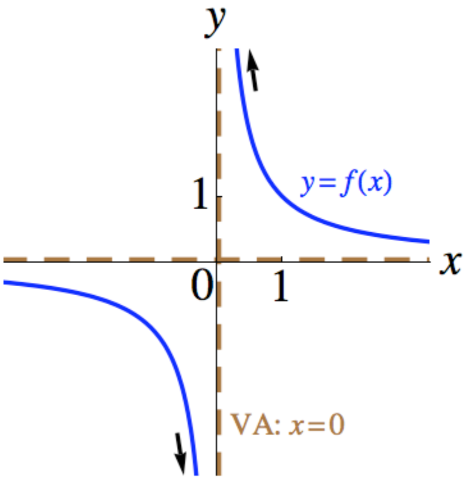
SECTION 2.4: LIMITS AND INFINITY II

PART A: "EXPLODING GRAPHS": VERTICAL ASYMPTOTES (VAs) and INFINITE LIMITS AT A POINT

In Section 2.1, we discussed $\lim_{x\to a} f(x)$, which is referred to as a limit at a point, the "point" being the real constant *a*. (*a* can be thought of as a point on the real number line.) This limit might not exist, although we can sometimes use ∞ and $-\infty$ to describe **why** the limit does not exist.

Example 1 (Reciprocal Function; Revisiting Example 1 from Section 2.3)

Let $f(x) = \frac{1}{x}$. Consider the graph of $y = f(x)$.



(Figure 2.4.a)

Let's experiment with a table:

x	-1	$-\frac{1}{10}$	$-\frac{1}{100}$	$\to 0^-$	$0^+ \leftarrow$	$\frac{1}{100}$	$\frac{1}{10}$	1
$f(x)=\frac{1}{x}$	-1	-10	-100	$\to -\infty$	$\infty \leftarrow$	100	10	1

Here, we can write: $\lim_{x\to 0^+} f(x) = \infty$.

In words: "the limit of $f(x)$ as *x* approaches 0 from the right is infinity." That is, as *x* approaches 0 from greater numbers, the function values $f(x)$ (generally) increase without bound.

Similarly, $\lim\limits_{x\to 0^-} f(x) = -\infty$.

> In words: "the limit of $f(x)$ as *x* approaches 0 from the left is negative infinity." That is, as *x* approaches 0 from lesser numbers, the function values $f(x)$ (generally) decrease without bound.

Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist (DNE), not even as ∞ or $-\infty$.
(See Footnote 1.)

As a consequence of $\lim\limits_{x\to 0^+} f(x) = \infty$ or $\lim\limits_{x\to 0^-} f(x) = -\infty$ (either alone would have been sufficient), the graph of $x = 0$ (i.e., the *y*-axis) is a <u>vertical asymptote (VA)</u> for the graph of $y = f(x)$.

<u>Using Infinite Limits at a Point to Find Vertical Asymptotes (VAs)</u>

The graph of $y = f(x)$ has a vertical asymptote (VA) at $x = a$ (for a real constant *a*) $\Leftrightarrow$

$$\left(\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty, \text{ or} \right.$$
$$\left. \lim_{x\to a^-} f(x) = \infty \text{ or } -\infty \right).$$

As *x* approaches *a* from the left or the right, $f(x)$ "explodes" in the sense that it approaches ∞ or $-\infty$.

- The graph of $y = f(x)$ for a function f can have any nonnegative integer number of VAs, or infinitely many (remember the graph of $y = \tan x$, for example).

- A polynomial graph has no VAs.

- The graph of a rational function has a nonnegative integer number of VAs.

<u>Note</u>: The graph of $y = f(x)$ for some function f cannot cross over a VA, but it can cross over an HA (see Example 5 in Section 2.3).

PART B : THE LIMIT FORMS $\frac{1}{0^+}$ AND $\frac{1}{0^-}$

Example 1 showed us that $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$, and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

More generally, it is true that, for functions *N* and *D*,

$$\frac{N(x)}{D(x)} \to \infty \text{ if } \left[N(x) \to 1, \text{ and } D(x) \to 0^+ \right].$$

This is true whether we are considering all of the indicated limits as $x \to \infty$, as $x \to -\infty$, or as $x \to a$ (or as $x \to a^+$, or as $x \to a^-$) for some real constant *a*. Also,

$$\frac{N(x)}{D(x)} \to -\infty \text{ if } \left[N(x) \to 1, \text{ and } D(x) \to 0^- \right].$$

<u>In summary</u>:

Limit Form $\frac{1}{0^+} \Rightarrow \infty$.

Limit Form $\frac{1}{0^-} \Rightarrow -\infty$.

<u>More generally; *c* is a real constant</u>:

Limit Form $\frac{c}{0^+} \Rightarrow \infty$, if $c > 0$.

Limit Form $\frac{c}{0^-} \Rightarrow -\infty$, if $c > 0$.

<u>The Sign Variants are as Expected</u>:

Limit Form $\frac{-1}{0^+} \Rightarrow -\infty$.

Limit Form $\frac{-1}{0^-} \Rightarrow \infty$.

<u>More generally; *c* is a real constant</u>:

Limit Form $\frac{c}{0^+} \Rightarrow -\infty$, if $c < 0$.

Limit Form $\frac{c}{0^-} \Rightarrow \infty$, if $c < 0$.

PART C: RATIONAL FUNCTIONS

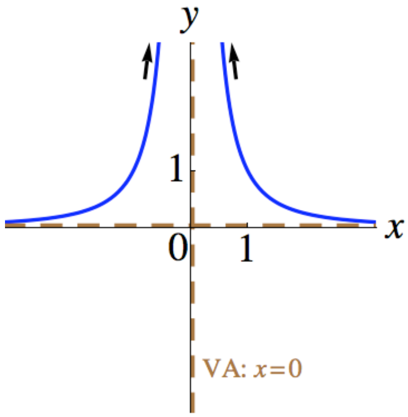
Example 2 (Contrast with Example 1)

Evaluate $\lim_{x\to 0^+} \frac{1}{x^2}$, $\lim_{x\to 0^-} \frac{1}{x^2}$, and $\lim_{x\to 0} \frac{1}{x^2}$.

Solution

Because $x^2 > 0$ for all nonzero values of *x*, all three give the Limit Form $\frac{1}{0^+} \Rightarrow \infty$.

Here is the graph of $y = \frac{1}{x^2}$:



(Figure 2.4.b)

Example 3

Let $f(x) = \frac{x+1}{x^2+4x}$. Evaluate $\lim_{x\to -4^+} f(x)$, $\lim_{x\to -4^-} f(x)$, and $\lim_{x\to -4} f(x)$.

Solution Method

$\lim_{x\to -4} (x+1) = -4+1 = -3$. This is also true for the one-sided variants.

$\lim_{x\to -4} (x^2+4x) = (-4)^2 + 4(-4) = 0$.

All three problems give the Limit Form $\frac{-3}{0}$. However, we need to know **how** the denominator approaches 0. Since it is easier to analyze signs of products than of sums (for example, do you automatically know the sum of *a* and *b* if $a > 0$ and $b < 0$?), we will factor the denominator.

Solution

$$\lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} \frac{x+1}{x^2+4x}$$

$$= \lim_{x\to -4^+} \frac{\overbrace{x+1}^{\to -3}}{\underbrace{x}_{\to -4}\underbrace{(x+4)}_{\to 0^+}} \quad \left(\text{Limit Form } \frac{-3}{0^-}\right)$$

Regarding the denominator: Remember that "negative times positive equals negative."

$$= \infty$$

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{x+1}{x^2+4x}$$

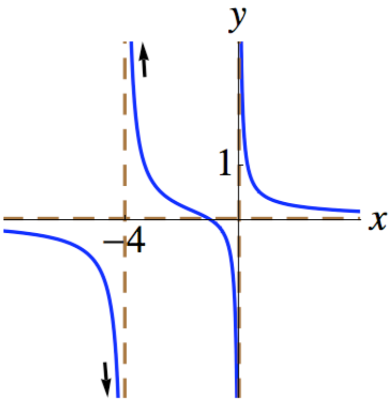
$$= \lim_{x\to -4^-} \frac{\overbrace{x+1}^{\to -3}}{\underbrace{x}_{\to -4}\underbrace{(x+4)}_{\to 0^-}} \quad \left(\text{Limit Form } \frac{-3}{0^+}\right)$$

Regarding the denominator: Remember that "negative times negative equals positive."

$$= -\infty$$

Therefore, the two-sided limit $\lim_{x\to -4} f(x)$ does not exist (DNE), not even as ∞ or $-\infty$. (See Footnote 1.)

Here is the graph of $y = \frac{x+1}{x^2+4x}$:



(Figure 2.4.c)

Finding VAs for Graphs of Rational Functions (Expressed in Simplified Form)

If:

- $f(x)$ is rational and written in the form $f(x)=\dfrac{N(x)}{D(x)}$,
- $N(x)$ and $D(x)$ are polynomials,
- $D(x)\neq 0$ (i.e., the zero polynomial), and
- $N(x)$ and $D(x)$ have no real zeros in common;

  i.e., they have no variable factors in common.

Then:

The graph of $y=f(x)$ has a VA at $x=a$

(and $\lim\limits_{x\to a^+} f(x)=\infty$ or $-\infty$, and $\lim\limits_{x\to a^-} f(x)=\infty$ or $-\infty$) $\Leftrightarrow$

a is a real zero of $D(x)$.

Example 4 (Revisiting Example 3)

Let $f(x)=\dfrac{x+1}{x^2+4x}$. Find the equations of the vertical asymptotes (VAs) of the graph of $y=f(x)$ in the *xy*-plane. Justify your answer using limits.

Solution Method

$\dfrac{x+1}{x^2+4x}=\dfrac{x+1}{x(x+4)}$. Observe that the numerator and the denominator have no variable factors (and no real zeros) in common. Therefore, the real zeros of the denominator correspond to the VAs of the graph. The VAs are at $x=0$ and $x=-4$.

To justify the VA at $x=0$ using limits, show one of the following:

- $\lim\limits_{x\to 0^+} f(x)=\infty$ or $-\infty$ (it turns out to be ∞), or
- $\lim\limits_{x\to 0^-} f(x)=\infty$ or $-\infty$ (it turns out to be $-\infty$).

To justify the VA at $x=-4$ using limits, show one of the following:

- $\lim\limits_{x\to -4^+} f(x)=\infty$ or $-\infty$ (it turns out to be ∞), or
- $\lim\limits_{x\to -4^-} f(x)=\infty$ or $-\infty$ (it turns out to be $-\infty$).

FOOTNOTES

1. **Infinite limits.** Different books handle infinite limits differently. Refer to Footnote 1 in Section 2.3. Sometimes, ∞ and $-\infty$ are treated as the same (we collapse them together and identify them with one another as ∞), and adjoining them to the usual real number system gives us the one-point compactification of the real numbers, also known as the real projective line. Then, the Limit Form $\frac{1}{0} \Rightarrow \infty$, and we can state that $\lim\limits_{x\to 0} \frac{1}{x} = \infty$, and $\lim\limits_{x\to -4} \frac{x+1}{x^2+4x} = \infty$. The limit still does not exist as a real number.

SECTION 2.5 : THE INDETERMINATE FORMS $\frac{0}{0}$ AND $\frac{\infty}{\infty}$

PART A: WHAT ARE INDETERMINATE FORMS?

Roughly speaking, an indeterminate form is a Limit Form that could yield a variety of real values; the limit might not exist. Further analysis is required to know what the limit is. The seven "classic" indeterminate forms are:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ \infty^0,\ 0^0,\ 1^\infty$$

Although, for example, the indeterminate form $\frac{\infty}{\infty}$ may be more precisely written as $\frac{\pm\infty}{\pm\infty}$, we rarely indicate signs like that, because further analysis beyond the sign issue is typically required, anyway. We simply write $\frac{\infty}{\infty}$ out of convenience.

Why is $\frac{0}{0}$ an Indeterminate Form?

Some examples demonstrate this:

$$\lim_{x\to 0}\frac{2x}{x}\quad\left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0} 2$$
$$= 2$$

Observe that $\frac{2x}{x} = 2$ for all nonzero real values of x, but the limit as $x \to 0$ does not require $\frac{2x}{x}$ to be defined for $x = 0$.

$$\lim_{x\to 0}\frac{-\pi x}{x}\quad\left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0}\left(-\pi\right)$$
$$= -\pi$$

More generally, $\lim_{x\to 0} \frac{cx}{x} = c$ for any real constant c.

$$\lim_{x\to 0} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0} \right) = \lim_{x\to 0} \frac{1}{x}, \text{ which does not exist (DNE).}$$

Note: The "=" sign is then technically inappropriate here, but we often leave it in, anyway.

Why is $\frac{\infty}{\infty}$ an Indeterminate Form?

Some examples demonstrate this:

If c is a nonzero real constant, then:

$$\begin{aligned} \lim_{x\to\infty} \frac{cx}{x} \quad \left(\text{Limit Form } \frac{\pm\infty}{\infty} \text{ or just } \frac{\infty}{\infty} \right) &= \lim_{x\to\infty} c \\ &= c \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2}{x} \quad \left(\text{Limit Form } \frac{\infty}{\infty} \right) &= \lim_{x\to\infty} x \\ &= \infty \end{aligned}$$

Why Isn't $\frac{1}{0}$ an Indeterminate Form?

As it stands, this Limit Form actually tells us a lot. The corresponding limit cannot exist as a real number; it could be ∞, $-\infty$, or "DNE." It is similar to an indeterminate form in that further analysis is required; we need to know **how** the denominator is approaching 0.

PART B : HOW DO WE RESOLVE THE $\frac{0}{0}$ FORM? ALSO, GRAPHS OF RATIONAL FUNCTIONS

When analyzing a rational function with rule $f(x) = \dfrac{N(x)}{D(x)}$, the Factor Theorem from Precalculus can help us factor the polynomials $N(x)$ and $D(x)$.

Factor Theorem

If $D(x)$ is a nonzero polynomial and a is a real constant, then:
a is a zero of $D(x) \Leftrightarrow (x-a)$ is a factor of $D(x)$.

The same goes for $N(x)$.

When considering $\lim\limits_{x\to a} f(x)$, if $(x-a)$ is a "common" factor of both $N(x)$ and $D(x)$, then we are dealing with a $\frac{0}{0}$ Limit Form, and simplifying $f(x)$ leads to factoring and canceling / dividing pairs of $(x-a)$ factors. We then investigate the simplified form.

Example 1 (Factoring and Canceling / Dividing)

Assume $f(x)=\dfrac{x^2-1}{x^2-x}$. Evaluate: a) $\lim\limits_{x\to 1} f(x)$, and b) $\lim\limits_{x\to 0} f(x)$.

Solution to a)

$$\begin{aligned}
\lim_{x\to 1} f(x) &= \lim_{x\to 1}\frac{x^2-1}{x^2-x} \quad \left(\text{Limit Form } \frac{0}{0};\ (x-1) \text{ is a common factor}\right)\\
&= \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}\\
&= \lim_{x\to 1}\frac{x+1}{x}\\
&= \frac{(1)+1}{(1)} \quad \left(\underline{\text{Warning 1}}\text{: Don't remove } \lim_{x\to 1} \text{ until this substitution and evaluation phase.}\right)\\
&= 2
\end{aligned}$$

Commentary on a)

- We see that: $\dfrac{x^2-1}{x^2-x}=\dfrac{x+1}{x}\ (x\neq 1)$.

Therefore, $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2-x}=\lim\limits_{x\to 1}\dfrac{x+1}{x}$.

See Section 2.1, Part C on the "Ignore *a*" Theorems.

- $\lim\limits_{x\to 1} f(x)=2$, yet 1 is not in the domain of f.

As a consequence, the graph of f has a "hole" at the point $(1, 2)$.

- Why does the limit exist here? As $x\to 1$, the factor $(x-1)\to 0$.

When we simplify $f(x)$, we are able to cancel / divide out all of the $(x-1)$ factors in the denominator. The new denominator, x, no longer approaches 0.

Solution to b)

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x^2-1}{x^2-x} \quad \left(\text{Limit Form } \frac{-1}{0} \right) \\ &= \lim_{x\to 0} \frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}} \\ &= \lim_{x\to 0} \frac{x+1}{x} \end{aligned}$$

Observe: $\lim\limits_{x\to 0^+} \dfrac{x+1}{x} \left(\text{Limit Form } \dfrac{1}{0^+} \right) = \infty$, and

$\lim\limits_{x\to 0^-} \dfrac{x+1}{x} \left(\text{Limit Form } \dfrac{1}{0^-} \right) = -\infty$.

Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist (DNE), not even as ∞ or $-\infty$.

Commentary on b)

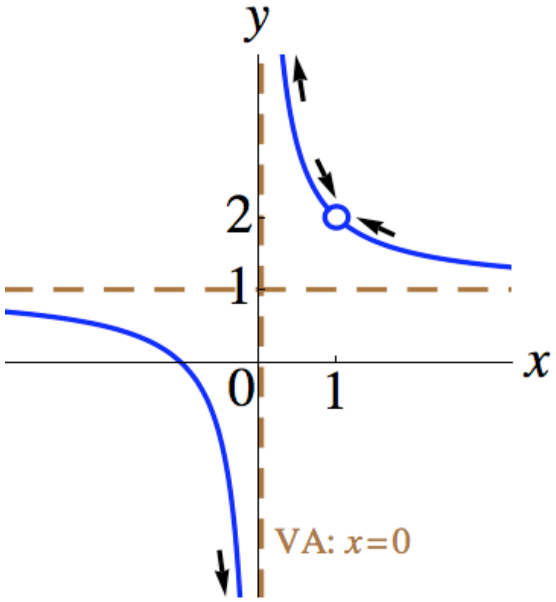
- Here, the cancellation / dividing out of the $(x-1)$ factors merely makes it more convenient when we analyze the limit as $x \to 0$.

- Why does the limit not exist here as a real number? As $x \to 0$, the new denominator of our simplified form, $\dfrac{x+1}{x}$, still approaches 0. We were unable to cancel / divide out all of the $(x-0)$, or x, factors in the denominator in the simplification process.

- Because of the infinite one-sided limits, the graph of f has a VA at $x = 0$ (the *y*-axis).

(See graph on next page.)

Here is the graph of $y = \dfrac{x^2 - 1}{x^2 - x}$:



(Figure 2.5.a)

The Graph of a Rational Function at a Point *a*

The graph of $y = f(x)$, where f is a rational function on its implied domain, can have **one** of the following at $x = a$, where a is a real constant:

1) the **point** $\left(a,\ f(a)\right)$, if f is defined at *a*.

2) a **VA**, if simplifying $f(x)$ yields the Limit Form $\dfrac{c}{0}$ as $x \to a$ for some nonzero real constant *c*. There are $(x - a)$ factors in the denominator that cannot be canceled out.
   - Then, $\lim\limits_{x \to a^+} f(x) = \infty$ or $-\infty$, and $\lim\limits_{x \to a^-} f(x) = \infty$ or $-\infty$.

3) a **hole** at the point $\left(a,\ \lim\limits_{x \to a} f(x)\right)$, if f is undefined at *a*, but $\lim\limits_{x \to a} f(x)$ exists. We get this if the original form of $f(x)$ gives us a Limit Form of $\dfrac{0}{0}$ as $x \to a$, but all $(x - a)$ factors in the denominator are canceled out by the ones in the numerator.

Example 2 (Rationalizing a Numerator or Denominator)

Evaluate: $\lim\limits_{x\to 0} \dfrac{\sqrt{9-x}-3}{x}$.

Solution Method

We will rationalize the numerator by multiplying the numerator and the denominator by the conjugate of the numerator; we are really multiplying by 1 in an effective way. The algebraic rule $(A-B)(A+B)=A^2-B^2$ allows us to eliminate the radicals in the numerator.

Note: $\sqrt{9-x}$ is defined as a real quantity on an open interval containing 0. Therefore, it is possible for the two-sided limit to exist, and the conjugate of the numerator is also appropriate to use.

Solution

$$\lim_{x\to 0} \frac{\sqrt{9-x}-3}{x} \quad \left(\text{Limit Form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \left[\frac{\left(\sqrt{9-x}-3\right)}{x} \cdot \frac{\left(\sqrt{9-x}+3\right)}{\left(\sqrt{9-x}+3\right)} \right]$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{9-x}\right)^2-(3)^2}{x\left(\sqrt{9-x}+3\right)}$$

(Warning 2: Write the entire denominator! It's not just *x*.)

$$= \lim_{x\to 0} \frac{(9-x)-9}{x\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x\to 0} \frac{\cancelto{-1}{-x}}{\cancel{x}\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x\to 0} \frac{-1}{\sqrt{9-x}+3}$$

Remember not to remove the $\lim\limits_{x\to 0}$ notation until the next step.

$$= \frac{-1}{\sqrt{9-(0)}+3}$$

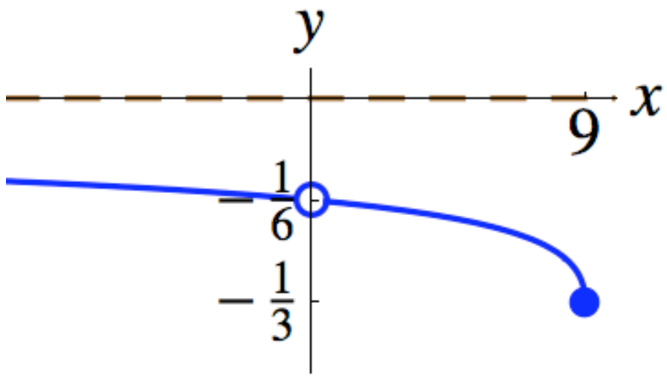
$$= -\frac{1}{6}$$

<u>Commentary</u>

- $\lim_{x \to 0} f(x) = -\frac{1}{6}$, yet 0 is not in the domain of f, where $f(x) = \frac{\sqrt{9-x}-3}{x}$. As a consequence, the graph of f has a "hole" at the point $\left(0, -\frac{1}{6}\right)$.

- <u>Why does the limit exist here?</u> We were able to cancel / divide out the *x*-factor in the denominator. The new denominator no longer approaches 0 as $x \to 0$.

Here is the graph of $y = \frac{\sqrt{9-x}-3}{x}$:



(Figure 2.5.b)

<u>L'Hôpital's Rule</u> will be discussed in a later chapter.

PART C : HOW DO WE RESOLVE THE $\frac{\infty}{\infty}$ FORM?

Example 3 (Rewriting as a 0/0 Limit Form)

$$\begin{aligned}\lim_{x\to 0^+} \frac{1/x}{1/x^2} \quad \left(\text{Limit Form } \frac{\infty}{\infty}\right) &= \lim_{x\to 0^+} \left(\frac{1}{x}\cdot x^2\right)\\ &= \lim_{x\to 0^+} \frac{x^2}{x} \quad \left(\text{Limit Form } \frac{0}{0}\right)\\ &= \lim_{x\to 0^+} x\\ &= 0\end{aligned}$$

Methods from Section 2.3

See Examples 13-16 in Section 2.3, Part H and Example 19 in Part I.

L'Hôpital's Rule will be discussed in a later chapter.

SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

PART A: EXAMPLES

We will formally state the Squeeze (Sandwich) Theorem later.

Example 1 below is one of many basic examples in which we use the Squeeze (Sandwich) Theorem to prove that the limit of a function is 0, where the function includes a trig function.

Example 1

Prove that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.

Idea

It makes sense that (something approaching 0) times (something bounded between two values) will approach 0.

Solution

Show how $\cos\left(\frac{1}{x}\right)$ is bounded.

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1, \text{ if } x \ne 0 \ \Rightarrow$$

Multiply all three parts by x^2 so that the middle part becomes the expression we want to take the limit of. If $x \ne 0$, then $x^2 > 0$, and so we have an appropriate step where none of the inequality symbols have to be reversed.

$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2, \text{ if } x \ne 0 \ \Rightarrow$$

As $x \to 0$, the left and right parts approach 0. Therefore, by the Squeeze (Sandwich) Theorem, the middle part is forced to approach 0, as well. The middle part is “squeezed” or “sandwiched” between the left and right parts, and it must approach the same limit that the other two are approaching.

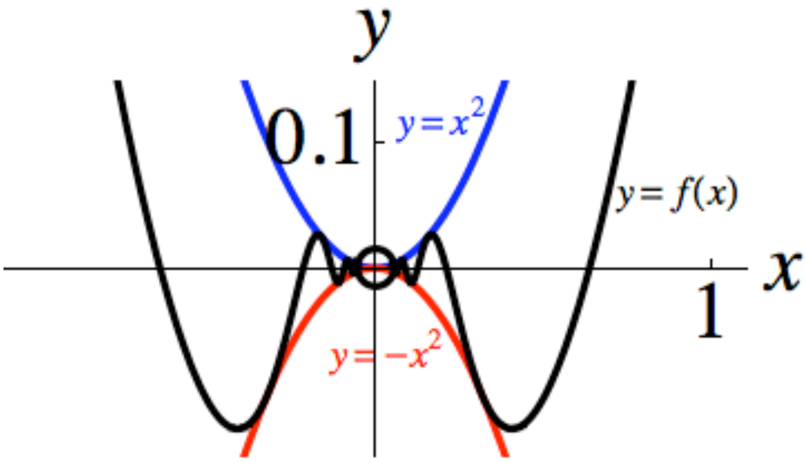
$$\lim_{x\to 0}\left(-x^2\right)=0\text{, and }\lim_{x\to 0}x^2=0\,.$$

$$\text{Therefore, }\lim_{x\to 0}x^2\cos\left(\frac{1}{x}\right)=0\,.$$

Shorthand:

$$\text{As } x\to 0,\quad \underbrace{-x^2}_{\to 0}\le \underbrace{x^2\cos\left(\frac{1}{x}\right)}_{\substack{\text{Therefore,}\\ \to 0}}\le \underbrace{x^2}_{\to 0}\qquad \left(x\ne 0\right)$$

Here is the graph of $f\left(x\right)=x^2\cos\left(\frac{1}{x}\right)$, together with the squeezing graphs of $y=-x^2$ and $y=x^2$:



(Figure 2.6.a)

Example 2

Find $\lim\limits_{x \to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$, and prove it.

Solution 1 (Using Absolute Value)

Show how $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$ is bounded.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le 1, \text{ if } x \ne 0 \Rightarrow$$

The problem with multiplying all three parts by x^3 is that, when $x < 0$, $x^3 < 0$. The inequality symbols would have to be reversed for negative values of *x*. One way to avoid this problem is to use absolute value.

$$\left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le 1, \text{ if } x \ne 0 \Rightarrow$$

Note: More precisely, we could write $0 \le \left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le 1$, if $x \ne 0$, but we take for granted that the absolute value of any real quantity is nonnegative.

We will now multiply both sides of the inequality by $\left|x^3\right|$.

We know $\left|x^3\right| > 0$, if $x \ne 0$.

$$\left|x^3\right| \left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le \left|x^3\right|, \text{ if } x \ne 0 \Rightarrow$$

“The product of absolute values equals the absolute value of the product.”

$$\left|x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le \left|x^3\right|, \text{ if } x \ne 0 \Rightarrow$$

If, say, $\left| a \right| \le 4$, then $-4 \le a \le 4$. Similarly,

$$-\left| x^3 \right| \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le \left| x^3 \right|, \text{ if } x \ne 0 \ \Rightarrow$$

Now, we apply the Squeeze (Sandwich) Theorem.

$$\lim_{x\to 0}\left(-\left| x^3 \right|\right) = 0, \text{ and } \lim_{x\to 0}\left| x^3 \right| = 0.$$

$$\text{Therefore, } \lim_{x\to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0.$$

Shorthand:

$$\text{As } x \to 0, \quad \underbrace{-\left| x^3 \right|}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{\left| x^3 \right|}_{\to 0} \quad \left(x \ne 0\right)$$

<u>Solution 2 (Analyze Right-Hand and Left-Hand Limits Separately)</u>

Analyze: $\lim\limits_{x\to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$.

We may assume $x > 0$ in our analysis, since we are considering a limit as $x \to 0^+$.

Show how $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$ is bounded.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le 1, \text{ if } x > 0 \ \Rightarrow$$

We will now multiply all three parts by x^3.
We know $x^3 > 0$, if $x > 0$.

$$-x^3 \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le x^3, \text{ if } x > 0 \ \Rightarrow$$

Now, we apply the Squeeze (Sandwich) Theorem.

$$\lim_{x\to 0^+}\left(-x^3\right)=0, \text{ and } \lim_{x\to 0^+} x^3=0.$$

$$\text{Therefore, } \lim_{x\to 0^+} x^3\sin\left(\frac{1}{\sqrt[3]{x}}\right)=0.$$

Shorthand:

$$\text{As } x\to 0^+, \quad \underbrace{-x^3}_{\to 0} \le \underbrace{x^3\sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore,}\\ \to 0}} \le \underbrace{x^3}_{\to 0} \quad \left(x>0\right)$$

Analyze: $\displaystyle\lim_{x\to 0^-} x^3\sin\left(\frac{1}{\sqrt[3]{x}}\right)$.

We may assume $x<0$ in our analysis, since we are considering a limit as $x\to 0^-$.

Show how $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$ is bounded.

$$-1\le\sin\left(\frac{1}{\sqrt[3]{x}}\right)\le 1, \text{ if } x<0 \ \Rightarrow$$

We will now multiply all three parts by x^3.
We know $x^3<0$, if $x<0$.
Therefore, we must reverse the ≤ inequality symbols.

$$-x^3\ge x^3\sin\left(\frac{1}{\sqrt[3]{x}}\right)\ge x^3, \text{ if } x<0 \ \Rightarrow$$

Reversing the compound inequality will make it easier to read:

$$x^3\le x^3\sin\left(\frac{1}{\sqrt[3]{x}}\right)\le -x^3, \text{ if } x<0 \ \Rightarrow$$

Now, we apply the Squeeze (Sandwich) Theorem.

$\lim_{x\to 0^-} x^3 = 0$, and $\lim_{x\to 0^-} \left(-x^3\right) = 0$.

Therefore, $\lim_{x\to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$.

Shorthand:

$$\text{As } x \to 0^-,\quad \underbrace{x^3}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore,}\\ \to 0}} \le \underbrace{-x^3}_{\to 0} \quad \left(x < 0\right)$$

Now, we apply the Squeeze (Sandwich) Theorem.

$\lim_{x\to 0^+} \left(-x^3\right) = 0$, and $\lim_{x\to 0^+} x^3 = 0$.

Therefore, $\lim_{x\to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$.

Shorthand:

$$\text{As } x \to 0^+,\quad \underbrace{-x^3}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore,}\\ \to 0}} \le \underbrace{x^3}_{\to 0} \quad \left(x > 0\right)$$

Now, $\lim_{x\to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$, and $\lim_{x\to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$, and so $\lim_{x\to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$.

Example 3 (Limits are Local)

The open x-interval $(-1, 1)$ contains 0. On this interval,

$$\text{As } x \to 0, \quad \underbrace{x^6}_{\to 0} \le \underbrace{x^4}_{\substack{\text{Therefore,} \\ \to 0}} \le \underbrace{x^2}_{\to 0} \quad (-1 < x < 1)$$

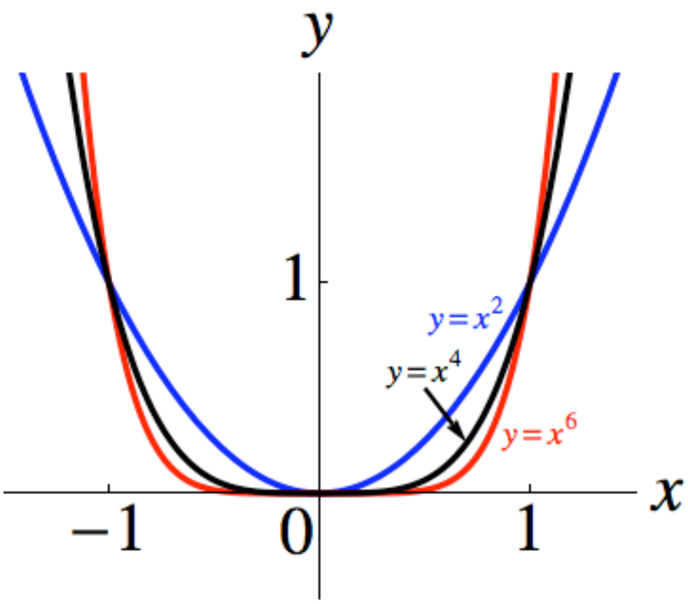
Note: Observe that $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{16} < \frac{1}{4}$.

We conclude: $\lim_{x \to 0} x^4 = 0$, which we knew already.

We do **not** need the compound inequality to hold true for **all** nonzero values of x. We only need it to hold true on some open x-interval containing 0 (possibly excluding 0) so that we can immediately discuss the two-sided limit $\lim_{x \to 0} x^4$.

This is because "Limits are Local."

As seen below, the graphs of $y = x^6$ and $y = x^2$ squeeze (from below and above, respectively) the graph of $y = x^4$ on the x-interval $(-1, 1)$. In fact, this is the case on the closed x-interval $[-1, 1]$.



(Figure 2.6.b)

PART B: THE SQUEEZE (SANDWICH) THEOREM

We will call B the "bottom" function and T the "top" function.

The Squeeze (Sandwich) Theorem

Let B and T be functions such that $B(x) \le f(x) \le T(x)$ on an open x-interval containing a, except possibly at a itself.

If $\lim\limits_{x \to a} B(x) = L$, and $\lim\limits_{x \to a} T(x) = L$, where L is a real constant, then $\lim\limits_{x \to a} f(x) = L$.

Variations for One-Sided Limits

- To show that $\lim\limits_{x \to a^+} f(x) = L$, we show that $\lim\limits_{x \to a^+} B(x) = L$, and $\lim\limits_{x \to a^+} T(x) = L$, and we require $B(x) \le f(x) \le T(x)$ on an x-interval of the form (a, c) for some real constant c, where $c > a$.

- To show that $\lim\limits_{x \to a^-} f(x) = L$, we show that $\lim\limits_{x \to a^-} B(x) = L$, and $\lim\limits_{x \to a^-} T(x) = L$, and we require $B(x) \le f(x) \le T(x)$ on an x-interval of the form (c, a) for some real constant c, where $c < a$.

PART C: MODIFICATIONS FOR "LONG-RUN" LIMITS

Example 4 (Revisiting Example 5 from Section 2.3)

Find $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$, and prove it.

Solution

Show how $\sin x$ is bounded.

$$-1 \le \sin x \le 1, \text{ for all real } x \Rightarrow$$

We may assume $x > 0$ in our analysis, since we are considering a limit as $x \to \infty$.

Divide all three parts by x^2 so that the middle part becomes the expression we want to take the limit of. We assume $x > 0$, so none of the inequality symbols have to be reversed.

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}, \text{ if } x > 0 \Rightarrow$$

Now, we apply the Squeeze (Sandwich) Theorem.

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0, \text{ and } \lim_{x\to\infty}\frac{1}{x} = 0.$$

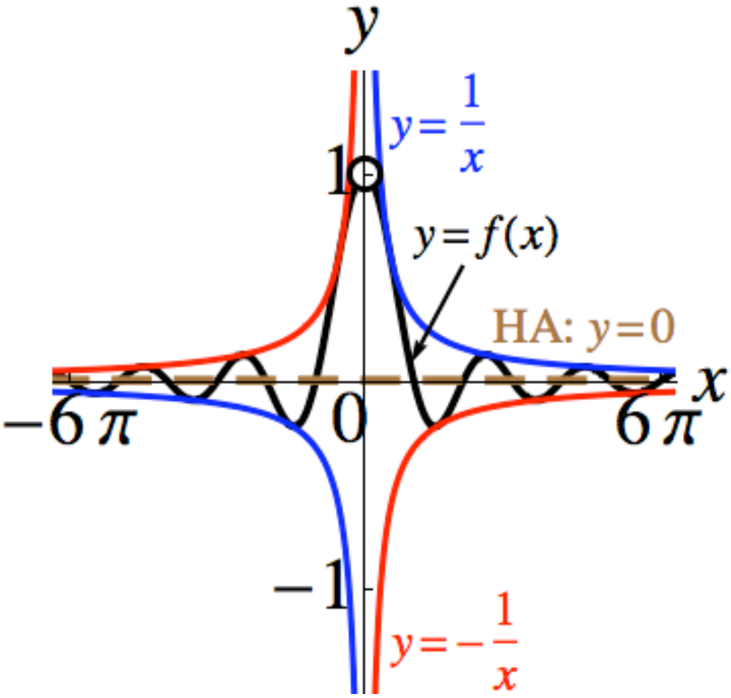
$$\text{Therefore, } \lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

Shorthand:

$$\text{As } x \to \infty, \quad \underbrace{-\frac{1}{x}}_{\to 0} \le \underbrace{\frac{\sin x}{x}}_{\substack{\text{Therefore,}\\ \to 0}} \le \underbrace{\frac{1}{x}}_{\to 0} \quad (x > 0)$$

(See the graph on the next page.)

Here is the graph of $f(x)=\frac{\sin x}{x}$, together with the squeezing graphs of $y=-\frac{1}{x}$ and $y=\frac{1}{x}$:



(Figure 2.6.c)

Modifications of the Squeeze (Sandwich) Theorem for "Long-Run" Limits

- To show that $\lim_{x\to\infty} f(x)=L$, we show that $\lim_{x\to\infty} B(x)=L$, and $\lim_{x\to\infty} T(x)=L$, and we require $B(x)\le f(x)\le T(x)$ on an x-interval of the form (c,∞) for some real constant c.

  Note: In Example 4, we used $c=0$. We need the compound inequality to hold "forever" after some point c.

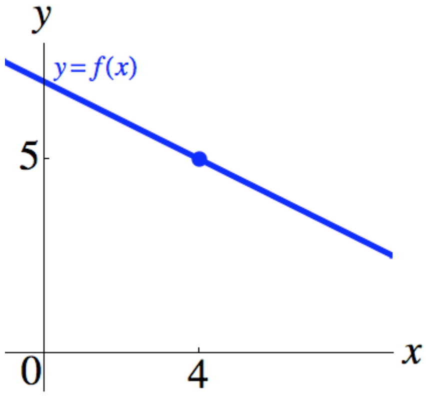
- To show that $\lim_{x\to-\infty} f(x)=L$, we show that $\lim_{x\to-\infty} B(x)=L$, and $\lim_{x\to-\infty} T(x)=L$, and we require $B(x)\le f(x)\le T(x)$ on an x-interval of the form $(-\infty,c)$ for some real constant c.

SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

PART A: AN EXAMPLE

We will formally prove that: $\lim_{x \to 4} \left(7 - \frac{1}{2}x \right) = 5$.

The statement is of the form $\lim_{x \to a} f(x) = L$, where $f(x) = 7 - \frac{1}{2}x$, $a = 4$, and $L = 5$.



(Figure 2.7.a)

The informal idea is that, as x "approaches" or "gets closer to" 4, $f(x)$ "approaches" or "gets closer to" 5. This informal approach represents a "dynamic" view of limits. (See Footnote 2 in Section 2.1.)

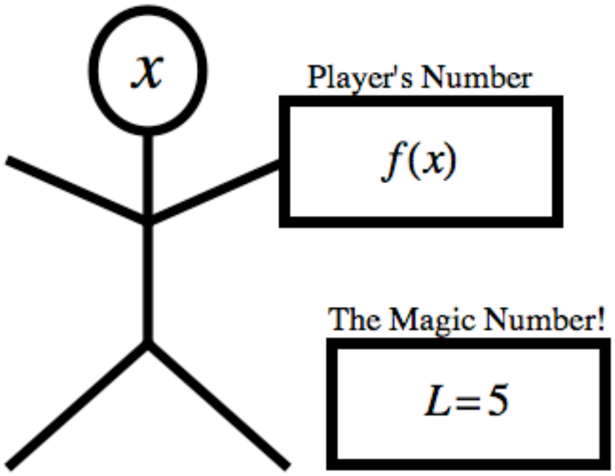
The precise approach takes on a more "static" view. The idea is that, if x is close to 4, then $f(x)$ is close to 5.

The Lottery Image (Intro)

Imagine a lottery in which every x-value in the domain of f represents a player. However, we disqualify $x = a$ (here, $x = 4$), because that person manages the lottery. Refer to the "Ignore *a* Theorems" in Section 2.1, Part C.

Each player is assigned a lottery number by the rule $f(x) = 7 - \frac{1}{2}x$.

The "exact" winning lottery number turns out to be 5.



(Figure 2.7.b)

<u>When Does Player x Win?</u>

This lottery is different from most in that more than one player can win, and it is sufficient for a player to be "close enough" to the exact winning number ($L = 5$ here) in order to win. In particular, we say that player x wins $(x \neq a)$ $\Leftrightarrow$ that player's lottery number, $f(x)$, is strictly within ε units of L, where $\varepsilon > 0$. The Greek letter ε ("epsilon") often represents a small positive quantity. Here, we can think of ε as a tolerance level that measures how liberal the lottery is in determining winners.

Symbolically:

$$\text{Player } x \text{ wins } (x \neq a) \;\Leftrightarrow\; L - \varepsilon < f(x) < L + \varepsilon$$

Subtract L from all three parts.

$$\Leftrightarrow \; -\varepsilon < f(x) - L < \varepsilon$$

Remember, for example:
$-1 < r < 1 \;\Leftrightarrow\; |r| < 1.$
Similarly:

$$\Leftrightarrow \; \left| f(x) - L \right| < \varepsilon$$

This makes sense, because $\left| f(x) - L \right|$ represents the distance between the lottery number $f(x)$ and the exact winning number L. Player x wins $\Leftrightarrow$ this distance is less than ε.

Where Do We Look For Winners?

We only care about players that are "close" to $x = a$ (here, $x = 4$).
We ignore *a* itself, and we say in this context that *a* is not "close" to itself.
In particular, we only care about players that are strictly between 0 and δ units of *a*, where $\delta > 0$. Like ε, the Greek letter δ ("delta") often represents a small positive quantity. Here, we can think of δ as the half-width of a "symmetric punctured interval" that is symmetric about $x = a$, though we delete *a* itself as a "puncture."

Symbolically:

$$\text{Player } x \text{ is "close" to } a \iff a - \delta < x < a + \delta \ \left(x \neq a \right)$$

Subtract *a* from all three parts.

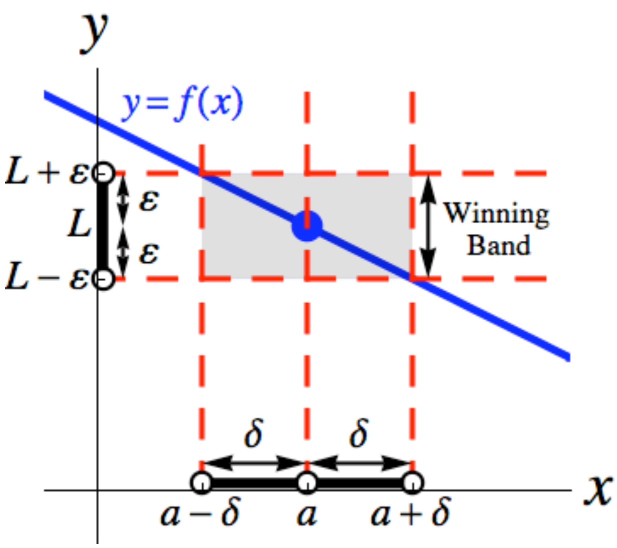
$$\iff -\delta < x - a < \delta \ \left(x \neq a \right)$$

$$\iff 0 < \left| x - a \right| < \delta$$

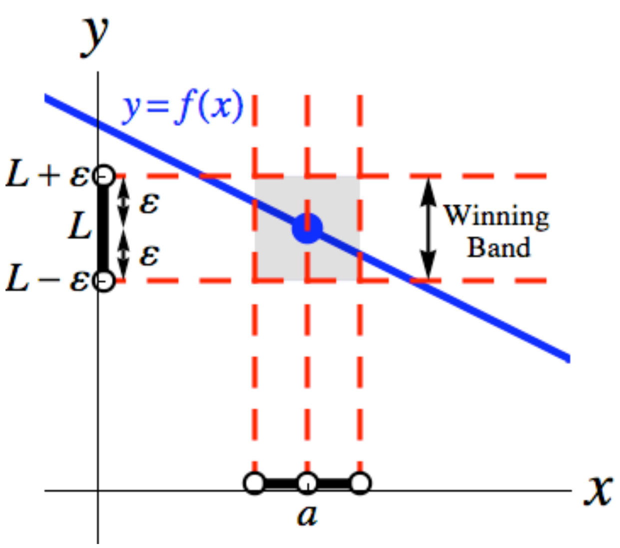
This makes sense, because $\left| x - a \right|$ represents the distance between Player *x* and *a*. Player *x* is "close" to *a* $\Leftrightarrow$ this distance is strictly between 0 and δ. (If the distance is 0, then we have $x = a$, which is disqualified.)

In the figure below, the value for δ is giving us a "symmetric punctured winning interval," in which all the players win; we exclude *a*.
In this sense, if *x* is close to *a*, then $f\left(x\right)$ is close to *L*.

Observe that any smaller positive value for δ could also have been chosen. (See Figure 2.7.d below.) The dashed lines are **not** asymptotes; they correspond to the boundaries of the open intervals and the puncture at $x = a$.



(Figure 2.7.c)



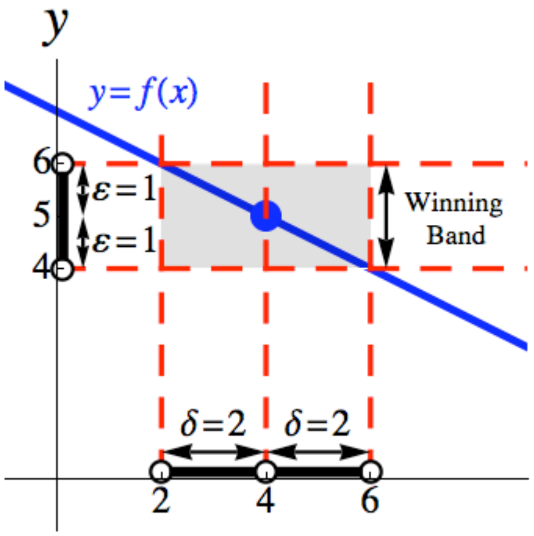
(Figure 2.7.d)

How Does the Static Approach to Limits Meld with the Dynamic Approach?

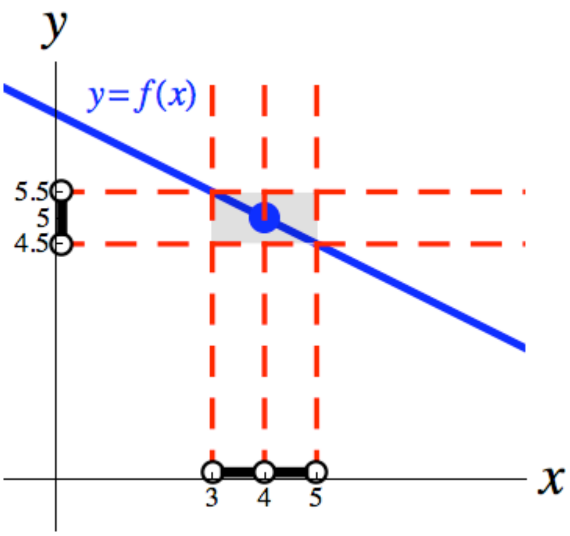
Why is $\lim_{x\to 4}\left(7-\frac{1}{2}x\right)=5$? Because, regardless of how small we make the tolerance level ε and how tight we make the lottery for the players, there is a value for δ for which the corresponding "symmetric punctured interval" is a winning one; in other words, the corresponding shaded region traps the graph of f on the symmetric punctured interval.

As $\varepsilon\to 0^+$, we can choose values for δ in such a way that the shaded region always traps the graph and zooms in, or collapses in, on the point $(4,5)$. This would have been the case even if that point alone had been deleted from the graph.

If $\varepsilon=1$, we can choose $\delta=2$. If $\varepsilon=0.5$, we can choose $\delta=1$.



(Figure 2.7.e)



(Figure 2.7.f)

More generally for this example, if ε is any positive real constant, we can choose $\delta=2\varepsilon$. Why is that?

- Graphically, we can exploit the fact that the slope of the line $y=7-\frac{1}{2}x$ is $-\frac{1}{2}$. Remember: $\text{slope}=\frac{\text{rise}}{\text{run}}$. Along the line, an *x*-run of 2 units corresponds to a *y*-drop of 1 unit.

- We will now demonstrate this rigorously.

PART B : THE PRECISE ε - δ DEFINITION OF A LIMIT AT A POINT

The Precise ε-δ Definition of a Limit at a Point
(Version 1)

For real constants *a* and *L*, and for a function f defined on an open interval containing *a*, possibly excluding *a* itself,

$\lim_{x \to a} f(x) = L \Leftrightarrow$ for every $\varepsilon > 0$, there exists a $\delta > 0$ such that,
if $0 < |x - a| < \delta$ (that is, if *x* is “close” to *a*),
then $|f(x) - L| < \varepsilon$ (that is, $f(x)$ is “close” to *L*).

Variation Using Interval Form

We can replace $0 < |x - a| < \delta$ with: *x* is in $(a - \delta, a + \delta)$, $x \neq a$.
We can replace $|f(x) - L| < \varepsilon$ with: $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$.

Related Notation

$a \in \mathbb{R}$	*a* is a member of the set of real numbers
$a, L \in \mathbb{R}$	*a* and *L* are members of the set of real numbers
$\forall$	for all / for every (universal quantifier)
$\exists$	there exists / there is (existential quantifier)
$\ni$	such that
$\Rightarrow$	implies

The Precise ε-δ Definition of a Limit at a Point
(Version 2: More Symbolic Version)

For $a, L \in \mathbb{R}$, and for a function f defined on an open interval containing *a*, possibly excluding *a* itself,

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni$$
$$\left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\right).$$

How Do We Formally Prove the Limit Statement in Part A?

Prove: $\lim_{x\to 4}\left(7-\frac{1}{2}x\right)=5$.

We have:

$$a=4,\ f\left(x\right)=7-\frac{1}{2}x, \text{ and } L=5.$$

We need to show:

$$\forall\varepsilon>0,\ \exists\delta>0 \ni \left(0<\left|x-a\right|<\delta \Rightarrow \left|f\left(x\right)-L\right|<\varepsilon\right);\text{ i.e.,}$$

$$\forall\varepsilon>0,\ \exists\delta>0 \ni \left(0<\left|x-4\right|<\delta \Rightarrow \left|\left(7-\frac{1}{2}x\right)-\left(5\right)\right|<\varepsilon\right).$$

Rewrite $\left|f\left(x\right)-L\right|$ in terms of $\left|x-a\right|$; here, $\left|x-4\right|$:

$$\left|f\left(x\right)-L\right|=\left|\left(7-\frac{1}{2}x\right)-\left(5\right)\right|$$

$$=\left|-\frac{1}{2}x+2\right|$$

Factor out $-\frac{1}{2}$, the coefficient of *x*.

To divide the $+2$ term by $-\frac{1}{2}$, we multiply it by -2 and obtain -4.

$$=\left|-\frac{1}{2}\left(x-4\right)\right|$$

$$=\left|-\frac{1}{2}\right|\left|x-4\right|$$

This is because, if *m* and *n* represent real quantities, then $\left|mn\right|=\left|m\right|\left|n\right|$.

$$=\frac{1}{2}\left|x-4\right|$$

We have: $\left|f\left(x\right)-L\right|=\frac{1}{2}\left|x-4\right|$; call this statement *.

Assuming ε is fixed $\left(\varepsilon > 0\right)$, find an appropriate value for δ.

Assume that ε is a fixed positive real constant.

We will find a value for δ that corresponds to a "symmetric punctured interval" about $a = 4$ that is "winning," meaning that all of the following statements are true for every player *x* in that interval:

$$\begin{aligned} \left| f\left(x\right) - L \right| < \varepsilon \quad &\Leftrightarrow \\ \frac{1}{2}\left| x - 4 \right| < \varepsilon \quad \left(\text{by } *\right) \quad &\Leftrightarrow \\ \left| x - 4 \right| < 2\varepsilon \end{aligned}$$

We choose $\delta = 2\varepsilon$. We will formally justify this choice in our verification step.

Observe that, since $\varepsilon > 0$, then our $\delta > 0$.

Verify that our choice for δ is appropriate.

We need to show that, given ε and our choice for δ $\left(\delta = 2\varepsilon\right)$, $0 < \left| x - a \right| < \delta \Rightarrow \left| f\left(x\right) - L \right| < \varepsilon$.

$$\begin{aligned} &0 < \left| x - a \right| < \delta \Rightarrow \\ &0 < \left| x - 4 \right| < \delta \Rightarrow \\ &0 < \left| x - 4 \right| < 2\varepsilon \Rightarrow \\ &0 < \frac{1}{2}\left| x - 4 \right| < \varepsilon \Rightarrow \\ &\left| f\left(x\right) - L \right| < \varepsilon \quad \left(\text{by } *\right) \end{aligned}$$

Note: It is true that: $0 < \left| f\left(x\right) - L \right| < \varepsilon$, but the first inequality $\left(0 < \left| f\left(x\right) - L \right|\right)$ does not help us.

Q.E.D.

("Quod erat demonstrandum" – Latin for "which was to be demonstrated." This is a formal end to a proof.)

PART C: ONE-SIDED LIMITS

The precise definition of $\lim_{x\to a} f(x) = L$ can be modified for left-hand and right-hand limits. The **only** changes are the *x*-intervals where we look for winners. (See red type.) Our *x*-intervals of interest will no longer be symmetric about *a*.

- Therefore, we will use interval form instead of absolute value notation when describing these *x*-intervals.
- Also, we will let δ represent the **entire** width of an *x*-interval, not just half the width of a punctured *x*-interval.

The Precise ε-δ Definition of a Left-Hand Limit at a Point

For $a, L \in \mathbf{R}$, and for a function f defined on an interval of the form (c, a), where *c* is a real constant and $c < a$,

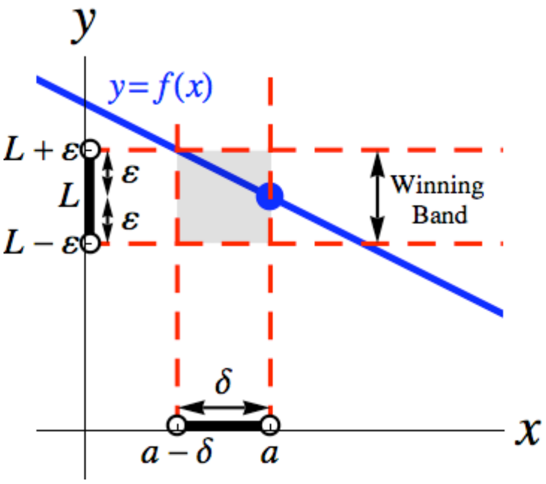
$$\lim_{x\to a^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni$$
$$\left[x \in (a-\delta, a) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

The Precise ε-δ Definition of a Right-Hand Limit at a Point

For $a, L \in \mathbf{R}$, and for a function f defined on an interval of the form (a, c), where *c* is a real constant and $c > a$,

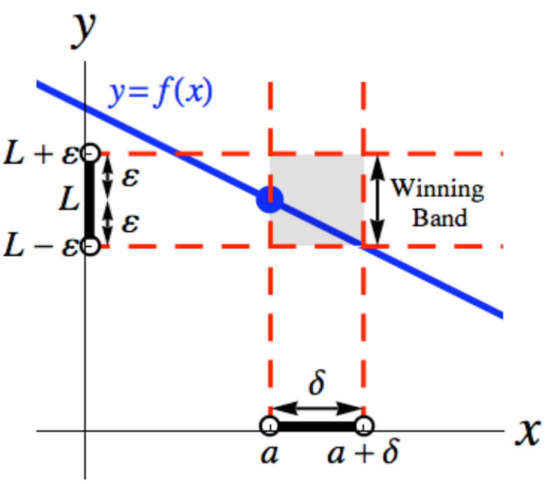
$$\lim_{x\to a^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni$$
$$\left[x \in (a, a+\delta) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

Left-Hand Limit



(Figure 2.7.g)

Right-Hand Limit



(Figure 2.7.h)

PART D: "LONG-RUN" LIMITS

The precise definition of $\lim\limits_{x\to a} f(x) = L$ can also be modified for "long-run" limits.

Again, the **only** changes are the *x*-intervals where we look for winners. (See red type.) Our *x*-intervals of interest will be unbounded.

- Therefore, we will use interval form instead of absolute value notation when describing these *x*-intervals.
- Also, instead of using δ, we will use *M* (think "Million") and *N* (think "very Negative") to denote "points of no return."

The Precise ε-M Definition of $\lim\limits_{x\to\infty} f(x) = L$

For $L \in \mathbb{R}$, and for a function f defined on an interval of the form (c, ∞), where *c* is a real constant,

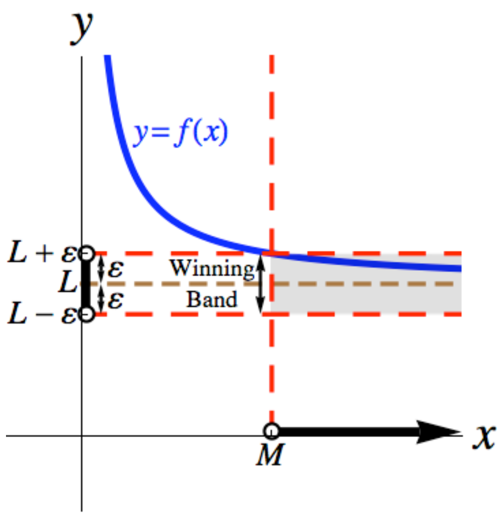
$$\lim_{x\to\infty} f(x) = L \iff \forall \varepsilon > 0,\ \exists M \in \mathbb{R} \ni$$
$$\left[x > M;\ \text{that is, } x \in (M, \infty) \right) \Rightarrow \left| f(x) - L \right| < \varepsilon \Big].$$

The Precise ε-N Definition of $\lim\limits_{x\to-\infty} f(x) = L$

For $L \in \mathbb{R}$, and for a function f defined on an interval of the form $(-\infty, c)$, where *c* is a real constant,

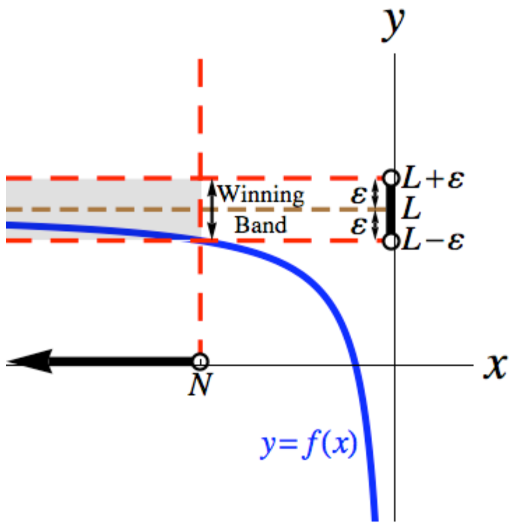
$$\lim_{x\to-\infty} f(x) = L \iff \forall \varepsilon > 0,\ \exists N \in \mathbb{R} \ni$$
$$\left[x < N;\ \text{that is, } x \in (-\infty, N) \right) \Rightarrow \left| f(x) - L \right| < \varepsilon \Big].$$

$\lim\limits_{x\to\infty} f(x) = L$; here, $f(x) = \dfrac{1}{x} + 2$



(Figure 2.7.i)

$\lim\limits_{x\to-\infty} f(x) = L$; here, $f(x) = \dfrac{1}{x} + 2$



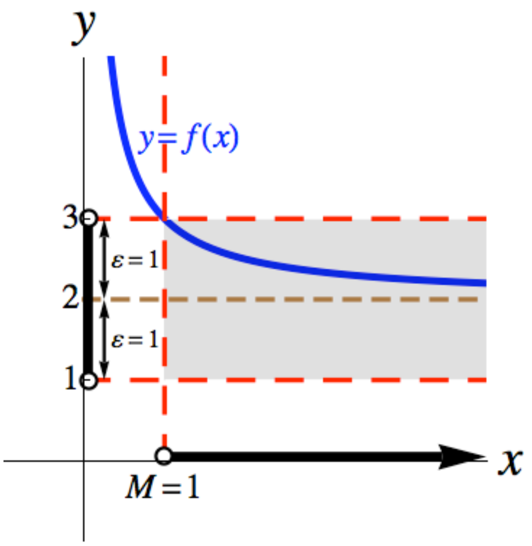
(Figure 2.7.j)

How Does the Static Approach to $\lim\limits_{x\to\infty} f(x) = L$ Meld with the Dynamic Approach?

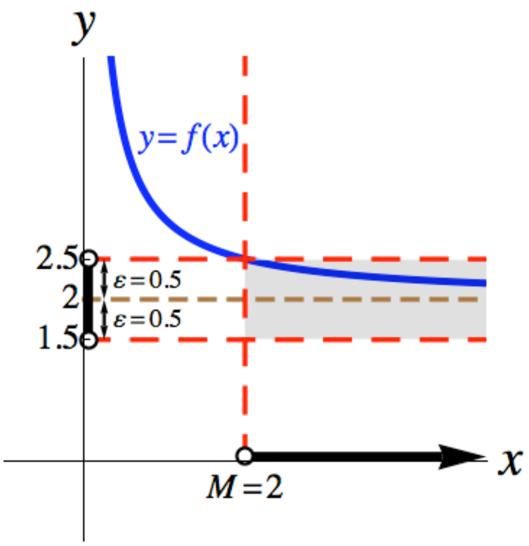
Why is $\lim\limits_{x\to\infty}\left(\frac{1}{x}+2\right)=2$? Because, regardless of how small we make the tolerance level ε and how tight we make the lottery for the players, there is a "point of no return" M after which all the players win; in other words, the corresponding shaded region traps the graph of f.

As $\varepsilon \to 0^{+}$, we can choose values for M in such a way that the shaded region always traps the graph and zooms in, or collapses in, on the HA $y = 2$.

If $\varepsilon = 1$, we can choose $M = 1$. If $\varepsilon = 0.5$, we can choose $M = 2$.



(Figure 2.7.k)



(Figure 2.7.letter l)

More generally for this example, if ε is any positive real constant, we can choose $M = \frac{1}{\varepsilon}$.

PART E: INFINITE LIMITS AT A POINT

Challenge to the reader:

Give precise "M-δ" and "N-δ" definitions of $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} f(x) = -\infty$, where a is a real constant, and the function f is defined on an open interval containing a, possibly excluding a itself.

SECTION 2.8: CONTINUITY

PART A: INTRO

Informally, a function f with domain $\mathbb{R}$ is continuous everywhere (i.e., continuous on $\mathbb{R}$) $\Leftrightarrow$ we can take a pencil and trace out the graph of f between any two distinct points on the graph without having to lift up our pencil.

We must make this idea more precise!

PART B: CONTINUITY AT A POINT

Assume that a is a real constant; it corresponds to a point on the real number line, usually as the x-axis.

Definition of Continuity at a Point, a

f is continuous at a $\Leftrightarrow$

1) $f(a)$ is defined,
2) $\lim_{x \to a} f(x)$ exists, and
3) $\lim_{x \to a} f(x) = f(a)$.

f is discontinuous at a $\Leftrightarrow$ f is not continuous at a.

Comments

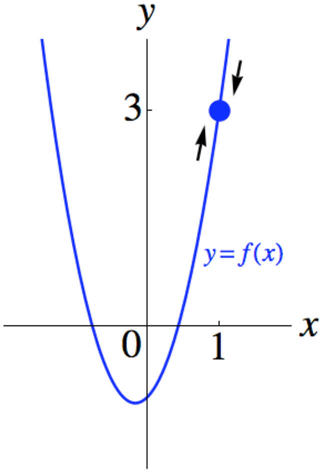
1) ensures that there is literally a point at a (usually, $x = a$).
2) constrains the behavior of f immediately around a.
3) then ensures "safe passage" through the point $\left(a,\ f(a)\right)$ on the graph of f.

Example 1 (Revisiting Example 1 in Section 2.1)

If $f(x) = 3x^2 + x - 1$, then f is continuous at 1. This is because:

1) $f(1) = 3$,
2) $\lim_{x \to 1} f(x) = 3$, and, therefore
3) $\lim_{x \to 1} f(x) = f(1)$.

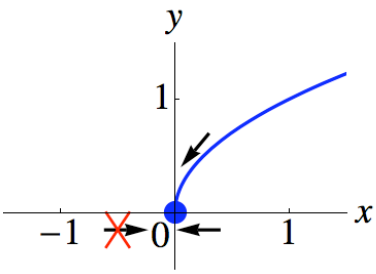
A graph can demonstrate this.



(Figure 2.8.a)

Example 2 (Revisiting Example 2 in Section 2.2)

Consider $f(x) = \sqrt{x}$.



(Figure 2.8.b)

- f is undefined (as a real number) at -1, so it is not continuous there.
- $f(0) = 0$, but $\lim_{x \to 0} \sqrt{x}$ does not exist (DNE), so f is not continuous at 0, either.

Many textbooks would not call -1 and 0 "discontinuities" of f, however.

PART C: CLASSIFYING DISCONTINUITIES

We will now consider cases where a function f is discontinuous at a, even though it is defined on an open interval containing a, possibly excluding a itself.

We will define removable, jump, and infinite discontinuities.
(See Footnote 1 for an example with discontinuities that are none of these.)

Removable Discontinuities

A function f has a removable discontinuity at a $\Leftrightarrow$

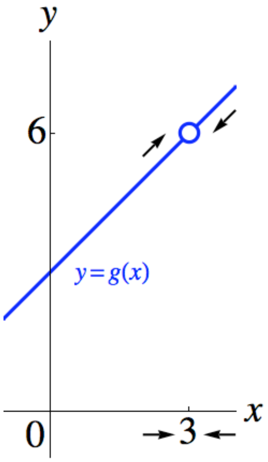
1) $\lim_{x\to a} f(x)$ exists, but

2) f is still discontinuous at a

Then, the graph of f has a "hole" at the point $\left(a, \lim_{x\to a} f(x)\right)$.

Example 3 (Revisiting Example 9 from Section 2.1)

Let $g(x) = x+3,\ (x \neq 3)$.



(Figure 2.8.c)

g has a removable discontinuity at 3, because:

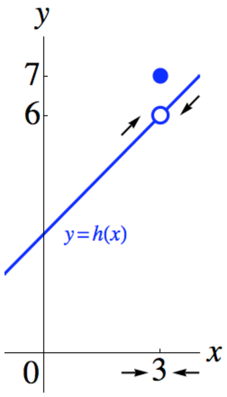
1) $\lim_{x\to 3} g(x) = 6$, but

2) g is still discontinuous at 3;
here, $g(3)$ is undefined.

The graph of g has a hole at the point $(3, 6)$.

Example 4 (Revisiting Example 10 from Section 2.1)

Let $h(x)=\begin{cases} x+3, & x\neq 3 \\ 7, & x=3 \end{cases}$



(Figure 2.8.d)

h has a removable discontinuity at 3, because:

1) $\lim\limits_{x\to 3} h(x)=6$, but

2) *h* is still discontinuous at 3;
here, $\lim\limits_{x\to 3} h(x)\neq h(3)$, because $h(3)=7$.

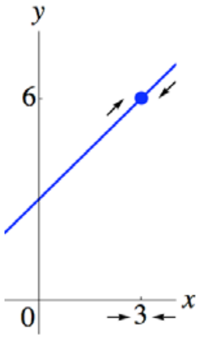
The graph of *h* also has a hole at the point $(3, 6)$.

Why are these discontinuities called removable?

The term “removable” is a bit of a misnomer here, since we have no business changing the function at hand.

The idea is that, in principle, a removable discontinuity at *a* can be removed by defining (or redefining) the function appropriately at *a*; then, the modified function will be continuous at *a*.

For example, if we were to define $g(3)=6$ in Example 3 and redefine $h(3)=6$ in Example 4, then we would remove the discontinuity at 3 in both situations. We would obtain this graph:



(Figure 2.8.e)

Jump Discontinuities

A function f has a jump discontinuity at a $\Leftrightarrow$

1) $\lim_{x\to a^-} f(x)$ exists, (call this limit L_1)

2) $\lim_{x\to a^+} f(x)$ exists, but (call this limit L_2)

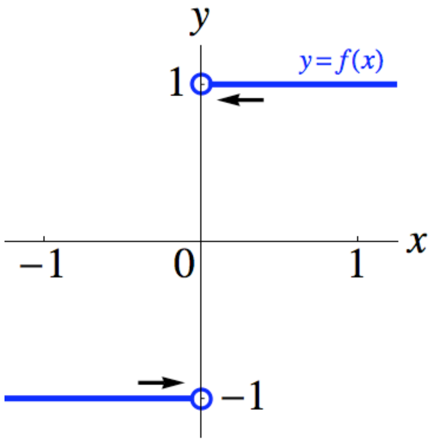
3) $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. ($L_1 \neq L_2$)

Therefore, $\lim_{x\to a} f(x)$ does not exist (DNE).

It is irrelevant how $f(a)$ is defined, or if it is at all.

Example 5 (Revisiting Example 13 from Section 2.1)

Let $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & \text{if } x > 0 \\ \dfrac{-x}{x} = -1, & \text{if } x < 0 \end{cases}$



(Figure 2.8.f)

$\lim_{x\to 0^-} f(x) = -1$, and

$\lim_{x\to 0^+} f(x) = 1$, but

$\lim_{x\to 0} f(x)$ does not exist (DNE), because $-1 \neq 1$.

f has a jump discontinuity at 0. We cannot remove this discontinuity by simply defining $f(0)$.

Infinite Discontinuities

A function f has an infinite discontinuity at $a \Leftrightarrow$

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty, \text{ or}$$

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

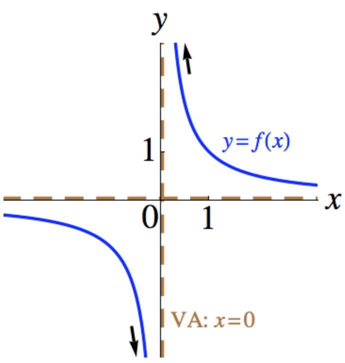
Then, the graph of f has the line $x = a$ as a VA.
It is irrelevant how $f(a)$ is defined, or if it is at all.

(See Footnote 2 for an alternate definition.)

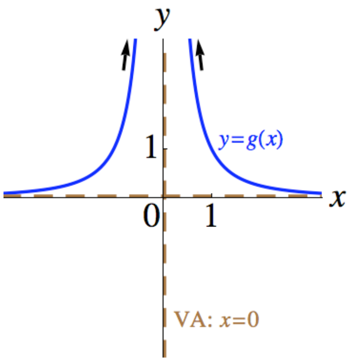
Example 6 (Revisiting Examples 1 and 2 from Section 2.4)

The three functions described below have infinite discontinuities at 0.
We will review $\ln x$ in a later chapter.

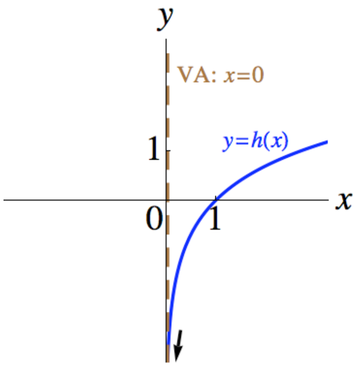
$f(x) = \dfrac{1}{x}$ $\qquad$ $g(x) = \dfrac{1}{x^2}$ $\qquad$ $h(x) = \ln x$



(Figure 2.8.g)



(Figure 2.8.h)



(Figure 2.8.i)

PART D: CONTINUITY ON AN OPEN INTERVAL

Definition of Continuity on an Open Interval

Assume that a and b are real constants such that $a < b$.

A function f is continuous on the open interval $\left(a, b\right) \Leftrightarrow$ f is continuous at every number (point) in $\left(a, b\right)$.

This extends to unbounded open intervals of the form $\left(a, \infty\right)$, $\left(-\infty, b\right)$, or $\left(-\infty, \infty\right)$.

In Example 6, all three functions are continuous on the interval $\left(0, \infty\right)$.
The first two functions are also continuous on the interval $\left(-\infty, 0\right)$.
We say that the continuity set (in interval form) of the first two functions is $\left(-\infty, 0\right) \cup \left(0, \infty\right)$, because that is the set of all real numbers at which those functions are continuous. (See Footnotes 3 and 5.)

PART E: CONTINUITY ON A CLOSED INTERVAL

Definition of Continuity on a Closed Interval (Version 1)

A function f is continuous on the closed interval $\left[a, b\right] \Leftrightarrow$

1) f is defined on $\left[a, b\right]$,
2) f is continuous on $\left(a, b\right)$,
3) $\lim_{x \to a^+} f\left(x\right) = f\left(a\right)$, and
4) $\lim_{x \to b^-} f\left(x\right) = f\left(b\right)$.

3) and 4) weaken the continuity requirements at the endpoints, *a* and *b*. Imagine taking limits as we "push outwards" towards the endpoints.

We will say that f is continuous from the right at *a*, and f is continuous from the left at *b*. (See Part F.)

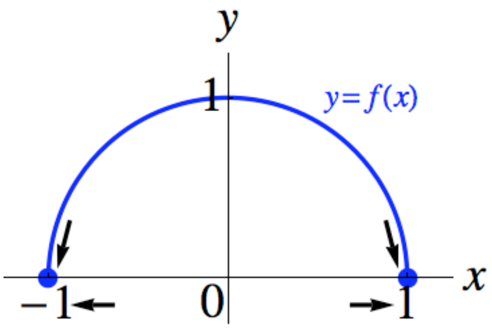
Example 7

Let $f(x)=\sqrt{1-x^2}$.

What is the graph of $y=f(x)$?

$$\begin{aligned} y &= \sqrt{1-x^2} && && \Leftrightarrow \\ y^2 &= 1-x^2 && (y \geq 0) && \Leftrightarrow \\ x^2+y^2 &= 1 && (y \geq 0) && \end{aligned}$$

The graph is the upper half of the unit circle centered at the origin, including the points $(-1,0)$ and $(1,0)$.



(Figure 2.8.j)

What are the "continuity intervals" here?

When giving what we will call the "continuity intervals" of a function, we give the most complete possible answer to the question, "Where is the function continuous?" We include brackets where appropriate. For the first two functions in Example 6, the continuity intervals are given by $(-\infty,0)\cup(0,\infty)$. We may get single-element singletons, also (see Footnote 4).

Here, the sole continuity interval is $[-1,1]$. Observe:

1) f is defined on $[-1,1]$,
2) f is continuous on $(-1,1)$,
3) $\lim_{x\to -1^+} f(x)=f(-1)$, and
4) $\lim_{x\to 1^-} f(x)=f(1)$.

Note: $f(-1)=0$, and $f(1)=0$, but they need not be equal.

Challenge to the Reader: Draw a graph where f is defined on $\left[a, b\right]$, and f is continuous on $\left(a, b\right)$, but f is not continuous on the closed interval $\left[a, b\right]$.

PART F: ONE-SIDED CONTINUITY and CONTINUITY ON HALF-OPEN, HALF-CLOSED INTERVALS

Definition of Right-Hand Continuity at a Point, *a*

f is continuous from the right at a $\Leftrightarrow$

1) $f\left(a\right)$ is defined,

2) $\lim\limits_{x\to a^+} f\left(x\right)$ exists, and

3) $\lim\limits_{x\to a^+} f\left(x\right) = f\left(a\right)$.

Definition of Left-Hand Continuity at a Point, *b*

f is continuous from the left at b $\Leftrightarrow$

1) $f\left(b\right)$ is defined,

2) $\lim\limits_{x\to b^-} f\left(x\right)$ exists, and

3) $\lim\limits_{x\to b^-} f\left(x\right) = f\left(b\right)$.

Continuity on Half-Open, Half-Closed Intervals

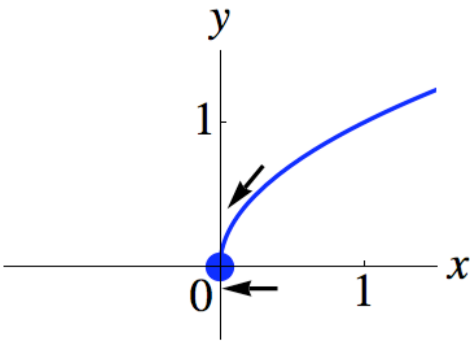
f is continuous on an interval of the form $\left[a, b\right)$ or $\left[a, \infty\right)$ $\Leftrightarrow$ it is continuous on $\left(a, b\right)$ or $\left(a, \infty\right)$, respectively; and it is continuous from the right at a.

f is continuous on an interval of the form $\left(a, b\right]$ or $\left(-\infty, b\right]$ $\Leftrightarrow$ it is continuous on $\left(a, b\right)$ or $\left(-\infty, b\right)$, respectively; and it is continuous from the left at b.

Example 8 (Revisiting Example 2)

Consider $f(x)=\sqrt{x}$. f is continuous on $(0,\infty)$, and it is continuous from the right at 0. The sole continuity interval for f is $[0,\infty)$.

Here is the graph of f:

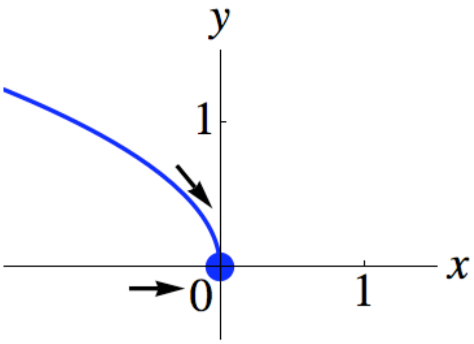


(Figure 2.8.k)

Example 9

Consider $f(x)=\sqrt{-x}$. f is continuous on $(-\infty,0)$, and it is continuous from the left at 0. The sole continuity interval for f is $(-\infty,0]$.

Here is the graph of f:



(Figure 2.8.letter l)

Definition of Continuity on a Closed Interval (Version 2)

A function f is continuous on the closed interval $[a,b]$ $\Leftrightarrow$

1) f is continuous on (a,b),

2) f is continuous from the right at a, and

3) f is continuous from the left at b.

(See Footnote 5 for continuity on sets in general.)

PART G: CONTINUITY THEOREMS AND EXAMPLES

Algebra of Continuity

If f and g are functions that are continuous at a, then so are the functions $f+g$, $f-g$, and fg.

- The function $\frac{f}{g}$ is, also, if $g(a) \neq 0$.
- The function f^n is, also, if n is a positive integer.
- The function $\sqrt[n]{f}$ is, also, if:
  - (n is an odd positive integer), or
  - (n is an even positive integer, and $f(a) > 0$.)

In a manner similar to the limit properties in Section 2.2, Part A, the theorem above, together with the fact that constant functions and the identity function (represented by $f(x) = x$) are everywhere continuous (on $\mathbb{R}$), justifies the following:

Continuity of Rational Functions

A rational function is continuous on its domain.

More precisely, the continuity interval(s) for a rational function f correspond(s) to the domain of f.

In particular, polynomial functions are everywhere continuous (on $\mathbb{R}$).

Note 1: We cannot quite say that this is true for all algebraic functions. (For a counterexample, see Footnote 4. Also see Footnote 5.)

Note 2: It is possible for a function to only be continuous at a single point (i.e., a singleton). (See Footnote 1.)

Example 10 (Revisiting Example 6)

If $f(x) = \frac{1}{x}$, then $\text{Dom}(f) = (-\infty, 0) \cup (0, \infty)$.

f is rational, so these are also the continuity intervals for f.

Example 11 (Revisiting Example 6 in Chapter 1)

Assuming $h(x)=\dfrac{\sqrt{x+3}}{x-10}$, then what are the continuity intervals of *h*?

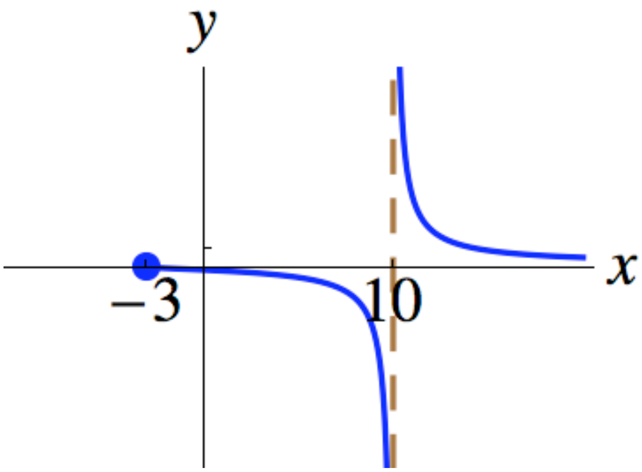
Solution

In Chapter 1, we found that $\mathrm{Dom}(h)=[-3,10)\cup(10,\infty)$.
These are also the continuity intervals.

By the Algebra of Continuity Theorems, we find that *h* is continuous on $(-3,10)\cup(10,\infty)$.

$\lim\limits_{x\to-3^{+}} h(x)=h(-3)$, because both sides equal 0.
Therefore, *h* is continuous from the right at -3, and the continuity intervals are: $[-3,10)\cup(10,\infty)$.

Here is the graph of *h*:



(Figure 2.8.m)

Continuity of Composite Functions

If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

(See Footnote 6.)

Continuity of Basic Trig Functions

The six basic trig functions (sine, cosine, tangent, cosecant, secant, and cotangent) are continuous on their domains.

More precisely, the continuity interval(s) for a basic trig function correspond(s) to its domain. Such intervals are all open.

Example 12

Assuming $h(x) = \sec\left(\frac{1}{x}\right)$, then where is h continuous?

Solution

Observe that $h(x) = (f \circ g)(x) = f(g(x))$, where the "inside" reciprocal function is given by $g(x) = \frac{1}{x}$, and the "outside" function f is given by $f(\theta) = \sec\theta$, where $\theta = \frac{1}{x}$.

g is continuous at all real numbers except 0 (i.e., $\forall x \neq 0$).

f is continuous on its domain.

$$\begin{aligned} \sec\theta \text{ is defined (as a real quantity)} &\Leftrightarrow \cos\theta \neq 0 \\ &\Leftrightarrow \theta \neq \frac{\pi}{2} + \pi n \quad \begin{pmatrix} n \text{ integer;} \\ \text{i.e., } n \in \mathbb{Z} \end{pmatrix} \end{aligned}$$

Now, let $\theta = \frac{1}{x}$, and we require $x \neq 0$.

We will find the continuity set for *h*. We require:

$$\theta \neq \frac{\pi}{2} + \pi n \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0$$

$$\frac{1}{x} \neq \frac{\pi}{2} + \pi n \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0$$

We can replace both sides of the inequality with their reciprocals, because we already exclude the case $x = 0$, and the right side is never 0.

$$x \neq \frac{1}{\frac{\pi}{2} + \pi n} \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0$$

$$x \neq \frac{1}{\left(\frac{\pi}{2} + \pi n\right)} \cdot \frac{2}{2} \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0$$

$$x \neq \frac{2}{\pi + 2\pi n} \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0$$

The continuity set is:

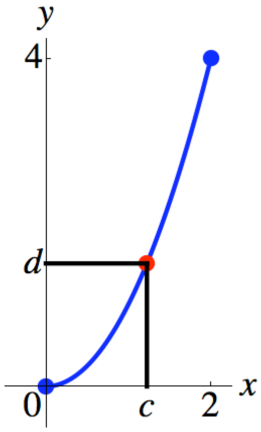
$$\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{2}{\pi + 2\pi n} \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0 \right\}, \text{ or}$$

$$\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{2}{\pi\left(2n + 1\right)} \ \left(n \in \mathbb{Z}\right), \text{ and } x \neq 0 \right\}.$$

PART H: THE INTERMEDIATE VALUE THEOREM (IVT)

Example 13

Examine the graph of $f(x) = x^2$ on the x-interval $[0, 2]$.



(Figure 2.8.n)

$f(0) = 0$, and $f(2) = 4$. Observe that every function value between 0 and 4 is taken on by f at some x-value in $[0, 2]$. This is guaranteed by the fact that f is continuous on $[0, 2]$. More generally …

The Intermediate Value Theorem (IVT): Informal Statement

If a function f is continuous on a closed interval $[a, b]$, then f takes on every real value between $f(a)$ and $f(b)$ on $[a, b]$.

The Intermediate Value Theorem (IVT): Precise Statement

Let $\min(f(a), f(b))$ be the smaller of $f(a)$ and $f(b)$; if they are equal, then we take their common value.

Let $\max(f(a), f(b))$ be the larger of $f(a)$ and $f(b)$; if they are equal, then we take their common value.

A function f is continuous on $[a, b]$ $\Rightarrow$

$$\forall d \in \left[\min(f(a), f(b)),\ \max(f(a), f(b))\right],\ \exists c \in [a, b] \ni f(c) = d.$$

Revisiting Example 13 (Verifying the Conclusion of the IVT)

Let $f(x) = x^2$. According to the IVT, for any value *d* in $[0, 4]$, there exists a value *c* in $[0, 2]$ such that $f(c) = d$. We try to find such a value for *c* in terms of *d*. Let *d* be any arbitrary value in $[0, 4]$.

$$
\begin{aligned}
f(c) = d \text{ and } c \in [0, 2] \quad &\Leftrightarrow \\
c^2 = d \text{ and } c \in [0, 2] \quad &\Leftrightarrow \\
c = \sqrt{d}\text{, a value in } [0, 2]&
\end{aligned}
$$

Observe:

$$
\begin{aligned}
0 \le d \le 4 \quad &\Leftrightarrow \\
0 \le \sqrt{d} \le 2&
\end{aligned}
$$

We do not write $c = \pm\sqrt{d}$, because either $d = 0$, or our value for *c* would fall outside of $[0, 2]$.

Verifying the Conclusion of the IVT: What to Write

f is continuous on $\mathbb{R}$; in particular, it is continuous on $[0, 2]$, so the IVT applies. $f(0) = 0$, $f(2) = 4$. Let $d \in [0, 4]$, and let $c = \sqrt{d}$.

(Particularly for harder problems, show the work leading to the choice for *c*).

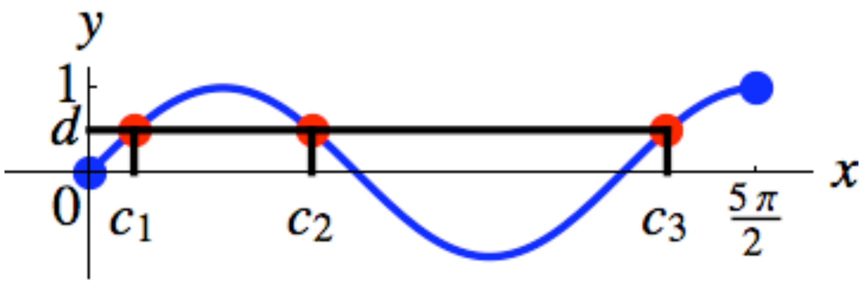
Observe: $0 \le d \le 4 \quad \Leftrightarrow \quad 0 \le \sqrt{d} \le 2$.

Then, $c \in [0, 2]$, and $f(c) = c^2 = \left(\sqrt{d}\right)^2 = d$.

Therefore, $\forall d \in [0, 4],\ \exists c \in [0, 2] \ni f(c) = d$.

Example 14

Examine the graph of $f(x) = \sin x$ on the x-interval $\left[0, \frac{5\pi}{2}\right]$.



(Figure 2.8.o)

$f(0) = 0$, and $f\left(\frac{5\pi}{2}\right) = 1$. Because f is continuous on $\left[0, \frac{5\pi}{2}\right]$, the IVT guarantees that every function value d between 0 and 1 is taken on by f at some x-value c in $\left[0, \frac{5\pi}{2}\right]$.

- Given an appropriate value for d, there may be more than one appropriate choice for c. The IVT does not forbid that.

- Also, there are function values outside of $[0, 1]$ that are taken on on the x-interval $\left[0, \frac{5\pi}{2}\right]$. The IVT does not forbid that, either.

PART I: THE BISECTION METHOD FOR APPROXIMATING ZEROS OF A FUNCTION

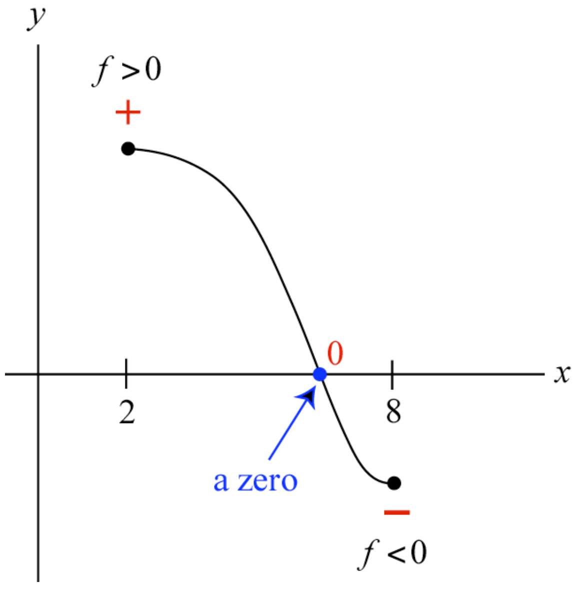
A dirty secret of mathematics is that we often have to use computer algorithms to help us approximate zeros of functions. While we do have (uglier) analogs of the Quadratic Formula for 3rd- and 4th-degree polynomial functions, it has actually been proven that there is no such formula for 5th- and higher-degree polynomial functions.

The Bisection Method for Approximating a Zero of a Continuous Function

Try to find *x*-values a_1 and b_1 such that $f\left(a_1\right)$ and $f\left(b_1\right)$ have opposite signs.

According to the IVT, there must be a zero of f somewhere between a_1 and b_1. If $a_1 < b_1$, then we can call $\left[a_1, b_1\right]$ our "search interval."

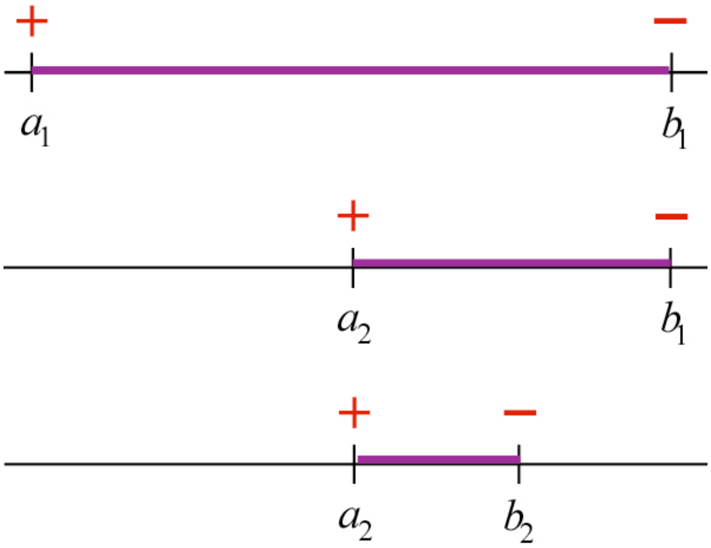
For example, our search interval below is $\left[2, 8\right]$.



(Figure 2.8.p)

If either $f\left(a_1\right)$ or $f\left(b_1\right)$ is 0, then we have a zero of f, and we can either stop or try to approximate another zero.

If neither is 0, then we can take the midpoint of the search interval and find out what sign f is there (in red below). We can then shrink the search interval (in purple below) and repeat the process.



(Figure 2.8.q)

We repeat the process until we either find a zero, or until the search interval is small enough so that we can be happy with simply taking the midpoint of the interval as our approximation.

A key drawback to the Bisection Method is that, unless we manage to find n distinct real zeros of an n^{th}-degree polynomial $f\left(x\right)$, we may need other techniques to be sure that we have found **all** of the real zeros, if we are looking for all of them.

FOOTNOTES

1. **A function with domain $\mathbb{R}$ that is only continuous at 0.** (Revisiting Footnote 1 in Section 2.1.) Consider the following function *f*.
$$f(x)=\begin{cases} 0, & \text{if } x \text{ is a rational value} \\ x, & \text{if } x \text{ is an irrational value} \end{cases}$$
f is continuous at 0, because $f(0)=0$, and we can use the Squeeze (Sandwich) Theorem to prove that $\lim_{x\to 0} f(x)=0$, also. The discontinuities at the nonzero real numbers are neither removable, jump, nor infinite.

2. **Infinite discontinuities: alternate definition.** Some textbooks only require that $\left|f(x)\right|\to\infty$ or $-\infty$ as $x\to a^{+}$ or $x\to a^{-}$. Such a definition would allow us to say that the following function has an infinite discontinuity at 0.
$$f(x)=\begin{cases} \dfrac{1}{x}, & \text{if } x \text{ is a rational value} \\ -\dfrac{1}{x}, & \text{if } x \text{ is an irrational value} \end{cases}$$
If we adopt the one-point compactification of the real numbers, also known as the real projective line, then we can state that $\lim_{x\to 0} f(x)=\infty$; see Footnote 1 in Section 2.4.

3. **Continuity sets and a nowhere continuous function.** See *Cardinality of the Set of Real Functions With a Given Continuity Set* by Jiaming Chen and Sam Smith. The 19th-century German mathematician Dirichlet came up with a nowhere continuous function, *D*:
$$D(x)=\begin{cases} 0, & \text{if } x \text{ is a rational value} \\ 1, & \text{if } x \text{ is an irrational value} \end{cases}$$

4. **An algebraic function that is not continuous on its domain.** Let $f(x)=\sqrt{x}+\sqrt{-x}$. $\text{Dom}(f)=\{0\}$, a singleton (a set consisting of a single element), but f is not continuous at 0, because $\lim_{x\to 0} f(x)$ does not exist (DNE). The same is true for $f(x)=\sqrt{-x^{2}}$.

5. **Continuity on a set.** A tricky thing to define! See "Continuity on a Set" by R. Bruce Crofoot, *The College Mathematics Journal*, Vol. 26, No. 1 (Jan. 1995) by the Mathematical Association of America (MAA). Also see Louis A. Talman, *The Teacher's Guide to Calculus* (free online). Talman suggests:

   Let S be a subset of the domain of a function f; i.e., $S \subseteq \text{Dom}(f)$. f is continuous on $S \Leftrightarrow$

   $\forall a \in S,\ \forall \varepsilon > 0,\ \exists \delta > 0 \ni \left[\left(x \in S \text{ and } \left|x - a\right| < \delta\right) \Rightarrow \left|f(x) - f(a)\right| < \varepsilon\right].$

   • The definition essentially states that, for every element a in the set of interest, its function value is arbitrarily close to the function values of nearby x-values in the set. Note that we use $f(a)$ instead of L, which we used to represent $\lim\limits_{x \to a} f(x)$, because we need

   $\lim\limits_{x \to a} f(x) = f(a)$ (or possibly some one-sided variation) in order to have continuity on S.

   • This definition covers / subsumes our definitions of continuity on open intervals; closed intervals; half-open, half-closed intervals; and unions (collections) thereof.

   • One possible criticism against this definition is that it implies that the functions described in Footnote 4 are, in fact, continuous on the singleton set $\{0\}$. Intuitively, that may seem odd. Perhaps we should require that f be defined on some interval of the form $[a, c)$ or $(c, a]$ for some real constant c.

   • Crofoot argues for the following definition: f is continuous on S if the restriction of f to S is continuous at each point of S. He acknowledges the use of one-sided continuity when dealing with closed intervals.

6. **Continuity and the limit properties in Section 2.2, Part A.** Assume a and K are real constants. If $\lim\limits_{x \to a} g(x) = K$, and f is continuous at K, then:

   $\lim\limits_{x \to a} (f \circ g)(x) = \lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right) = f(K)$. Basically, continuity allows f to commute with a limit operator: $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right)$. Think: "The limit of a (blank) is the (blank) of the limit." This relates to Property 5) The limit of a power and Property 6) The limit of a root in Section 2.2, Part A. For example, f could represent the squaring function.

7. **A function that is continuous at every irrational point and continuous at every rational point.** See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.27. Also see Tom Vogel, http://www.math.tamu.edu/~tvogel/gallery/node6.html. If x is rational, where $x = \dfrac{a}{b}$, where a and b are integers, $b > 0$, and the fraction is simplified, then let $f(x) = \dfrac{1}{b}$. If x is irrational, let $f(x) = 0$. Vogel calls this the "ruler function," appealing to the image of markings on a ruler. However, there does not exist a function that is continuous at every rational point and discontinuous at every irrational point.

8. **An everywhere continuous function that is nowhere monotonic (either increasing or decreasing).** See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.29. There is no open interval on which the function described there is either increasing or decreasing.