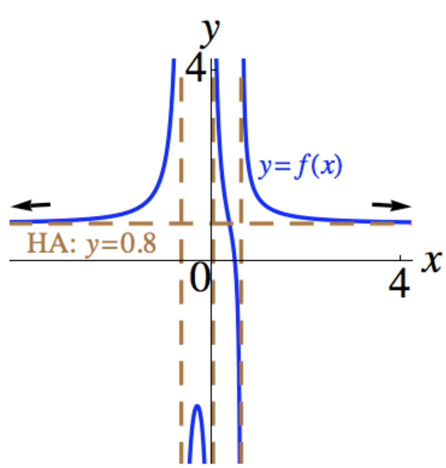
Evaluate $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$, where $f(x) = \dfrac{4x^3 + x - 1}{5x^3 - 2x}$.

§ Solution 1 (Using the Short Cut for Case 1)

Let $N(x) = 4x^3 + x - 1$ and $D(x) = 5x^3 - 2x$. They **both** have degree 3.

The **ratio of their leading coefficients** is $\dfrac{4}{5}$ (or 0.8), so $y = \dfrac{4}{5}$ is the **sole HA** for the graph of $y = f(x)$ below.

Also, $\lim\limits_{x \to \infty} f(x) = \dfrac{4}{5}$, and $\lim\limits_{x \to -\infty} f(x) = \dfrac{4}{5}$.



§

§ Solution 2 (Using the "Division Method" to Rigorously Justify the Short Cut)

$f(x) = \dfrac{\text{nonconstant polynomial in } x}{\text{nonconstant polynomial in } x}$, so both of its **"long-run"** limits will have Limit Form $\dfrac{\pm\infty}{\pm\infty}$. This is simply written as $\dfrac{\infty}{\infty}$, since **further analysis** is required, anyway. In Section 2.5, we will discuss indeterminate forms such as this.

The "Division Method" tells us to **divide** (each term of) the numerator and the denominator by x^3, the **highest power of *x* in the denominator**.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^3 + x - 1}{5x^3 - 2x} \qquad \left(\text{Indeterminate Limit Form } \frac{\infty}{\infty} \right)$$

$$= \lim_{x\to\infty} \frac{\dfrac{4x^3}{x^3} + \dfrac{x}{x^3} - \dfrac{1}{x^3}}{\dfrac{5x^3}{x^3} - \dfrac{2x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4 + \overbrace{\dfrac{1}{x^2}}^{\to 0} - \overbrace{\dfrac{1}{x^3}}^{\to 0}}{5 - \underbrace{\dfrac{2}{x^2}}_{\to 0}} \qquad \left(\begin{array}{l} \underline{\textbf{WARNING 5}}\textbf{:} \\ \text{When applying Part B,} \\ \textbf{remember the } \to \textbf{ arrows!} \end{array} \right)$$

$$= \frac{4}{5}$$

By the **"Twin (Long-Run) Limits" Property** of Rational Functions, $\lim\limits_{x\to-\infty} f(x) = \dfrac{4}{5}$, also, and the graph of $y = f(x)$ has its **sole HA** at $y = \dfrac{4}{5}$.

We can also show that $\lim\limits_{x\to-\infty} f(x) = \dfrac{4}{5}$ by using a very similar solution. §

§ Solution 3 (Using a Factoring Method to Rigorously Justify the Short Cut)

In Example 12, we **factored** the **leading term** out of a polynomial. Here, we will do the same to the **numerator** and the **denominator**.

$$\lim_{x\to\infty}\frac{4x^3+x-1}{5x^3-2x}=\lim_{x\to\infty}\frac{4x^3\left(1+\dfrac{x}{4x^3}-\dfrac{1}{4x^3}\right)}{5x^3\left(1-\dfrac{2x}{5x^3}\right)}$$

$$=\lim_{x\to\infty}\underbrace{\left(\frac{4\cancel{x^3}}{5\cancel{x^3}}\right)}_{\to\left(\frac{4}{5}\right)}\underbrace{\left(\frac{1+\overbrace{\dfrac{1}{4x^2}}^{\to 0}-\overbrace{\dfrac{1}{4x^3}}^{\to 0}}{1-\underbrace{\dfrac{2}{5x^2}}_{\to 0}}\right)}_{\to(1)}$$

$$=\frac{4}{5}$$

By the **"Twin (Long-Run) Limits" Property** of Rational Functions, $\lim_{x\to-\infty}f(x)=\frac{4}{5}$, also, and the graph of $y=f(x)$ has its **sole HA** at $y=\frac{4}{5}$.

The **"DTS"** short cut can be modified as follows. If we **replace** the numerator and the denominator with their **dominant terms**, $4x^3$ and $5x^3$, respectively, then we can simply take the "long-run" limits of the result.

• The idea is that $f(x)$ behaves like $\frac{4x^3}{5x^3}$, or $\frac{4}{5}$, in the **"long run."**

The following is **informal**, and **pitfalls** of this "DTS" short cut will be seen in Part E and Footnotes 5 and 6.

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{4x^3+x-1}{5x^3-2x}$$

$$=\lim_{x\to\infty}\frac{4x^3}{5x^3}$$

$$=\frac{4}{5}$$

$\lim_{x\to-\infty}f(x)=\frac{4}{5}$ by using a very similar solution.

§

Case 2: "Bottom-Heavy" in Degree

If $\deg\left(N(x)\right) < \deg\left(D(x)\right)$, then f is a **proper** rational function, and the **sole HA** of the graph of $y = f(x)$ is at $y = 0$, the x-axis. Also,

$$\lim_{x\to\infty} f(x) = 0, \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

Example 14 (Evaluating "Long-Run" Limits of a Rational Function; Case 2: "Bottom-Heavy" in Degree)

Evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$, where $f(x) = \dfrac{x^2-3}{x^3+4x^2+1}$.

§ Solution 1 (Using the Short Cut for Case 2)

Let $N(x) = x^2 - 3$ and $D(x) = x^3 + 4x^2 + 1$. $\deg\left(N(x)\right) < \deg\left(D(x)\right)$, because $2 < 3$, so $f(x)$ is **"bottom-heavy" in degree** and is **proper**. Therefore, $y = 0$ is the **sole HA** for the graph of $y = f(x)$.

Also, $\lim\limits_{x\to\infty} f(x) = 0$, and $\lim\limits_{x\to-\infty} f(x) = 0$. §

§ Solution 2 (Using the "Division Method" to Rigorously Justify the Short Cut)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-3}{x^3+4x^2+1} \qquad \left(\text{Indeterminate Limit Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\dfrac{x^2}{x^3} - \dfrac{3}{x^3}}{\dfrac{x^3}{x^3} + \dfrac{4x^2}{x^3} + \dfrac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\overbrace{\dfrac{1}{x}}^{\to 0} - \overbrace{\dfrac{3}{x^3}}^{\to 0}}{1 + \underbrace{\dfrac{4}{x}}_{\to 0} + \underbrace{\dfrac{1}{x^3}}_{\to 0}}$$

$$= 0$$

By the **"Twin (Long-Run) Limits" Property** of Rational Functions, $\lim\limits_{x\to-\infty} f(x) = 0$, also, and the graph of $y = f(x)$ has its **sole HA** at $y = 0$. §

§ Solution 3 (Using a Factoring Method to Rigorously Justify the Short Cut)

$$\lim_{x\to\infty}\frac{x^2-3}{x^3+4x^2+1}=\lim_{x\to\infty}\frac{\overbrace{\cancel{x^2}}^{(1)}\left(1-\frac{3}{x^2}\right)}{\underbrace{\cancel{x^3}}_{(x)}\left(1+\frac{4x^2}{x^3}+\frac{1}{x^3}\right)}$$

$$=\lim_{x\to\infty}\underbrace{\left(\frac{1}{x}\right)}_{\to(0)}\underbrace{\left(\frac{1-\overbrace{\frac{3}{x^2}}^{\to 0}}{1+\underbrace{\frac{4}{x}}_{\to 0}+\underbrace{\frac{1}{x^3}}_{\to 0}}\right)}_{\to(1)}$$

$$=0$$

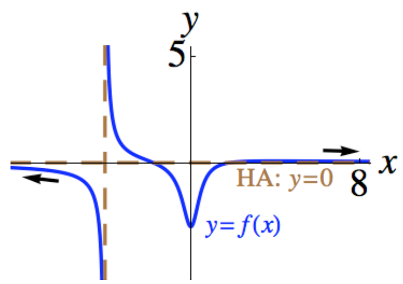
By the **"Twin (Long-Run) Limits" Property** of Rational Functions, $\lim_{x\to-\infty} f(x)=0$, also, and the graph of $y=f(x)$ has its **sole HA** at $y=0$.

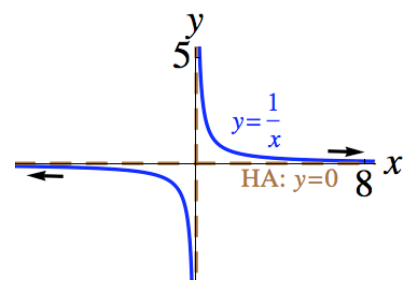
The **"DTS"** short cut suggests that $f(x)$ behaves like $\frac{1}{x}$ in the **"long run."**

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2-3}{x^3+4x^2+1}$$
$$=\lim_{x\to\infty}\frac{x^2}{x^3}$$
$$=\lim_{x\to\infty}\frac{1}{x}\quad\left(\text{Limit Form }\frac{1}{\infty}\right)$$
$$=0$$

$\lim_{x\to-\infty} f(x)=0$ by using a very similar solution.

The graph of $y=f(x)$ (on the left) behaves like that of $y=\frac{1}{x}$ (on the right) and approaches their **common HA** at $y=0$ in the **"long run."**





§

Case 3: "Top-Heavy" in Degree

If $\deg\left(N(x)\right) > \deg\left(D(x)\right)$, then the graph of $y = f(x)$ has **no HAs**. Also,

$$\lim_{x\to\infty} f(x) = \infty \text{ or } -\infty\text{, and}$$
$$\lim_{x\to-\infty} f(x) = \infty \text{ or } -\infty.$$

If we apply **"DTS"** to $N(x)$ and $D(x)$ by replacing them with their dominant terms, then the **"long-run"** limits of the result will be the **same** as the "long-run" limits of $f(x)$. This was true in Case 1 and Case 2, as well.

- If $\deg\left(N(x)\right) = \deg\left(D(x)\right) + 1$, then the graph of $y = f(x)$ has a slant asymptote ("SA"), also known as an oblique asymptote.

Example 15 (Evaluating "Long-Run" Limits of a Rational Function; Case 3: "Top-Heavy" in Degree)

Evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$, where $f(x) = \dfrac{-5+3x^2+6x^3}{1+3x^2}$.

§ Solution 1 (Using the "DTS" Short Cut)

For convenience, we will **rewrite** the numerator and the denominator in **descending powers** of x: $f(x) = \dfrac{6x^3+3x^2-5}{3x^2+1}$. Let $N(x) = 6x^3+3x^2-5$ and $D(x) = 3x^2+1$. $\deg\left(N(x)\right) > \deg\left(D(x)\right)$, because $3 > 2$, so $f(x)$ is **"top-heavy" in degree**. The graph of $y = f(x)$ has **no HAs**. We know that the "**long-run"** limits will be **infinite**; we now **specify** them as ∞ or $-\infty$.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{6x^3+3x^2-5}{3x^2+1} \\
&= \lim_{x\to\infty} \frac{6x^3}{3x^2} \\
&= \lim_{x\to\infty} 2x \quad \left(\text{L.F. } 2\cdot\infty\right) \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{6x^3+3x^2-5}{3x^2+1} \\
&= \lim_{x\to-\infty} \frac{6x^3}{3x^2} \\
&= \lim_{x\to-\infty} 2x \quad \left(\text{L.F. } 2\cdot(-\infty)\right) \\
&= -\infty
\end{aligned}$$

The idea is that $f(x)$ behaves like $2x$ in the **"long run."** §

§ Solution 2 (Using the "Division Method" to Rigorously Justify the Short Cut)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x^3+3x^2-5}{3x^2+1} \quad \left(\text{Indeterminate Limit Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{6x^3}{x^2}+\frac{3x^2}{x^2}-\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{6x+3-\overbrace{\frac{5}{x^2}}^{\to 0}}{3+\underbrace{\frac{1}{x^2}}_{\to 0}} \quad \left(\text{Limit Form } \frac{\infty}{3}\right)$$

$$= \infty$$

The "Twin (Long-Run) Limits" Property does **not** apply, because this limit is **not real**. When evaluating $\lim_{x\to-\infty} f(x)$, the initial algebra is identical.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{6x^3+3x^2-5}{3x^2+1} \quad \left(\text{Indeterminate Limit Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to-\infty} \frac{\frac{6x^3}{x^2}+\frac{3x^2}{x^2}-\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{1}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{6x+3-\overbrace{\frac{5}{x^2}}^{\to 0}}{3+\underbrace{\frac{1}{x^2}}_{\to 0}} \quad \left(\text{Limit Form } \frac{-\infty}{3}\right)$$

$$= -\infty$$

§

§ Solution 3 (Using a Factoring Method to Rigorously Justify the Short Cut)

$$\lim_{x\to\infty}\frac{6x^3+3x^2-5}{3x^2+1}=\lim_{x\to\infty}\frac{\overbrace{6}^{(2)}\overbrace{\cancel{x^3}}^{(x)}\left(1+\frac{3x^2}{6x^3}-\frac{5}{6x^3}\right)}{\underbrace{3}_{(1)}\underbrace{\cancel{x^2}}_{(1)}\left(1+\frac{1}{3x^2}\right)}$$

$$=\lim_{x\to\infty}\underbrace{(2x)}_{\to(\infty)}\underbrace{\left(\frac{1+\overbrace{\frac{1}{2x}}^{\to 0}-\overbrace{\frac{5}{6x^3}}^{\to 0}}{1+\frac{1}{\underbrace{3x^2}_{\to 0}}}\right)}_{\to(1)}\quad(\text{Limit Form } \infty\cdot 1)$$

$$=\infty$$

The "Twin (Long-Run) Limits" Property does **not** apply, because this limit is **not real**. When evaluating $\lim\limits_{x\to-\infty} f(x)$, the initial algebra is identical.

$$\lim_{x\to-\infty}\frac{6x^3+3x^2-5}{3x^2+1}=\lim_{x\to-\infty}\frac{\overbrace{6}^{(2)}\overbrace{\cancel{x^3}}^{(x)}\left(1+\frac{3x^2}{6x^3}-\frac{5}{6x^3}\right)}{\underbrace{3}_{(1)}\underbrace{\cancel{x^2}}_{(1)}\left(1+\frac{1}{3x^2}\right)}$$

$$=\lim_{x\to-\infty}\underbrace{(2x)}_{\to(-\infty)}\underbrace{\left(\frac{1+\overbrace{\frac{1}{2x}}^{\to 0}-\overbrace{\frac{5}{6x^3}}^{\to 0}}{1+\frac{1}{\underbrace{3x^2}_{\to 0}}}\right)}_{\to(1)}\quad(\text{Limit Form } (-\infty)\cdot 1)$$

$$=-\infty$$

§

Example 16 (Finding a Slant Asymptote (SA); Revisiting Example 15)

Find the slant asymptote (SA) for the graph of $y = f(x)$, where

$$f(x) = \frac{-5 + 3x^2 + 6x^3}{1 + 3x^2}, \text{ or } \frac{6x^3 + 3x^2 - 5}{3x^2 + 1}.$$

§ Solution

$\deg(N(x)) = \deg(D(x)) + 1$, since $3 = 2 + 1$.
Therefore, the graph of $y = f(x)$ has a **slant asymptote (SA)**.

Unfortunately, our previous methods for evaluating "long-run" limits are **not** guaranteed to give us the equation of the SA. We will use **Long Division** (see Section 2.3 of the Precalculus notes) to re-express $f(x)$ and find the SA. We begin with the **"descending powers"** form $f(x) = \dfrac{6x^3 + 3x^2 - 5}{3x^2 + 1}$ and insert missing terms by using zero coefficients (helpful but optional).

$$\begin{array}{r|l} & \quad 2x + 1 \\ 3x^2 + 0x + 1 & \quad 6x^3 + 3x^2 + 0x - 5 \\ & \quad \cancel{6x^3 + 0x^2 + 2x} \\ & \quad \underline{-6x^3 - 0x^2 - 2x} \\ & \qquad\quad 3x^2 - 2x - 5 \\ & \qquad\quad \cancel{3x^2 + 0x + 1} \\ & \qquad\quad \underline{-3x^2 - 0x - 1} \\ & \qquad\qquad\quad -2x - 6 \end{array}$$

We **stop** the division process here, because the degree of the remainder $(-2x - 6)$ is **less than** the degree of the divisor $(3x^2 + 1)$; that is, $1 < 2$.

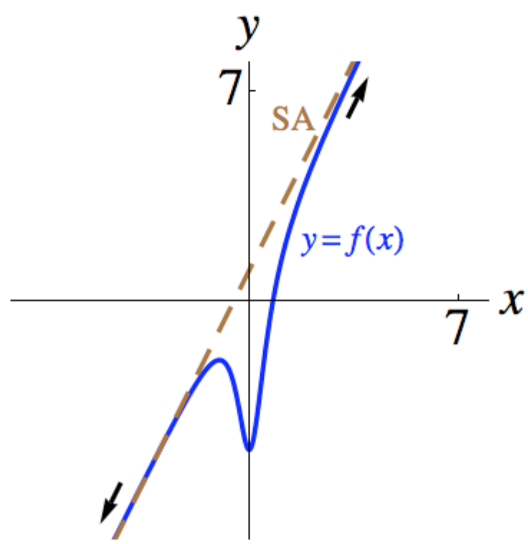
We can now re-express $f(x)$ in the form: $(\text{quotient}) + \dfrac{(\text{remainder})}{(\text{divisor})}$.

$$f(x) = \underbrace{2x + 1}_{\substack{\text{polynomial} \\ \text{part, } p(x)}} + \underbrace{\frac{-2x - 6}{3x^2 + 1}}_{\substack{\text{proper rational} \\ \text{part, } r(x)}}$$

$$= 2x + 1 - \frac{2x + 6}{3x^2 + 1}$$

$r(x) = \dfrac{-2x-6}{3x^2+1}$ represents a **proper** rational function. As a result, $r(x) \to 0$ as $x \to \infty$ **and** as $x \to -\infty$; see Case 1. In the "long run," $r(x)$ "decays" in magnitude. Therefore, the graph of $y = f(x)$ approaches the graph of $y = p(x)$ as $x \to \infty$ **and** as $x \to -\infty$.

The graph of $y = f(x)$ below (in blue) approaches its SA, $y = 2x+1$ (dashed in brown), in the **"long run."**



> "Zoom Out" Property of HAs and SAs
>
> A graph with **an HA or SA** will resemble the HA or SA in the **"long run."**
>
> • If we keep expanding the scope of a grapher's window, then a graph with **an HA or SA** will generally look more and more like the HA or SA.

Note: In Example 15, we said that $f(x)$ behaves like $2x$ in the **"long run."** In fact, $2x$ approaches $2x+1$ in a **relative** sense, in that $\dfrac{2x}{2x+1} \to 1$ in the **"long run."** However, $2x+1$ is more accurate in an **absolute** sense, in that $2x$ always differs from it by 1. §

Example 17 (Evaluating "Long-Run" Limits of a Rational Function; Case 3: "Top-Heavy" in Degree)

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and analyze the "long-run" behavior of the graph of $y = f(x)$, where $f(x) = \dfrac{-4x^7 + 12x^6 + 5x^4 - 23x^3 + 11}{4x^3 - 5}$.

§ Solution 1 (Using the "DTS" Short Cut)

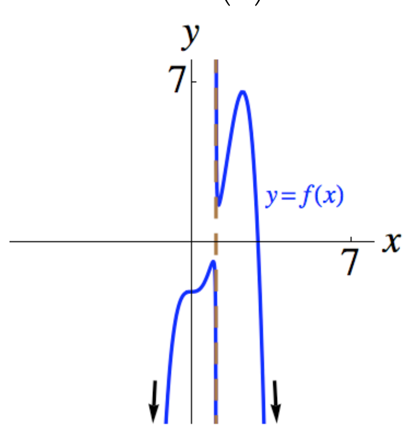
$\deg(N(x)) > \deg(D(x))$, because $7 > 3$, so $f(x)$ is **"top-heavy" in degree**. The graph of $y = f(x)$ has **no HAs** (and **no SAs**).

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{-4x^7}{4x^3} \\ &= \lim_{x\to\infty}\left(-x^4\right) \\ &= -\infty\end{aligned} \qquad \begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{-4x^7}{4x^3} \\ &= \lim_{x\to-\infty}\left(-x^4\right) \\ &= -\infty\end{aligned}$$

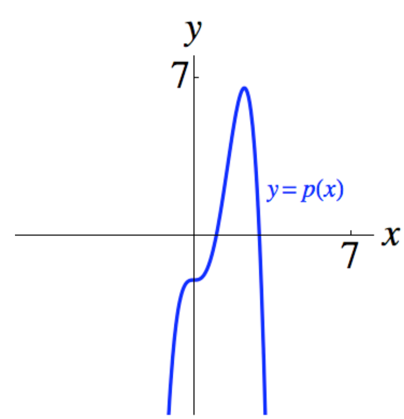
Long Division gives us: $f(x) = -x^4 + 3x^3 - 2 \; + \; \dfrac{1}{4x^3 - 5}$.

The graph of $y = -x^4 + 3x^3 - 2$ (on the right below) is a nonlinear asymptote that the graph of $y = f(x)$ approaches in the **"long run."** Observe that the **leading term** is $-x^4$, which was the result of **"DTS."** Based on this alone, we know that the graph of $y = f(x)$ approaches the shape of a **downward-opening bowl** if we "zoom out in the long run."

Graph of $y = f(x)$



Graph of $y = -x^4 + 3x^3 - 2$



§

§ Solutions 2 and 3 (Using the "Division Method" and Factoring)

These are left for the reader. §

PART E: "LONG-RUN" LIMITS OF ALGEBRAIC FUNCTIONS

"DTS" can be applied **carefully** to **some "long-run"** limits of general algebraic functions and beyond. (See Footnote 4 on dominant terms and Footnotes 5 and 6 on pitfalls.)

Example 18 (Using "DTS" to Evaluate "Long-Run" Limits of an Algebraic Function)

$$\lim_{x \to \infty} \left(5x^{7/2} - 2x^3 + x^{1/4} + 1 + x^{-2}\right) = \lim_{x \to \infty} 5x^{7/2}$$
$$= \infty$$

• This is because $5x^{7/2}$ is the **dominant term** as $x \to \infty$. (Think of 1 as x^0, even though 0^0 is controversial.)

$\lim_{x \to -\infty} \left(5x^{7/2} - 2x^3 + x^{1/4} + 1 + x^{-2}\right)$ does not exist (DNE).

• This is because $5x^{7/2}$, also written as $5\left(\sqrt{x}\right)^7$, and $x^{1/4}$, also written as $\sqrt[4]{x}$, are **not real** if $x < 0$. §

"Dominant Term Substitution (DTS)" Short Cut for Algebraic Functions

Let f be an **algebraic** function. When evaluating **"long-run"** limits of $f(x)$, **"DTS"** can be applied to:

- sums and differences of terms of the form cx^k $\left(c \in \mathbb{R},\ k \in \mathbb{Q}\right)$,
- numerators and denominators, and
- radicands and bases of powers,

if $f(x)$ is **real** in the desired "long-run" direction(s), and
if there are no **"ties"** as described in Warning 6 below.

The **"long-run"** limit(s) of the result will be the **same** as those of $f(x)$.

$f(x)$ and $g(x)$ are of the same order $\Leftrightarrow$ their "long-run" ratio in the desired direction is a nonzero real number, and thus **neither dominates** the other.

• x^2, $5x^2$, $\sqrt{x^4+1}$, and $\sqrt{3x^4 - x}$ are of the **same order** as $x \to \infty$.

WARNING 6: Avoid using "DTS" in the event of "ties." Avoid using "DTS" if, at **any stage** of the evaluation process, you encounter a **sum** or **difference** of expressions that are of the **same order**, and simplification cannot resolve this.

WARNING 7: Showing work. Although "DTS" is a useful tool in calculus, you may be expected to give **rigorous** solutions to, say, Examples 19 and 20 on exams.

Example 19 (Evaluating "Long-Run" Limits of an Algebraic Function)

Evaluate: a) $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$ and b) $\lim_{x\to-\infty}\left(x-\sqrt{x^2+x}\right)$.

§ Solution to a) (Rationalizing a Numerator)

Observe that $x^2+x\geq 0$ for all $x\geq 0$, so $\left(x-\sqrt{x^2+x}\right)$ is **real** as $x\to\infty$.

It is sufficient to observe that $x^2+x\geq 0$ for all **"sufficiently high"** x-values.

We re-express $x-\sqrt{x^2+x}$ as $\dfrac{x-\sqrt{x^2+x}}{1}$ and **rationalize the numerator**.

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) \quad \left(\text{Indeterminate Limit Form } \infty-\infty\right)$$

$$=\lim_{x\to\infty}\left[\frac{\left(x-\sqrt{x^2+x}\right)}{1}\cdot\frac{\left(x+\sqrt{x^2+x}\right)}{\left(x+\sqrt{x^2+x}\right)}\right] \quad \left(\text{Assume } x>0.\right)$$

$$=\lim_{x\to\infty}\frac{x^2-\left(x^2+x\right)}{x+\sqrt{x^2+x}}$$

(**WARNING 8 :** Use **grouping symbols** when subtracting more than one term.)

$$=\lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$$

($\sqrt{x^2+x}$ is on the **order** of x, as is the entire denominator. Now apply the "Division Method" by dividing the numerator and the denominator by x.)

$$=\lim_{x\to\infty}\frac{\dfrac{-x}{x}}{\dfrac{x}{x}+\dfrac{\sqrt{x^2+x}}{x}}$$

$$= \lim_{x\to\infty} \frac{-1}{1+\sqrt{\dfrac{x^2+x}{x^2}}}$$

$$\left(\sqrt{x^2} = |x| = x, \text{ since } x > 0.\right)$$

$$= \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+\underbrace{\dfrac{1}{x}}_{\to 0}}}$$

$$= -\frac{1}{2}$$

§

§ Solution to b) (Rationalizing a Numerator)

Observe that $x^2 + x \geq 0$ for all $x \leq -1$, so $\left(x - \sqrt{x^2+x}\right)$ is **real** as $x \to -\infty$.

It is sufficient to observe that $x^2 + x \geq 0$ for all **"sufficiently low"** x-values. We assume $x \leq -1$, and then only the last few steps effectively differ from our solution to a).

$$\lim_{x\to-\infty} \left(x - \sqrt{x^2+x}\right)$$

$$\left[\text{This turns out to be: } \left(\text{Limit Form } -\infty - \infty\right) \Rightarrow -\infty.\right]$$

$$= \lim_{x\to-\infty} \left[\frac{\left(x - \sqrt{x^2+x}\right)}{1} \cdot \frac{\left(x + \sqrt{x^2+x}\right)}{\left(x + \sqrt{x^2+x}\right)}\right]$$

$$= \lim_{x\to-\infty} \frac{x^2 - \left(x^2+x\right)}{x + \sqrt{x^2+x}}$$

$$= \lim_{x\to-\infty} \frac{-x}{x + \sqrt{x^2+x}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{-x}{x}}{\dfrac{x}{x} + \dfrac{\sqrt{x^2+x}}{x}}$$

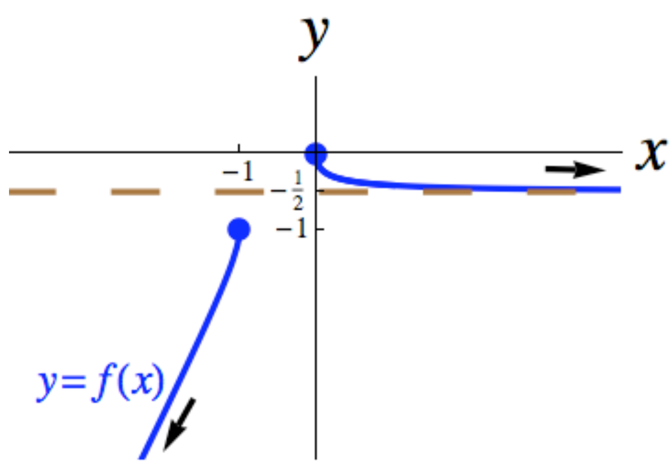
$$= \lim_{x \to -\infty} \frac{-1}{1 - \sqrt{\frac{x^2 + x}{x^2}}}$$

$$\left(\sqrt{x^2} = |x| = -x, \text{ since } x \leq -1, \text{ so } x = -\sqrt{x^2}. \right)$$

$$= \lim_{x \to -\infty} \frac{-1}{\underbrace{1 - \sqrt{\underbrace{1 + \underbrace{\frac{1}{x}}_{\to 0^-}}_{\to 1^-}}}_{\to 1^-}} \quad \left(\text{Limit Form } \frac{-1}{0^+}; \text{ see Section 2.4} \right)$$

$$= -\infty$$

The graph of $y = f(x)$, where $f(x) = x - \sqrt{x^2 + x}$, is below.
The "Twin (Long-Run) Limits" Property does **not** apply, because f is **not rational**.



§

§ ("DTS" Can Fail in the Event of "Ties")

If we try to apply **"DTS"** to a), we obtain:

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 + x} \right) \overbrace{=}^{?} \lim_{x \to \infty} \left(x - \sqrt{x^2} \right) = \lim_{x \to \infty} \left(x - |x| \right) = \lim_{x \to \infty} (x - x) = 0,$$

which is **incorrect**.

"DTS" fails here because **neither** x nor $-\sqrt{x^2 + x}$ is **dominant**; they are **both** on the order of x. §

Example 20 (Evaluating "Long-Run" Limits of an Algebraic Function)

Evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$, where $f(x)=\dfrac{\sqrt{x^{10}-5}}{(x+3)^2}$.

§ Solution 1 (Using the "DTS" Short Cut)

$x^{10}-5\geq 0$ on $\left(-\infty,-\sqrt[10]{5}\right]\cup\left[\sqrt[10]{5},\infty\right)$, and $\left[(x+3)^2=0 \iff x=-3\right]$,
so $f(x)$ is **real** for **"sufficiently high"** and **"sufficiently low"** values of x.

• In the radicand, $x^{10}-5$, x^{10} dominates -5.

• In the power-base, $x+3$, x dominates 3.

$$\lim_{x\to\infty}\frac{\sqrt{x^{10}-5}}{(x+3)^2}=\lim_{x\to\infty}\frac{\sqrt{x^{10}}}{(x)^2}$$

$$\left(\text{Now, } \sqrt{x^{10}}=\left|x^5\right|=x^5 \text{ for } x\geq 0.\right)$$

$$=\lim_{x\to\infty}\frac{x^5}{x^2}$$

$$=\lim_{x\to\infty}x^3$$

$$=\infty$$

$$\lim_{x\to-\infty}\frac{\sqrt{x^{10}-5}}{(x+3)^2}=\lim_{x\to-\infty}\frac{\sqrt{x^{10}}}{(x)^2}$$

$$\left(\text{Now, } \sqrt{x^{10}}=\left|x^5\right|=-x^5 \text{ for } x\leq 0.\right)$$

$$=\lim_{x\to-\infty}\frac{-x^5}{x^2}$$

$$=\lim_{x\to-\infty}\left(-x^3\right)$$

$$=\infty$$

§

§ *Solution 2 (Using the "Division Method")*

The denominator really has degree 2, so we will divide the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{\sqrt{x^{10}-5}}{(x+3)^2} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{x^{10}-5}}{x^2}}{\dfrac{(x+3)^2}{x^2}}$$

$$\left(\sqrt{x^4} = x^2 \text{ for } x \in \mathbb{R}.\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{\dfrac{x^{10}-5}{x^4}}}{\left(\dfrac{x+3}{x}\right)^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\overbrace{x^6}^{\to\infty} - \overbrace{\dfrac{5}{x^4}}^{\to 0}}}{\left(1 + \underbrace{\dfrac{3}{x}}_{\to 0}\right)^2}$$

$$\left(\text{Limit Form } \frac{\infty}{1}\right)$$

$$= \infty$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^{10}-5}}{(x+3)^2} = \lim_{x\to-\infty} \frac{\dfrac{\sqrt{x^{10}-5}}{x^2}}{\dfrac{(x+3)^2}{x^2}}$$

$$\left(\sqrt{x^4} = x^2 \text{ for } x \in \mathbb{R}.\right)$$

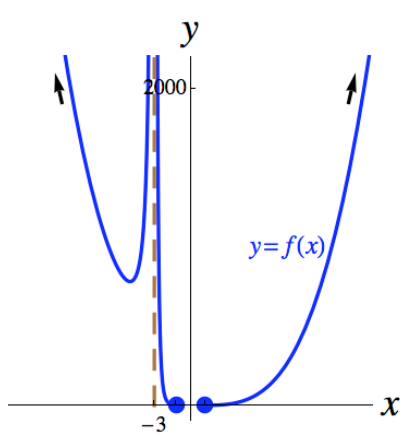
$$= \lim_{x\to-\infty} \frac{\sqrt{\dfrac{x^{10}-5}{x^4}}}{\left(\dfrac{x+3}{x}\right)^2}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\overbrace{x^6}^{\to\infty} - \overbrace{\dfrac{5}{x^4}}^{\to 0}}}{\left(1 + \underbrace{\dfrac{3}{x}}_{\to 0}\right)^2}$$

$$\left(\text{Limit Form } \frac{\infty}{1}\right)$$

$$= \infty$$

The graph of $y = f(x)$ is below.



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PART F: A "WORD PROBLEM"

Example 21 (Pond Problem)

A freshwater pond contains 1000 gallons of **pure water** at noon. Starting at noon, a **saltwater mixture** is poured into the pond at the **rate** of 2 gallons per minute. The mixture has a **salt concentration** of 0.3 pounds of salt per gallon. (Ignore issues such as evaporation.)

a) Find an expression for $C(t)$, the **salt concentration** in the pond t minutes after noon, where $t \geq 0$.

b) Find $\lim_{t \to \infty} C(t)$, and **interpret** the result. Discuss the **realism** of all this.

§ Solution to a)

Let $V(t)$ be the **volume** (in gallons) of the pond t minutes after noon $(t \geq 0)$.

• t minutes after noon, $2t$ gallons of the incoming mixture have been **poured** into the pond. The pond **started** with 1000 gallons of pure water, so the **total volume** in the pond is given by:

$$V(t) = 1000 + 2t \text{ (in gal).}$$

Let $S(t)$ be the weight (in pounds) of the **salt** in the pond t minutes after noon $(t \geq 0)$.

• All of the salt in the pond at any moment had been **poured** in, so:

$$S(t) = \left(\frac{0.3 \text{ lb}}{\cancel{\text{gal}}}\right)\left(2t \ \cancel{\text{gal}}\right) = 0.6t \text{ (in lb).}$$

Then,
$$C(t) = \frac{S(t)}{V(t)}$$
$$= \frac{0.6t}{1000 + 2t} \quad \left(\frac{\leftarrow \text{Multiply by 10, though 5 is better.}}{\leftarrow \text{Multiply by 10, though 5 is better.}}\right)$$
$$= \frac{6t}{10{,}000 + 20t}$$
$$= \frac{3t}{5000 + 10t} \quad \left(\text{in } \frac{\text{lb}}{\text{gal}}\right) \quad §$$

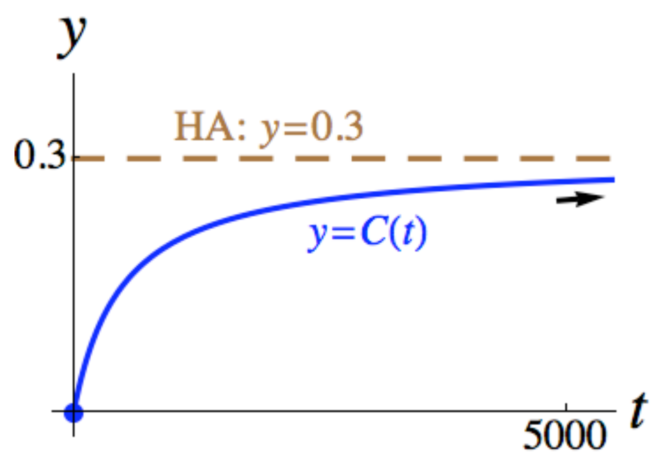
§ Solution to b)

We will use Case 1 in Part D to find the desired **"long-run"** limit.

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{3t}{5000+10t}$$
$$= \frac{3}{10}\,\frac{\text{lb}}{\text{gal}}, \text{ or } 0.3\,\frac{\text{lb}}{\text{gal}}$$

In the **"long run,"** the **salt concentration** in the pond approaches $0.3\,\frac{\text{lb}}{\text{gal}}$, the **same** as for the incoming mixture. However, this calculation assumes that the pond can approach **infinite** volume, which is unrealistic. Also, it assumes an **unlimited** supply of the incoming saltwater mixture.

The graph of $y = C(t)$ is below.



Think About It: Was the **initial volume** of the pond relevant in the "long-run" analysis? §

FOOTNOTES

1. Irrational exponents; Roots of negative real numbers. It is true that $x^k \to \infty$ and $\lim_{x\to\infty} \frac{c}{x^k} = 0$ $\left(c \in \mathbb{R},\ k \in \mathbb{R}^+\right)$. But what if k is (positive and) irrational $\left(k \in \mathbb{R}^+ \setminus \mathbb{Q}\right)$? For example, if $k = \pi$, then how do we define something like 2^π when $x = 2$? Remember that $\pi = 3.14159\ldots$. Consider the corresponding sequence:

$$\begin{aligned} 2^3 &= 8 \\ 2^{3.1} &= 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.57419 \\ 2^{3.14} &= 2^{\frac{314}{100}} = 2^{\frac{157}{50}} = \sqrt[50]{2^{157}} \approx 8.81524 \\ &\vdots \end{aligned}$$

The limit of this sequence (as the number of digits of π approaches ∞) is taken to be 2^π. It turns out that $2^\pi \approx 8.82498$. However, defining $\left(-2\right)^\pi$ is more problematic. For example, $\left(-2\right)^{3.1} = \left(-2\right)^{\frac{31}{10}}$. We are looking for a 10th root of $\left(-2\right)^{31}$. From the Precalculus notes (Section 6.5), we know that $\left(-2\right)^{31}$ has ten distinct 10th roots in $\mathbb{C}$, the set of complex numbers, none of them real. Refer to DeMoivre's Theorem for the complex roots of a complex number. See The Math Forum @ Drexel: Ask Dr. Math, *Meaning of Irrational Exponents.*

2. $\left(\textbf{Limit Form } \dfrac{\mathbf{1}}{\textbf{DNE}}\right) \Rightarrow \textbf{0 or DNE}$**.** (The notation here is highly informal.)

The desired limit must be either 0 or nonexistent (DNE), not even in the sense of ∞ or $-\infty$. Otherwise, the denominator would have had to approach 0 in the latter cases or (if the desired limit were a nonzero real number L) the real reciprocal $1/L$; the Limit Form is contradicted.

- $\lim_{x\to\infty} \dfrac{1}{D(x)} = 0$, where $D(x) = \begin{cases} x, & \text{if } x \text{ is a rational number } \left(x \in \mathbb{Q}\right) \\ -x, & \text{if } x \text{ is an irrational number } \left(x \notin \mathbb{Q};\ \text{really, } x \in \mathbb{R} \setminus \mathbb{Q}\right) \end{cases}$.
- $\lim_{x\to\infty} \dfrac{1}{\sin x} = \lim_{x\to\infty} \csc x$ does not exist (DNE). (The "$=$" sign here is informal.)

In Example 10, we saw the Limit Form $\dfrac{2}{\text{DNE}}$ also yield "DNE."

3. Computer science and function growth. "Big O" notation is used in theoretical computer science to compare the growth of functions. The analysis of algorithms deals with the "long-run" efficiency of computer algorithms with respect to memory, time, and space requirements as, say, the input size approaches infinity.

4. Dominant terms. We say that x^d "dominates" x^n as $x \to \infty \Leftrightarrow d > n \left(d, n \in \mathbb{R} \right)$.

If $d > n$, the (absolute value of) x^d "explodes more dramatically" and makes the growth of the (absolute value) of x^n seem negligible by comparison as $x \to \infty$. More precisely,

$\lim_{x\to\infty} \frac{x^n}{x^d} = \lim_{x\to\infty} x^{n-d} = \lim_{x\to\infty} x^{-(d-n)} = \lim_{x\to\infty} \frac{1}{x^{d-n}} = 0 \Leftrightarrow d > n$, in which case $d - n > 0$; see Part B and Footnote 1.

• Also, $\lim_{x\to-\infty} \frac{x^n}{x^d} = 0 \Leftrightarrow$ ($d > n$, and x^n and x^d are real for all $x < 0$).

• This dominant term analysis can be extended to non-algebraic (or transcendental) expressions. For example, we will see the exponential expression e^x in Chapter 7. However, the identification of a dominant term in a "long-run" analysis may well depend on whether we are considering a limit as $x \to \infty$ or a limit as $x \to -\infty$, beyond the "DNE" issue. It turns out that $\lim_{x\to\infty} \frac{1}{e^x} = 0$, so e^x dominates 1 as $x \to \infty$. However, it also turns out that $\lim_{x\to-\infty} \frac{e^x}{1} = 0$, so 1 dominates e^x as $x \to -\infty$.

• In $\sin x + \cos x$, neither term dominates in the "long run." Any nonconstant polynomial will dominate either term, in either "long-run" direction.

5. A pitfall of "DTS." 2^x is another exponential expression we will see in Chapter 7.

$\lim_{x\to\infty} \frac{2^{x+3}}{2^x} = \lim_{x\to\infty} \frac{\cancel{2^x} \cdot 2^3}{\underbrace{\cancel{2^x}}_{(1)}} = 8$. If we try to apply "DTS" locally and replace $x+3$ with x in the exponent of 2^{x+3}, we obtain: $\lim_{x\to\infty} \frac{2^{x+3}}{2^x} \overset{?}{=} \lim_{x\to\infty} \frac{2^x}{2^x} = 1$, which is incorrect.

• It is risky to apply "DTS" to exponents, particularly when an exponential expression is a piece of a larger expression. (It is true, however, that $2^{x+3} \to \infty$ and $2^x \to \infty$ as $x \to \infty$.)

6. Another pitfall of "DTS." $\lim_{x\to\infty} \left[\sin x - \sin\left(x + \pi\right) \right] = \lim_{x\to\infty} \left[\sin x + \sin x \right]$ (See Note.) $=$ $\lim_{x\to\infty} 2\sin x$ does not exist (DNE). (The "$=$" signs here are informal.)

• Note: This is justified by the Sum Identity for the sine function, or by exploiting symmetry along the Unit Circle.

• If we try to apply "DTS" locally and replace $x + \pi$ with x in the argument of $\sin\left(x+\pi\right)$, we obtain: $\lim_{x\to\infty} \left[\sin x - \sin\left(x+\pi\right) \right] \overset{?}{=} \lim_{x\to\infty} \left[\sin x - \sin x \right] = 0$, which is incorrect.

• As we saw in Part E and Footnote 5, it is risky to apply "DTS" locally to pieces of $f(x)$, the expression we are finding a "long-run" limit for.

• As we see here and in Footnote 5, it is especially dangerous to apply "DTS" to the argument of a non-algebraic (or transcendental) function. (We did safely apply "DTS" to entire numerators and entire denominators of $f(x)$ in Part D on rational functions.)