SECTION 2.4: LIMITS AND INFINITY II

LEARNING OBJECTIVES

• Understand infinite limits at a point and relate them to vertical asymptotes of graphs.
• Be able to evaluate infinite limits at a point, particularly for rational functions expressed in simplified form, and use a short cut to find vertical asymptotes of their graphs.
• Be able to use informal Limit Form notation to analyze infinite limits at a point.

PART A: VERTICAL ASYMPTOTES ("VA"s) and INFINITE LIMITS AT A POINT

In Section 2.1, we discussed finite limits at a point *a*.
We saw (two-sided) limits where $\lim_{x\to a} f(x) = L \ \left(a, L \in \mathbb{R}\right)$.

In Section 2.3, we discussed finite and infinite limits at $(\pm)$ infinity.
We saw examples where $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is $L \ \left(L \in \mathbb{R}\right),\ \infty,$ or $-\infty$.

Now, if $a \in \mathbb{R}$:

f has an infinite limit at a point *a* $\Leftrightarrow \lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is ∞ or $-\infty$.

• We read $\lim_{x\to a} f(x) = \infty$ as "the limit of $f(x)$ as x approaches a is infinity."

• See Footnote 1 for an alternate definition.

A vertical asymptote, which we will denote by "VA," is a vertical line that a graph **approaches in an "explosive" sense**. (See Section 2.1, Example 11.)

Using Infinite Limits at a Point to Find Vertical Asymptotes (VAs)

The graph of $y = f(x)$ has a **vertical asymptote (VA)** at $x = a \ \left(a \in \mathbb{R}\right)$

$\Leftrightarrow \lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is ∞ or $-\infty$.

• That is, the graph has a VA at $x = a \Leftrightarrow$ there is an infinite limit there from one or both sides.

The **number of VAs** the graph has can be a nonnegative integer (0, 1, 2, …), or it can have infinitely many VAs (consider $f(x) = \tan x$).

- If f is **rational**, then the graph **cannot have infinitely many VAs**.
- If f is **polynomial**, then the graph has **no VAs**.

Note: The graph of $y = f(x)$ **cannot cross over a VA**, but it can cross over an HA (see Section 2.3, Example 6).

Example 1 (The Graph of the Reciprocal Function has an "Odd VA"; Revisiting Section 2.3, Example 1)

Let $f(x) = \dfrac{1}{x}$. Evaluate $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^-} f(x)$, and show that the graph of $y = f(x)$ has a vertical asymptote (VA) at $x = 0$.

§ Solution

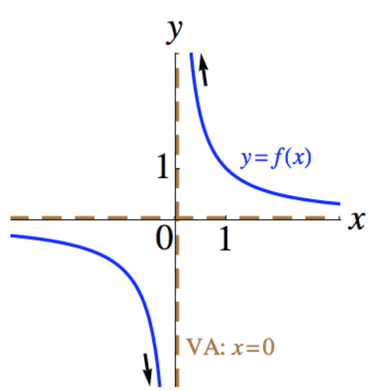
Let's use the **numerical / tabular approach**:

x	-1	$-\dfrac{1}{10}$	$-\dfrac{1}{100}$	$\to 0^-$	$0^+ \leftarrow$	$\dfrac{1}{100}$	$\dfrac{1}{10}$	1
$f(x) = \dfrac{1}{x}$	-1	-10	-100	$\to -\infty$	$\infty \leftarrow$	100	10	1

- Apparently, as x approaches 0 from the **right**, $f(x)$ **increases without bound**. That is, $\lim\limits_{x \to 0^+} f(x) = \infty$.

- Also, as x approaches 0 from the **left**, $f(x)$ **decreases without bound**. That is, $\lim\limits_{x \to 0^-} f(x) = -\infty$.

- Either limit statement implies that the graph of $y = f(x)$ below has a **vertical asymptote ("VA")** at $x = 0$, the y-axis.



- $\lim\limits_{x \to 0} f(x)$ does not exist (DNE).

(See Footnote 2.) §

Example 1 gave us the most basic cases of the following Limit Forms.

$$\boxed{\left(\text{Limit Form } \frac{1}{0^+}\right) \Rightarrow \infty, \text{ and } \left(\text{Limit Form } \frac{1}{0^-}\right) \Rightarrow -\infty}$$

• These Limit Forms can be **rescaled**, as described in Section 2.3, Part A.

<u>"Odd and Even VAs"</u>

Assume that the graph of $y = f(x)$ has a VA at $x = a$.
(The following terminology is informal and nonstandard.)

• If the two one-sided limits at $x = a$ are ∞ and $-\infty$, in either order, then the VA is an "<u>odd VA</u>."

• If those limits are **both** ∞ or **both** $-\infty$, then the VA is an "<u>even VA</u>."

• In Example 1, the *y*-axis was an **"odd VA,"** partly due to the fact that *f* was an odd function. The graph of $y = f(x)$ "shot off" in **different** directions around the VA.

• In Example 2 below, the *y*-axis is an **"even VA,"** partly due to the fact that *g* is an even function, where $g(x) = \dfrac{1}{x^2}$. The graph of $y = g(x)$ "shoots off" in the **same** direction around the VA.

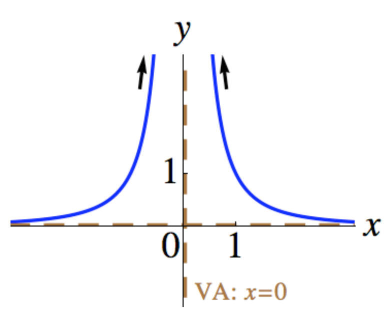
<u>Example 2 (A Graph With an "Even VA")</u>

Evaluate $\displaystyle\lim_{x\to 0^+} \frac{1}{x^2}$, $\displaystyle\lim_{x\to 0^-} \frac{1}{x^2}$, and $\displaystyle\lim_{x\to 0} \frac{1}{x^2}$.

<u>§ Solution</u>

Because $x^2 > 0$ for all $x \neq 0$, **all three** give: $\left(\text{Limit Form } \dfrac{1}{0^+}\right) \Rightarrow \infty$.

The graph of $y = \dfrac{1}{x^2}$ is below.



$\displaystyle\lim_{x\to 0^+} \frac{1}{x^2} = \infty$, $\displaystyle\lim_{x\to 0^-} \frac{1}{x^2} = \infty$, and $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$. §

PART B: EVALUATING INFINITE LIMITS FOR RATIONAL FUNCTIONS

Example 3 (Evaluating Infinite Limits at a Point for a Rational Function)

Let $f(x)=\dfrac{x+1}{x^3+4x^2}$. Evaluate $\lim\limits_{x\to-4^+} f(x)$, $\lim\limits_{x\to-4^-} f(x)$, and $\lim\limits_{x\to-4} f(x)$.

§ Solution

$\lim\limits_{x\to-4}(x+1)=-4+1=-3$, and $\lim\limits_{x\to-4}\left(x^3+4x^2\right)=(-4)^3+4(-4)^2=0$.

All three problems give the Limit Form $\dfrac{-3}{0}$. For each, we must know **how** the denominator approaches 0. Since it is easier to analyze **signs of products** than of sums (for example, do we automatically know the sum of a and b if $a>0$ and $b<0$?), we will **factor the denominator**.

WARNING 1: Many students **improperly use methods** such as the "Division Method" and "DTS" from Section 2.3. Those methods are designed to evaluate **"long-run"** limits, **not** limits at a point.

$$\lim_{x\to-4^+} f(x)=\lim_{x\to-4^+}\frac{x+1}{x^3+4x^2}$$

$$=\lim_{x\to-4^+}\frac{\overbrace{x+1}^{\to-3}}{\underbrace{x^2}_{\to16}\underbrace{(x+4)}_{\to0^+}}\quad\left(\text{Limit Form } \frac{-3}{0^+}\right)$$

WARNING 2: Write 0^+ and 0^- as necessary.

In the denominator: Remember that "positive times positive equals positive."

$$=-\infty$$

$$\lim_{x\to-4^-} f(x)=\lim_{x\to-4^-}\frac{x+1}{x^3+4x^2}$$

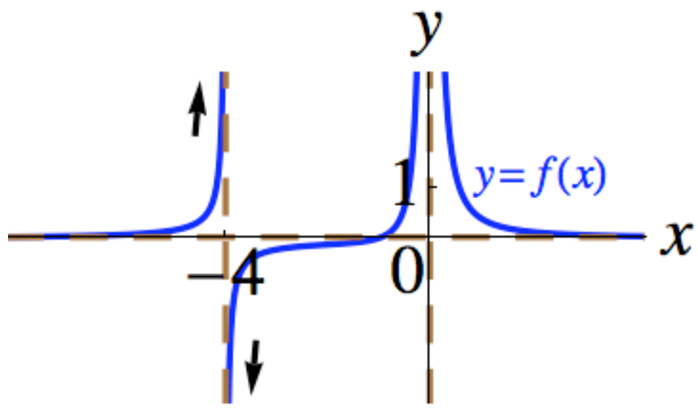
$$=\lim_{x\to-4^-}\frac{\overbrace{x+1}^{\to-3}}{\underbrace{x^2}_{\to16}\underbrace{(x+4)}_{\to0^-}}\quad\left(\text{Limit Form } \frac{-3}{0^-}\right)$$

In the denominator: Remember that "positive times negative equals negative."

$$=\infty$$

$\lim_{x \to -4} f(x)$ does not exist (DNE). (See Footnote 2.)

The graph of $y = f(x)$ is below. Observe the **"odd VA"** at $x = -4$.
(Why is there an HA at the *x*-axis?)



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<u>Finding VAs for Graphs of "Simplified" Rational Functions</u>

Let $f(x) = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are nonzero polynomials in x with **no real zeros in common**; this is guaranteed (by the Factor Theorem from Precalculus) if they have **no variable factors in common**, up to constant multiples. Then,

The graph of $y = f(x)$ has a **VA** at $x = a \Leftrightarrow a$ is a **real zero** of $D(x)$.

<u>Note</u>: The numerator and the denominator of $\dfrac{x - \frac{1}{3}}{3x - 1}$ are **common factors up to constant multiples** (the denominator is 3 times the numerator); observe that $\dfrac{1}{3}$ is a **real zero of both**.

Example 4 (Finding VAs for the Graph of a "Simplified" Rational Function; Revisiting Example 3)

Let $f\left(x\right)=\dfrac{x+1}{x^3+4x^2}$. Find the equations of the vertical asymptotes (VAs) of the graph of $y=f\left(x\right)$. **Justify** using limits.

§ Solution

$f\left(x\right)=\dfrac{x+1}{x^3+4x^2}=\dfrac{x+1}{x^2\left(x+4\right)}$, which is **simplified**. The **VAs** have **equations** $x=0$ and $x=-4$, corresponding to the **real zeros of the denominator**.

To **justify** the VA at $x=0$, show there is an infinite limit there. **Either** of the following will suffice:

$$\begin{aligned}\lim_{x\to 0^+} f\left(x\right) &= \lim_{x\to 0^+} \frac{x+1}{x^3+4x^2}\\ &= \lim_{x\to 0^+} \frac{\overbrace{x+1}^{\to 1}}{\underbrace{x^2}_{\to 0^+}\underbrace{\left(x+4\right)}_{\to 4}} \qquad \left(\text{Limit Form } \frac{1}{0^+}\right)\\ &= \infty\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-} f\left(x\right) &= \lim_{x\to 0^-} \frac{x+1}{x^3+4x^2}\\ &= \lim_{x\to 0^-} \frac{\overbrace{x+1}^{\to 1}}{\underbrace{x^2}_{\to 0^+}\underbrace{\left(x+4\right)}_{\to 4}} \qquad \left(\text{Limit Form } \frac{1}{0^+}\right)\\ &= \infty\end{aligned}$$

- Since 0 is a real zero of $D\left(x\right)$ with **multiplicity** 2 (an **even** number), there is an **"even VA"** at $x=0$.

To **justify** the VA at $x=-4$, show there is an infinite limit there, as we did in Example 3, by showing **either** $\lim\limits_{x\to -4^+} f\left(x\right)=-\infty$, or $\lim\limits_{x\to -4^-} f\left(x\right)=\infty$.

- Since -4 is a real zero of $D\left(x\right)$ with **multiplicity** 1 (an **odd** number), there is an **"odd VA"** at $x=-4$. §

<u>FOOTNOTES</u>

1. **Alternate definition of an infinite limit at a point.** If we say that f has an infinite limit at a $\Leftrightarrow \left(\lim_{x \to a^+} \left| f(x) \right| = \infty \text{ or } \lim_{x \to a^-} \left| f(x) \right| = \infty \right)$, we then extend the idea of an "infinite limit" to examples such as the following:
$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \text{ is a rational number } (x \in \mathbb{Q}) \\ -\frac{1}{x}, & \text{if } x \text{ is an irrational number } (x \notin \mathbb{Q}; \text{ really, } x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$
as $x \to 0$. In this work, we will not use this definition.

2. **Infinity and the real projective line.**
• The <u>affinely extended real number system</u>, denoted by $\overline{\mathbb{R}}$ or $\left[-\infty, \infty\right]$, includes two points of infinity, one referred to as ∞ (or $+\infty$) and the other referred to as $-\infty$. (We are "adjoining" them to the real number system.) We obtain the <u>two-point compactification of the real numbers</u>. We never refer to ∞ and $-\infty$ as real numbers, though.
• In the <u>projectively extended real number system</u>, denoted by $\mathbb{R}^*$, ∞ and $-\infty$ are treated as the same (we collapse them together and identify them with one another as ∞), and we then obtain the <u>one-point compactification of the real numbers</u>, also known as the <u>real projective line</u>. Then, $\frac{1}{0} = \infty$, the slope of a vertical line is ∞, $\lim_{x \to 0} \frac{1}{x} = \infty$, and $\lim_{x \to -4} \frac{x+1}{x^2 + 4x} = \infty$.
• A point at infinity is sometimes added to the complex plane, and it typically corresponds to the "north pole" of a Riemann sphere that the complex plane is wrapped around.
• See "Projectively Extended Real Numbers" in *MathWorld* (web) and "Real projective line" in *Wikipedia* (web).

SECTION 2.5: THE INDETERMINATE FORMS $\frac{\mathbf{0}}{\mathbf{0}}$ AND $\frac{\infty}{\infty}$

LEARNING OBJECTIVES

- Understand what it means for a Limit Form to be indeterminate.
- Recognize indeterminate forms, and know what other Limit Forms yield.
- Learn techniques for resolving indeterminate forms when evaluating limits, including factoring, rationalizing numerators and denominators, and (in Chapter 10) L'Hôpital's Rule.

PART A: WHAT ARE INDETERMINATE FORMS?

An indeterminate form is a Limit Form that could yield a variety of real values; the limit might not exist. **Further analysis** is required to know what the limit is.

The seven "classic" **indeterminate forms** are:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ \infty^0,\ 0^0, \text{ and } 1^\infty.$$

- Observe that the first six forms involve 0 and/or $(\pm)\infty$, while the seventh involves 1 and ∞.

In Section 2.3, Parts D and E, we encountered the indeterminate form $\frac{\pm\infty}{\pm\infty}$.

This is simply written as $\frac{\infty}{\infty}$, since **further analysis** is required, anyway.

(Sometimes, signs matter in the forms. For example, $\left(\text{Limit Form } \infty-\infty\right)$ is indeterminate, while $\left(\text{Limit Form } \infty+\infty\right)\Rightarrow\infty$. See Part D.)

Example 1 (0/0 is an Indeterminate Form)

If $c\in\mathbb{R}$,

$$\lim_{x\to 0}\frac{cx}{x}\quad\left(\text{Limit Form } \frac{0}{0}\right)=\lim_{x\to 0} c$$
$$= c$$

- We are taking a **limit** as $x\to 0$, so the fact that $\frac{cx}{x}$ is undefined at $x=0$ is irrelevant. (See Section 2.1, Part C.)

c could be 2, $-\pi$, etc. $\left(\text{Limit Form } \frac{0}{0}\right)$ can yield **any real number**.

We already know that $\left(\text{Limit Form } \frac{0}{0}\right)$ is **indeterminate**, but we can further show that it can yield **nonexistent limits**:

$$\lim_{x\to 0^+} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0^+} \frac{1}{x} \quad \left(\text{Limit Form } \frac{1}{0^+}\right) = \infty.$$

$$\lim_{x\to 0^-} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0^-} \frac{1}{x} \quad \left(\text{Limit Form } \frac{1}{0^-}\right) = -\infty.$$

$$\lim_{x\to 0} \frac{x}{x^2} \quad \left(\text{Limit Form } \frac{0}{0}\right) = \lim_{x\to 0} \frac{1}{x}, \text{ which does not exist (DNE).}$$

- We will use $\left(\text{Limit Form } \frac{0}{0}\right)$ when we define <u>derivatives</u> in Chapter 3.
- In turn, <u>L'Hôpital's Rule</u> will use **derivatives** to resolve indeterminate forms, particularly $\frac{0}{0}$ and $\frac{\infty}{\infty}$. (See Chapter 10.) §

Example 2 (∞/∞ is an Indeterminate Form)

If $c \neq 0$,

$$\lim_{x\to \infty} \frac{cx}{x} \quad \left(\text{Limit Form } \frac{\infty}{\infty}\right) = \lim_{x\to\infty} c$$

$$= c$$

c could be 2, $-\pi$, etc. $\left(\text{Limit Form } \frac{\infty}{\infty}\right)$ can yield **any real number**.

(In the Exercises, you will demonstrate how it can yield 0 and ∞.) §

Example 3 (1/0 is Not an Indeterminate Form)

$\left(\text{Limit Form } \frac{1}{0}\right) \Rightarrow \infty$, $-\infty$, or "DNE." We know a lot! The form is **not indeterminate**, although we must know **how** the denominator approaches 0.

$\left(\text{Limit Form } \frac{1}{0^+}\right) \Rightarrow \infty$. $\left(\text{Limit Form } \frac{1}{0^-}\right) \Rightarrow -\infty$. $\displaystyle\lim_{x\to\infty} \frac{1}{\frac{\sin x}{x}}$ "DNE"; see Section 2.3, Example 6. §

PART B: RESOLVING THE $\frac{0}{0}$ FORM BY FACTORING AND CANCELING; GRAPHS OF RATIONAL FUNCTIONS

Let $f(x)=\dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are nonzero polynomials in *x*.

We do **not** require simplified form, as we did in Section 2.4. If *a* is a **real zero** of **both** $N(x)$ and $D(x)$, then we can use the Factor Theorem from Precalculus to help us **factor** $N(x)$ and $D(x)$ and **simplify** $f(x)$.

> Factor Theorem
>
> a is a **real zero** of $D(x)$ $\Leftrightarrow$ $(x-a)$ is a **factor** of $D(x)$.
>
> • This also applied to Section 2.4, but it now helps that the same goes for $N(x)$.

Example 4 (Factoring and Canceling/Dividing to Resolve a 0/0 Form)

Let $f(x)=\dfrac{x^2-1}{x^2-x}$. Evaluate: a) $\lim\limits_{x\to 1} f(x)$ and b) $\lim\limits_{x\to 0^+} f(x)$.

§ Solution to a)

The Limit Form is $\dfrac{0}{0}$:

$$\lim_{x\to 1} N(x)=\lim_{x\to 1}\left(x^2-1\right)=(1)^2-1=0\text{, and}$$
$$\lim_{x\to 1} D(x)=\lim_{x\to 1}\left(x^2-x\right)=(1)^2-(1)=0\text{.}$$

1 is a **real zero** of **both** $N(x)$ and $D(x)$, so $(x-1)$ is a **common factor**.

We will **cancel** $(x-1)$ factors and **simplify** $f(x)$ to resolve the $\dfrac{0}{0}$ form.

WARNING 1: Some instructors prefer "divide out" to "cancel."

TIP 1: It often saves time to **begin by factoring** and worry about Limit Forms later.

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2-1}{x^2-x} \quad \left(\text{Limit Form } \frac{0}{0}\right)\\ &= \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}\\ &= \lim_{x\to 1} \frac{x+1}{x}\\ &= \frac{(1)+1}{(1)}\\ &= 2\end{aligned}$$

§

§ *Solution to b)*

The Limit Form is $\dfrac{-1}{0}$:

$$\lim_{x\to 0^+} N(x) = \lim_{x\to 0^+} \left(x^2-1\right) = (0)^2 - 1 = -1, \text{ and}$$
$$\lim_{x\to 0^+} D(x) = \lim_{x\to 0^+} \left(x^2-x\right) = (0)^2 - (0) = 0.$$

Here, when we **cancel** $(x-1)$ factors and **simplify** $f(x)$, it is a matter of **convenience**. It takes work to see that $\lim_{x\to 0^+} D(x) = \lim_{x\to 0^+} \left(x^2-x\right) = 0^-$, and then $\left(\text{Limit Form } \dfrac{-1}{0^-}\right) \Rightarrow \infty$.

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{x^2-1}{x^2-x} \quad \left(\text{Limit Form } \frac{-1}{0}\right)\\ &= \lim_{x\to 0^+} \frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}\\ &= \lim_{x\to 0^+} \frac{x+1}{x} \quad \left(\text{Limit Form } \frac{1}{0^+}\right)\\ &= \infty\end{aligned}$$

§

The Graph of a Rational Function f at a Point a

The graph of $y = f(x)$ has **one** of the following at $x = a$ $\left(a \in \mathbb{R}\right)$:

1) The **point** $\left(a, f(a)\right)$, if $f(a)$ is real $\left(a \in \mathrm{Dom}(f)\right)$.

In 2) and 3) below,

- $(x-a)$ is a **factor** of the denominator, $D(x)$.
- That is, a is a **real zero** of $D(x)$, and $a \notin \mathrm{Dom}(f)$.

2) A **VA**, if simplifying $f(x)$ yields the Limit Form $\dfrac{c}{0}$ as $x \to a$ $(c \neq 0)$.

- That is, there is at least one $(x-a)$ **factor** of $D(x)$ that **cannot be canceled/divided out**. It will still **force** the denominator towards 0 as $x \to a$.

3) A **hole** at the point (a, L), if $f(a)$ is undefined $\left(a \notin \mathrm{Dom}(f)\right)$, **but** $\lim\limits_{x \to a} f(x) = L$ $\left(L \in \mathbb{R}\right)$.

- That is, $(x-a)$ is a **factor** of $D(x)$, **but all such factors** can be **canceled/divided out** by $(x-a)$ factor(s) in the numerator. Then, the denominator is **no longer forced** towards 0.
- A hole can only occur if we start with the Limit Form $\dfrac{0}{0}$, because a **denominator** approaching 0 can only be **prevented** from "exploding" $f(x)$ if the **numerator** approaches 0, as well. (If the numerator **fails** to prevent this, we get a **VA**.)

Example 5 (VAs and Holes on the Graph of a Rational Function; Revisiting Example 4)

Let $f(x)=\dfrac{x^2-1}{x^2-x}$. Identify any **vertical asymptotes (VAs)** and **holes** on the graph of $y=f(x)$.

§ Solution

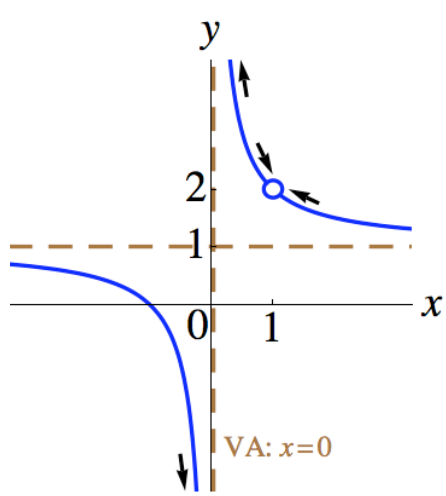
In Example 4, we saw that: $f(x)=\dfrac{x^2-1}{x^2-x}=\dfrac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}=\dfrac{x+1}{x}\ (x\neq 1)$.

The **real zeros** of x^2-x are 0 and 1, so they correspond to **VAs or holes**.

• In 4a, we found that: $\lim\limits_{x\to 1} f(x)=2$, even though $1\notin \mathrm{Dom}(f)$, so the graph has a **hole** at the point $(1,2)$. As $x\to 1$, the factor $(x-1)\to 0$. When we simplify $f(x)$, we **cancel (divide out) all** of the $(x-1)$ factors in the **denominator**. The new denominator, *x*, **no longer approaches 0**, and the overall limit **exists**.

• In 4b, we found that: $\lim\limits_{x\to 0^+} f(x)=\infty$, so the graph has a **VA** at $x=0$ (the *y*-axis). When we simplify $f(x)$, we **cannot cancel** (divide out) the *x* factor in the **denominator**. As $x\to 0$, the new denominator, *x*, **still approaches 0**.

The graph of $y=\dfrac{x^2-1}{x^2-x}$ $\left(\text{or } y=\dfrac{x+1}{x}\ (x\neq 1),\ \text{or } y=1+\dfrac{1}{x}\ (x\neq 1)\right)$ is below.



Since 0 is a zero of the new denominator, *x*, with multiplicity 1, the VA at $x=0$ is an **"odd VA."**

Why is there an **HA** at $y=1$? §

PART C: RESOLVING THE $\frac{0}{0}$ FORM BY RATIONALIZING; GRAPHS OF ALGEBRAIC FUNCTIONS

Graphs of algebraic functions can also have points, VAs, and holes. Unlike graphs of rational functions, they can also have **"blank spaces"** where there are no points for infinitely many real values of *x*.

Example 6 (Rationalizing a Numerator to Resolve a 0/0 Form)

Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{9-x}-3}{x}$.

§ Solution

Observe that $\sqrt{9-x}$ is **real** on a punctured neighborhood of 0.
We assume $x \approx 0$.

$$\lim_{x \to 0} \frac{\sqrt{9-x}-3}{x} \quad \left(\text{Limit Form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left[\frac{\left(\sqrt{9-x}-3\right)}{x} \cdot \frac{\left(\sqrt{9-x}+3\right)}{\left(\sqrt{9-x}+3\right)} \right] \quad \left(\text{Rationalizing the numerator}\right)$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{9-x}\right)^2 - (3)^2}{x\left(\sqrt{9-x}+3\right)} \quad \left(\textbf{WARNING 2: } \text{Write the entire denominator! It's not just } x.\right)$$

$$= \lim_{x \to 0} \frac{(9-x)-9}{x\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x \to 0} \frac{\overset{(-1)}{\cancel{-x}}}{\cancel{x}\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x \to 0} \frac{-1}{\sqrt{9-x}+3}$$

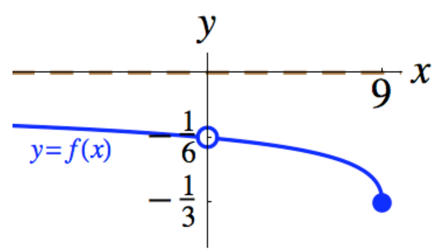
$$= \frac{-1}{\sqrt{9-(0)}+3}$$

$$= -\frac{1}{6}$$

$\lim_{x \to 0} f(x) = -\frac{1}{6}$, even though $0 \notin \text{Dom}(f)$, where $f(x) = \frac{\sqrt{9-x}-3}{x}$.

Therefore, the graph of $y = f(x)$ has a **hole** at the point $\left(0, -\frac{1}{6}\right)$.

The graph of $y = f(x)$ is below. What is $\text{Dom}(f)$?



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PART D: LIMIT FORMS THAT ARE NOT INDETERMINATE

Cover up the "Yields" columns below and guess at the results of the Limit Forms $(c \in \mathbb{R})$. Experiment with sequences of numbers and with extreme numbers.

For example, for $\infty^{-\infty}$, or $\frac{1}{\infty^{\infty}}$, look at $(1000)^{-10,000} = \frac{1}{(1000)^{10,000}}$.

Fractions

Limit Form	Yields
$\frac{1}{\infty}$	0^+
$\frac{1}{0^+}$	∞
$\frac{\infty}{1}$	∞
$\frac{\infty}{0^+}$	∞
$\frac{0^+}{\infty}$	0^+

Sums, Differences, Products

Limit Form	Yields
$\infty + c$	∞
$-\infty + c$	$-\infty$
$\infty + \infty$	∞
$-\infty - \infty$	$-\infty$
$\infty \cdot 1$	∞
$\infty \cdot \infty$	∞

With Exponents

Limit Form	Yields
∞^{∞}	∞
$\infty^{-\infty}$	0^+
0^{∞}	0
2^{∞}	∞
$\left(\frac{1}{2}\right)^{\infty}$	0

SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

LEARNING OBJECTIVES

• Understand and be able to rigorously apply the Squeeze (Sandwich) Theorem when evaluating limits at a point and "long-run" limits at $(\pm)$infinity.

PART A: APPLYING THE SQUEEZE (SANDWICH) THEOREM TO LIMITS AT A POINT

We will **formally state** the Squeeze (Sandwich) Theorem in Part B.

Example 1 below is one of many basic examples where we use the Squeeze (Sandwich) Theorem to show that $\lim_{x \to 0} f(x) = 0$, where $f(x)$ is the **product** of a **sine or cosine expression** and a **monomial of even degree**.

• The idea is that "something approaching 0" times "something bounded" (that is, trapped between two real numbers) will approach 0. Informally,

$$\boxed{(\text{Limit Form } \; 0 \cdot \text{bounded}) \Rightarrow 0 .}$$

Example 1 (Applying the Squeeze (Sandwich) Theorem to a Limit at a Point)

Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$. Prove that $\lim_{x \to 0} f(x) = 0$.

§ Solution

• We first **bound** $\cos\left(\frac{1}{x}\right)$, which is **real** for all $x \neq 0$.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad (\forall x \neq 0) \quad \Rightarrow$$

• **Multiply** all three parts by x^2 so that the middle part becomes $f(x)$.

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2 \quad (\forall x \neq 0) \quad \Rightarrow$$

WARNING 1: We **must** observe that $x^2 > 0$ for all $x \neq 0$, or at least on a **punctured neighborhood** of $x = 0$, so that we can multiply by x^2 **without reversing** inequality symbols.

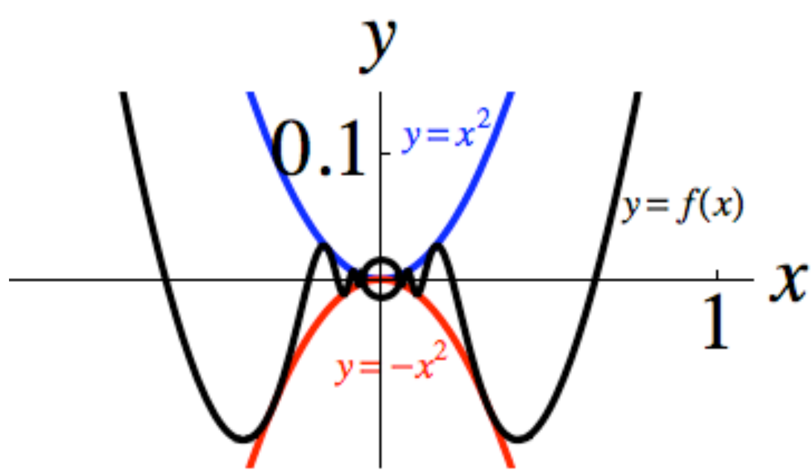
• As $x \to 0$, the **left and right parts approach 0**. Therefore, by the Squeeze (Sandwich) Theorem, the **middle part**, $f(x)$, is **forced to approach 0**, also. The middle part is "squeezed" or "sandwiched" between the left and right parts, so it **must approach the same limit** as the other two do.

$\lim\limits_{x\to 0}\left(-x^2\right)=0$, and $\lim\limits_{x\to 0} x^2 = 0$, so $\lim\limits_{x\to 0} x^2\cos\left(\frac{1}{x}\right)=0$ by the Squeeze Theorem.

Shorthand: As $x \to 0$,

$$\underbrace{-x^2}_{\to 0} \le \underbrace{x^2\cos\left(\frac{1}{x}\right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{x^2}_{\to 0} \quad \left(\forall x \ne 0\right).$$

The graph of $y = x^2\cos\left(\frac{1}{x}\right)$, together with the squeezing graphs of $y=-x^2$ and $y=x^2$, is below.



(The axes are scaled differently.)

§

In Example 2 below, $f(x)$ is the **product** of a **sine or cosine expression** and a **monomial of odd degree**.

Example 2 (Handling Complications with Signs)

Let $f(x) = x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$. Use the Squeeze Theorem to find $\lim_{x\to 0} f(x)$.

§ Solution 1 (Using Absolute Value)

• We first **bound** $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$, which is **real** for all $x \neq 0$.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le 1 \quad (\forall x \neq 0) \quad \Rightarrow$$

• **WARNING 2:** The problem with multiplying all three parts by x^3 is that $x^3 < 0$ when $x < 0$. The $\le$ inequality symbols would have to be **reversed** for $x < 0$.

Instead, we use **absolute value** here. We could write

$0 \le \left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le 1 \quad (\forall x \neq 0)$,

but we assume that absolute values are **nonnegative**.

$$\left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le 1 \quad (\forall x \neq 0) \quad \Rightarrow$$

• **Multiply** both sides of the inequality by $\left|x^3\right|$. We know $\left|x^3\right| > 0 \quad (\forall x \neq 0)$.

$$\left|x^3\right|\left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le \left|x^3\right| \quad (\forall x \neq 0) \quad \Rightarrow$$

• "The **product** of absolute values equals the **absolute value** of the product."

$$\left|x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \le \left|x^3\right| \quad (\forall x \neq 0) \quad \Rightarrow$$

• If, say, $|a| \le 4$, then $-4 \le a \le 4$. Similarly:

$$-\left|x^3\right| \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le \left|x^3\right| \quad (\forall x \neq 0) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x \to 0} \left(-\left| x^3 \right| \right) = 0$, and $\lim_{x \to 0} \left| x^3 \right| = 0$, so $\lim_{x \to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}} \right) = 0$ by the Squeeze Theorem.

Shorthand: As $x \to 0$,

$$\underbrace{-\left| x^3 \right|}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}} \right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{\left| x^3 \right|}_{\to 0} \quad \left(\forall x \ne 0 \right).$$ §

§ Solution 2 (Split Into Cases: Analyze Right-Hand and Left-Hand Limits Separately)

First, we analyze: $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}} \right)$.

Assume $x > 0$, since we are taking a limit as $x \to 0^+$.

• We first **bound** $\sin\left(\frac{1}{\sqrt[3]{x}} \right)$, which is **real** for all $x \ne 0$.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}} \right) \le 1 \quad \left(\forall x > 0 \right) \quad \Rightarrow$$

• **Multiply** all three parts by x^3 so that the middle part becomes $f(x)$. We know $x^3 > 0$ for all $x > 0$.

$$-x^3 \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}} \right) \le x^3 \quad \left(\forall x > 0 \right) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x \to 0^+} \left(-x^3 \right) = 0$, and $\lim_{x \to 0^+} x^3 = 0$, so $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}} \right) = 0$ by the Squeeze Theorem.

Shorthand: As $x \to 0^+$,

$$\underbrace{-x^3}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}} \right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{x^3}_{\to 0} \quad \left(\forall x > 0 \right).$$

Second, we analyze: $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$.

Assume $x < 0$, since we are taking a limit as $x \to 0^-$.

• We first **bound** $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$, which is **real** for all $x \neq 0$.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le 1 \quad (\forall x < 0) \quad \Rightarrow$$

• **Multiply** all three parts by x^3 so that the middle part becomes $f(x)$. We know $x^3 < 0$ for all $x < 0$, so we **reverse** the $\le$ inequality symbols.

$$-x^3 \ge x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \ge x^3 \quad (\forall x < 0) \quad \Rightarrow$$

• **Reversing the compound inequality** will make it easier to read.

$$x^3 \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le -x^3 \quad (\forall x < 0) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x \to 0^-} x^3 = 0$, and $\lim_{x \to 0^-} \left(-x^3\right) = 0$, so $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$ by the Squeeze Theorem.

Shorthand: As $x \to 0^-$,

$$\underbrace{x^3}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{-x^3}_{\to 0} \quad (\forall x < 0).$$

Now, $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$, and $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$, so $\lim_{x \to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$. §

Example 3 (Limits are Local)

Use $\lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to 0} x^6 = 0$ to show that $\lim\limits_{x\to 0} x^4 = 0$.

§ Solution

Let $I = (-1,1)\setminus\{0\}$. I is a **punctured neighborhood** of 0.
Shorthand: As $x \to 0$,

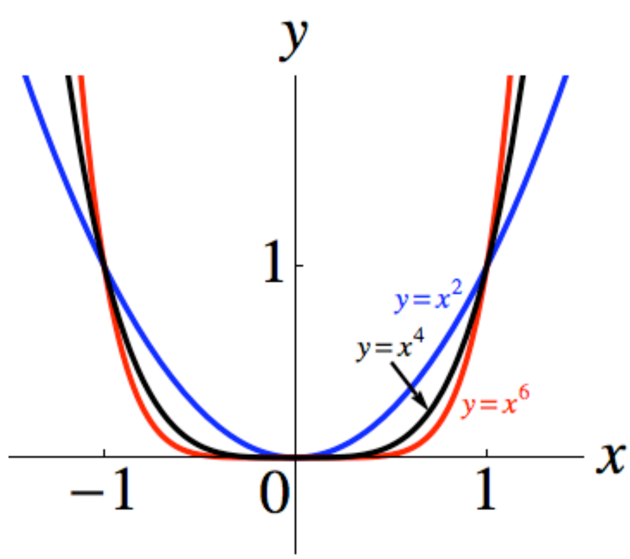
$$\underbrace{x^6}_{\to 0} \le \underbrace{x^4}_{\substack{\text{Therefore, } \to 0\\ \text{by the Squeeze}\\ \text{(Sandwich) Theorem}}} \le \underbrace{x^2}_{\to 0} \quad (\forall x \in I)$$

WARNING 3: The **direction** of the $\le$ inequality symbols may confuse students. Observe that $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{16} < \frac{1}{4}$.

We conclude: $\lim\limits_{x\to 0} x^4 = 0$.

We do **not** need the compound inequality to hold true for **all** nonzero values of *x*. We only need it to hold true on some **punctured neighborhood** of 0 so that we may apply the Squeeze (Sandwich) Theorem to the **two-sided** limit $\lim\limits_{x\to 0} x^4$. This is because **"Limits are Local."**

As seen below, the graphs of $y = x^6$ and $y = x^2$ squeeze (from below and above, respectively) the graph of $y = x^4$ on *I*. In Chapter 6, we will be able to find the **areas** of the bounded regions.



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PART B: THE SQUEEZE (SANDWICH) THEOREM

We will call B the "bottom" function and T the "top" function.

The Squeeze (Sandwich) Theorem

Let B and T be functions such that $B(x) \le f(x) \le T(x)$ on a **punctured neighborhood** of a.

If $\lim\limits_{x \to a} B(x) = L$ and $\lim\limits_{x \to a} T(x) = L$ $(L \in \mathbb{R})$, then $\lim\limits_{x \to a} f(x) = L$.

Variation for Right-Hand Limits at a Point

Let $B(x) \le f(x) \le T(x)$ on some **right-neighborhood** of a.

If $\lim\limits_{x \to a^+} B(x) = L$ and $\lim\limits_{x \to a^+} T(x) = L$ $(L \in \mathbb{R})$, then $\lim\limits_{x \to a^+} f(x) = L$.

Variation for Left-Hand Limits at a Point

Let $B(x) \le f(x) \le T(x)$ on some **left-neighborhood** of a.

If $\lim\limits_{x \to a^-} B(x) = L$ and $\lim\limits_{x \to a^-} T(x) = L$ $(L \in \mathbb{R})$, then $\lim\limits_{x \to a^-} f(x) = L$.

PART C: VARIATIONS FOR "LONG-RUN" LIMITS

In the upcoming Example 4, $f(x)$ is the **quotient** of a **sine or cosine expression** and a **polynomial**.

- The idea is that "something bounded" divided by "something approaching $(\pm)$ infinity" will approach 0. Informally,

$$\boxed{\left(\text{Limit Form } \frac{\text{bounded}}{\pm\infty}\right) \Rightarrow 0.}$$

Example 4 (Applying the Squeeze (Sandwich) Theorem to a "Long-Run" Limit; Revisiting Section 2.3, Example 6)

Evaluate: a) $\lim_{x\to\infty} f(x)$ and b) $\lim_{x\to-\infty} f(x)$, where $f(x)=\dfrac{\sin x}{x}$.

§ Solution to a)

Assume $x>0$, since we are taking a limit as $x\to\infty$.

• We first **bound** $\sin x$.

$$-1\le \sin x\le 1 \quad (\forall x>0) \quad\Rightarrow$$

• **Divide** all three parts by x $(x>0)$ so that the middle part becomes $f(x)$.

$$-\frac{1}{x}\le \frac{\sin x}{x}\le \frac{1}{x} \quad (\forall x>0) \quad\Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x\to\infty}\left(-\dfrac{1}{x}\right)=0$, and $\lim_{x\to\infty}\dfrac{1}{x}=0$, so

$\lim_{x\to\infty}\dfrac{\sin x}{x}=0$ by the Squeeze Theorem.

Shorthand: As $x\to\infty$,

$$\underbrace{-\frac{1}{x}}_{\to 0}\le \underbrace{\frac{\sin x}{x}}_{\substack{\text{Therefore, }\to 0\\ \text{by the Squeeze}\\ \text{(Sandwich) Theorem}}}\le \underbrace{\frac{1}{x}}_{\to 0} \quad (\forall x>0).$$ §

§ Solution to b)

Assume $x<0$, since we are taking a limit as $x\to-\infty$.

• We first **bound** $\sin x$.

$$-1\le \sin x\le 1 \quad (\forall x<0) \quad\Rightarrow$$

• **Divide** all three parts by x so that the middle part becomes $f(x)$. But $x<0$, so we must **reverse** the $\le$ inequality symbols.

$$-\frac{1}{x}\ge \frac{\sin x}{x}\ge \frac{1}{x} \quad (\forall x<0) \quad\Rightarrow$$

• **Reversing the compound inequality** will make it easier to read.

$$\frac{1}{x}\le \frac{\sin x}{x}\le -\frac{1}{x} \quad (\forall x<0) \quad\Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

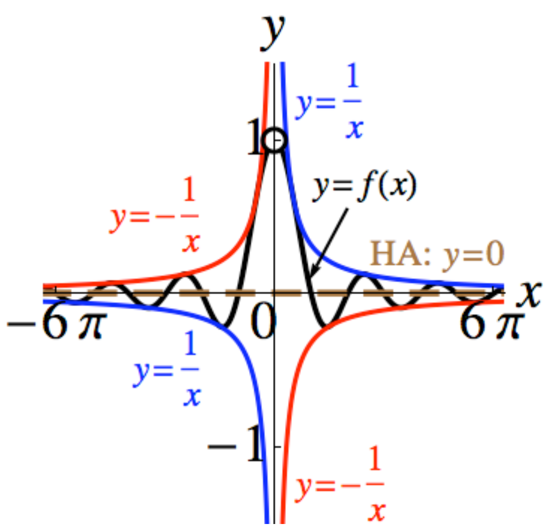
$$\lim_{x \to -\infty} \frac{1}{x} = 0, \text{ and } \lim_{x \to -\infty} \left(-\frac{1}{x} \right) = 0, \text{ so}$$

$$\lim_{x \to -\infty} \frac{\sin x}{x} = 0 \text{ by the Squeeze Theorem.}$$

Shorthand: As $x \to -\infty$,

$$\underbrace{\frac{1}{x}}_{\to 0} \le \underbrace{\frac{\sin x}{x}}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{-\frac{1}{x}}_{\to 0} \quad (\forall x < 0)$$

The graph of $y = \frac{\sin x}{x}$, together with the squeezing graphs of $y = -\frac{1}{x}$ and $y = \frac{1}{x}$, is below. We can now justify the **HA** at $y = 0$ (the x-axis).



(The axes are scaled differently.) §

Variation for "Long-Run" Limits to the Right

Let $B(x) \le f(x) \le T(x)$ on some x-interval of the form (c, ∞), $c \in \mathbb{R}$.

If $\lim_{x \to \infty} B(x) = L$ and $\lim_{x \to \infty} T(x) = L$ $(L \in \mathbb{R})$, then $\lim_{x \to \infty} f(x) = L$.

• In Example 4a, we used $c = 0$. We need the compound inequality to hold **"forever" after some point c**.

Variation for "Long-Run" Limits to the Left

Let $B(x) \le f(x) \le T(x)$ on some x-interval of the form $(-\infty, c)$, $c \in \mathbb{R}$.

If $\lim_{x \to -\infty} B(x) = L$ and $\lim_{x \to -\infty} T(x) = L$ $(L \in \mathbb{R})$, then $\lim_{x \to -\infty} f(x) = L$.

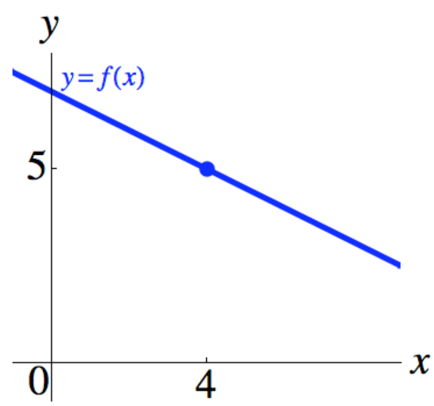
SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

LEARNING OBJECTIVES

• Know rigorous definitions of limits, and use them to rigorously prove limit statements.

PART A: THE "STATIC" APPROACH TO LIMITS

We will use the example $\lim_{x \to 4} \left(7 - \frac{1}{2}x \right) = 5$ in our quest to **rigorously define** what a **limit at a point** is. We consider $\lim_{x \to a} f(x) = L$, where $f(x) = 7 - \frac{1}{2}x$, $a = 4$, and $L = 5$. The graph of $y = f(x)$ is the line below.



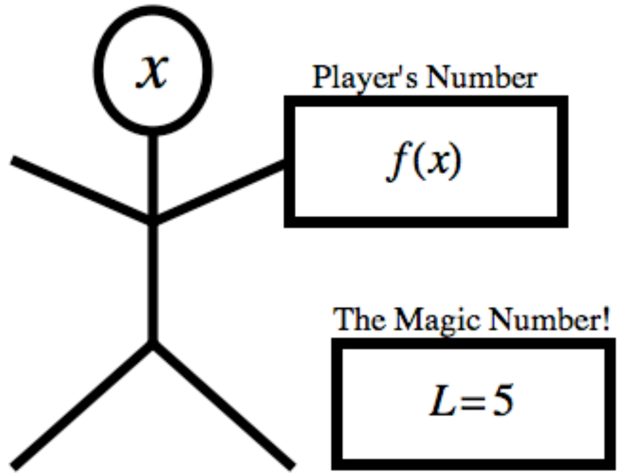
The **"dynamic" view** of limits states that, as *x* "approaches" or "gets closer to" 4, $f(x)$ "approaches" or "gets closer to" 5. (See Section 2.1, Footnote 2.)

The precise approach takes on a more **"static" view**. The idea is that, if *x* is close to 4, then $f(x)$ is close to 5.

The Lottery Analogy

Imagine a lottery in which every $x \in \text{Dom}(f)$ represents a player. However, we disqualify $x = a$ (here, $x = 4$), because that person manages the lottery. (See Section 2.1, Part C.)

Each player is assigned a lottery number by the rule $f(x) = 7 - \frac{1}{2}x$.
The "exact" winning lottery number (the "target") turns out to be $L = 5$.



When Does Player x Win?

In this lottery, more than one player can win, and it is sufficient for a player to be **"close enough"** to the "target" in order to win. In particular, Player x wins $(x \neq a)$ $\Leftrightarrow$ the player's lottery number, $f(x)$, is **strictly within** ε units of L, where $\varepsilon > 0$. The Greek letter ε ("epsilon") often represents a **small positive quantity**. Here, ε is a **tolerance level** that measures how liberal the lottery is in determining winners.

Symbolically:

$$\text{Player } x \text{ wins } (x \neq a) \Leftrightarrow L - \varepsilon < f(x) < L + \varepsilon$$

Subtract L from all three parts.

$$\Leftrightarrow -\varepsilon < f(x) - L < \varepsilon$$

$-1 < r < 1 \Leftrightarrow |r| < 1.$
Similarly:

$$\Leftrightarrow |f(x) - L| < \varepsilon$$

$|f(x) - L|$ is the **distance** (along the y-axis) between Player x's lottery number, $f(x)$, and the "target" L.

Player x **wins** $(x \neq a)$ $\Leftrightarrow$ this distance is less than ε.

Where Do We Look for Winners?

We only care about players that are "close" to $x = a$ (here, $x = 4$), excluding a itself. These players x are strictly between 0 and δ units of a, where $\delta > 0$. Like ε, the Greek letter δ ("delta") often represents a **small positive quantity**. δ is the **half-width** of a **punctured δ-neighborhood** of a.

Symbolically:

$$\text{Player } x \text{ is "close" to } a \;\Leftrightarrow\; a - \delta < x < a + \delta \;\; (x \neq a)$$

That is, $x \in (a - \delta,\ a + \delta) \setminus \{a\}$.
Subtract a from all three parts.

$$\Leftrightarrow\; -\delta < x - a < \delta \;\; (x \neq a)$$

$$\Leftrightarrow\; 0 < |x - a| < \delta$$

$|x - a|$ is the **distance** between Player x and a.

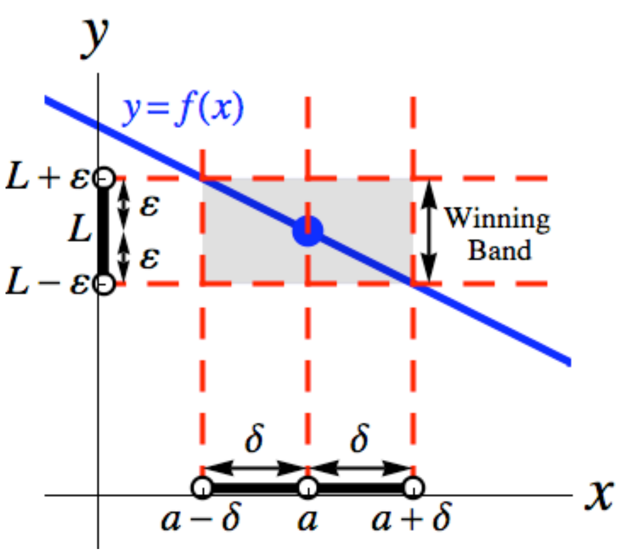
Player x is **"close" to *a*** $\Leftrightarrow$ this distance is strictly between 0 and δ.

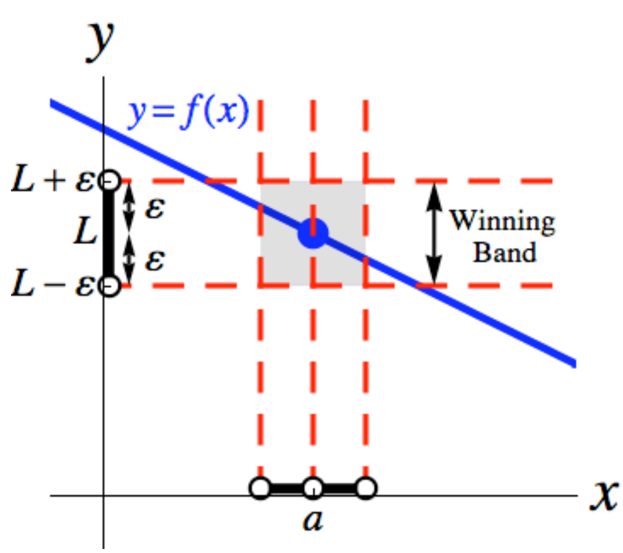
- If the distance is 0, we have $x = a$, which is disqualified.

In the figure on the left, the value for δ is giving us a **punctured δ-neighborhood** of a in which **everyone wins**.

- In this sense, if x is close to a, then $f(x)$ is close to L.

Observe that any **smaller** positive value for δ could also have been chosen. (See the figure on the right. The dashed lines are **not** asymptotes; they indicate the boundaries of the open intervals and the puncture at $x = a$.)





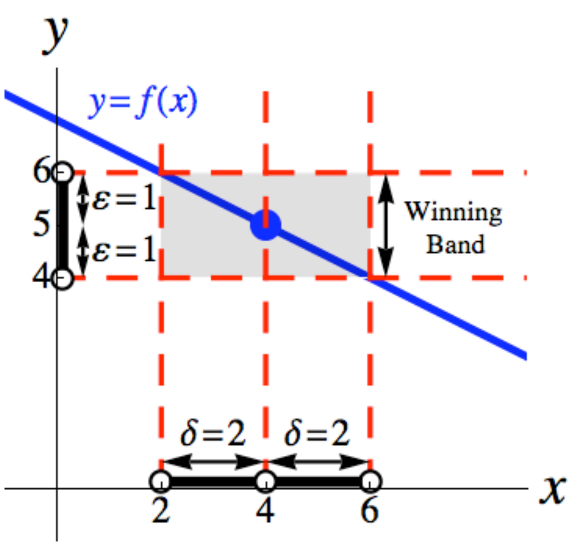
How Does the "Static" Approach to Limits Relate to the "Dynamic" Approach?

Why is $\lim_{x\to 4}\left(7-\frac{1}{2}x\right)=5$? Because, **regardless of how small** we make the **tolerance level** ε and how tight we make the lottery for the players, **there is** a value for δ for which the corresponding **punctured δ-neighborhood** of $a=4$ is made up **entirely of winners**. That is, the corresponding "punctured box" (see the shaded boxes in the figures) **traps the graph** of $y=f(x)$ on the punctured δ-neighborhood.

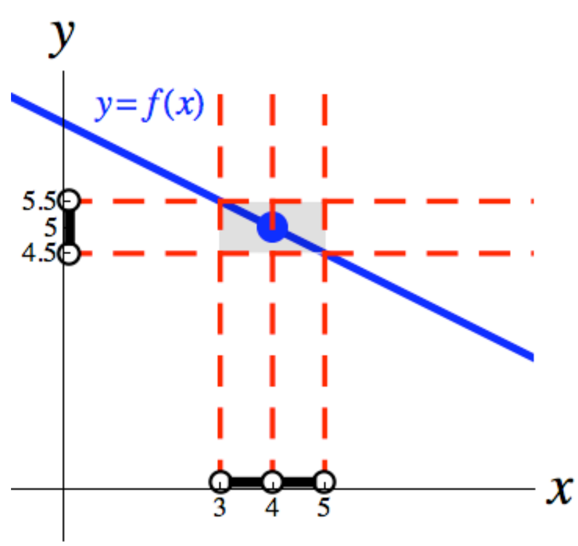
As $\varepsilon\to 0^+$, we can **choose values** for δ in such a way that the corresponding shaded "punctured boxes" **always trap the graph** and zoom in, or collapse in, on the **point** $(4,5)$. (This would have been the case even if that point had been **deleted** from the graph.) In other words, **there are always winners close to** $a=4$.

- As x gets arbitrarily close to a, $f(x)$ gets arbitrarily close to L.

If $\varepsilon=1$, we can choose $\delta=2$.



If $\varepsilon=0.5$, we can choose $\delta=1$.



For this example, if ε is **any positive real number**, we can choose $\delta=2\varepsilon$. Why is that?

- Graphically, we can exploit the fact that the **slope** of the line $y=7-\frac{1}{2}x$ is $-\frac{1}{2}$. Remember, slope $=\frac{\text{rise}}{\text{run}}$. Along the line, an x-run of 2 units corresponds to a y-drop of 1 unit.

- We will demonstrate this rigorously in Example 1.

PART B: THE PRECISE ε - δ DEFINITION OF A LIMIT AT A POINT

The Precise ε-δ Definition of a Limit at a Point (Version 1)

For $a, L \in \mathbb{R}$, if a function f is defined on a **punctured neighborhood** of a,

$\lim_{x \to a} f(x) = L \Leftrightarrow$ **for every** $\varepsilon > 0$, **there exists** a $\delta > 0$ such that,

if $0 < |x - a| < \delta$ (that is, if x is "close" to a, excluding a itself),

then $|f(x) - L| < \varepsilon$ (that is, $f(x)$ is "close" to L).

Variation Using Interval Form

We can replace $0 < |x - a| < \delta$ with: $x \in (a - \delta,\ a + \delta) \setminus \{a\}$.

We can replace $|f(x) - L| < \varepsilon$ with: $f(x) \in (L - \varepsilon,\ L + \varepsilon)$.

The Precise ε-δ Definition of a Limit at a Point (Version 2: More Symbolic)

For $a, L \in \mathbb{R}$, if a function f is defined on a **punctured neighborhood** of a,

$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni$

$\left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$.

Example 1 (Proving the Limit Statement from Part A)

Prove $\lim\limits_{x \to 4} \left(7 - \frac{1}{2}x\right) = 5$ using a precise ε-δ definition of a limit at a point.

§ Solution

We have: $f(x) = 7 - \frac{1}{2}x$, $a = 4$, and $L = 5$.

We need to show:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ni \left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\right); \text{ i.e.,}$$

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ni \left(0 < |x - 4| < \delta \Rightarrow \left|\left(7 - \frac{1}{2}x\right) - (5)\right| < \varepsilon\right).$$

Rewrite $|f(x) - L|$ in terms of $|x - a|$; here, $|x - 4|$:

$$|f(x) - L| = \left|\left(7 - \frac{1}{2}x\right) - (5)\right|$$

$$= \left|-\frac{1}{2}x + 2\right|$$

Factor out $-\frac{1}{2}$, the coefficient of *x*.

To divide the $+2$ term by $-\frac{1}{2}$, we multiply it by -2 and obtain -4.

$$= \left|-\frac{1}{2}(x - 4)\right|$$

$$= \left|-\frac{1}{2}\right| |x - 4|$$

This is because, if *m* and *n* represent real quantities, then $|mn| = |m||n|$.

$$= \frac{1}{2}|x - 4|$$

We have: $|f(x) - L| = \frac{1}{2}|x - 4|$; call this statement *.

Assuming ε is fixed $(\varepsilon > 0)$, find an appropriate value for δ.

We will find a value for δ that corresponds to a **punctured δ-neighborhood** of $a = 4$ in which **everyone wins**. This means that, for every player x in there:

$$\begin{aligned} \left| f(x) - L \right| < \varepsilon \quad &\Leftrightarrow \\ \frac{1}{2}\left| x - 4 \right| < \varepsilon \quad (\text{by } *) \quad &\Leftrightarrow \\ \left| x - 4 \right| < 2\varepsilon & \end{aligned}$$

We choose $\delta = 2\varepsilon$. We will formally justify this choice in our verification step.

Observe that, since $\varepsilon > 0$, then our $\delta > 0$.

Verify that our choice for δ is appropriate.

We will show that, given ε and our choice for δ $(\delta = 2\varepsilon)$,
$0 < \left| x - a \right| < \delta \Rightarrow \left| f(x) - L \right| < \varepsilon$.

$$\begin{aligned} 0 < \left| x - a \right| < \delta &\Rightarrow \\ 0 < \left| x - 4 \right| < \delta &\Rightarrow \\ 0 < \left| x - 4 \right| < 2\varepsilon &\Rightarrow \\ 0 < \frac{1}{2}\left| x - 4 \right| < \varepsilon &\Rightarrow \\ \left| f(x) - L \right| < \varepsilon \quad &(\text{by } *) \end{aligned}$$

<u>Note</u>: It is true that: $0 < \left| f(x) - L \right| < \varepsilon$, but the first inequality $\left(0 < \left| f(x) - L \right|\right)$ does not help us.

Q.E.D.

(“Quod erat demonstrandum” – Latin for “which was to be demonstrated / proven / shown.” This is a formal end to a proof.) §

PART C: DEFINING ONE-SIDED LIMITS AT A POINT

The precise definition of $\lim_{x \to a} f(x) = L$ can be modified for **left-hand** and **right-hand** limits. The **only** changes are the ***x*-intervals** where we look for winners. (See red type.) These *x*-intervals will no longer be symmetric about *a*.

- Therefore, we will use **interval form** instead of absolute value notation when describing these *x*-intervals.
- Also, we will let δ represent the **entire width** of an *x*-interval, not just half the width of a punctured *x*-interval.

The Precise ε-δ Definition of a Left-Hand Limit at a Point

For $a, L \in \mathbb{R}$, if a function f is defined on a **left-neighborhood** of a,

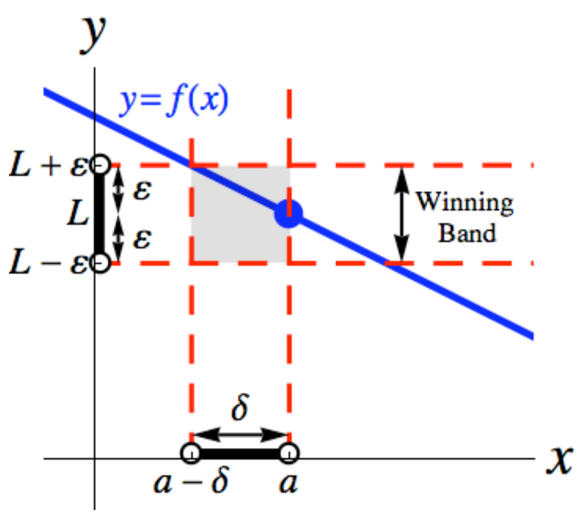
$$\lim_{x \to a^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni$$
$$\left[x \in (a - \delta, a) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

The Precise ε-δ Definition of a Right-Hand Limit at a Point

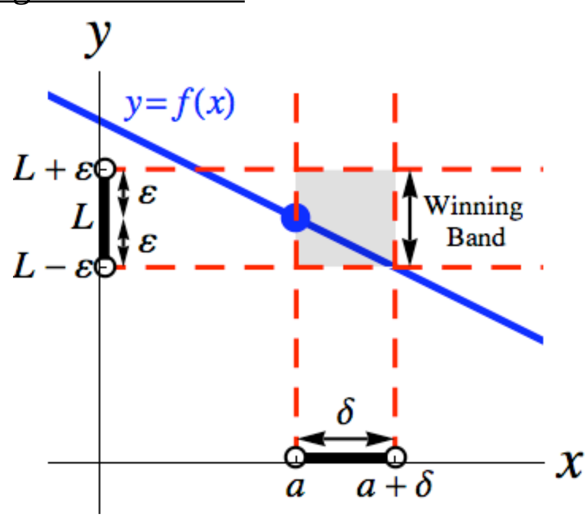
For $a, L \in \mathbb{R}$, if a function f is defined on a **right-neighborhood** of a,

$$\lim_{x \to a^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni$$
$$\left[x \in (a, a + \delta) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

Left-Hand Limit



Right-Hand Limit



PART D: DEFINING "LONG-RUN" LIMITS

The precise definition of $\lim_{x \to a} f(x) = L$ can also be modified for **"long-run"** limits. Again, the **only** changes are the ***x*-intervals** where we look for winners. (See red type.) These *x*-intervals will be unbounded.

• Therefore, we will use **interval form** instead of absolute value notation when describing these *x*-intervals.

• Also, instead of using δ, we will use *M* (think "Million") and *N* (think "Negative million") to denote "points of no return."

The Precise ε-M Definition of $\lim_{x \to \infty} f(x) = L$

For $L \in \mathbb{R}$, if a function f is defined on some interval (c, ∞), $c \in \mathbb{R}$.

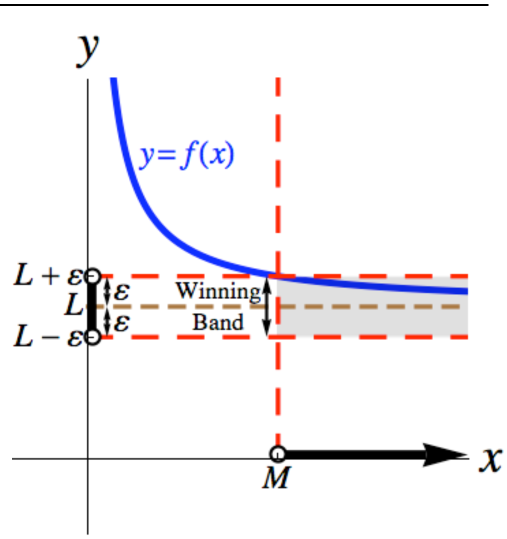
$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists M \in \mathbb{R} \ni$$
$$\left[x > M;\ \text{that is, } x \in (M, \infty) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

The Precise ε-N Definition of $\lim_{x \to -\infty} f(x) = L$

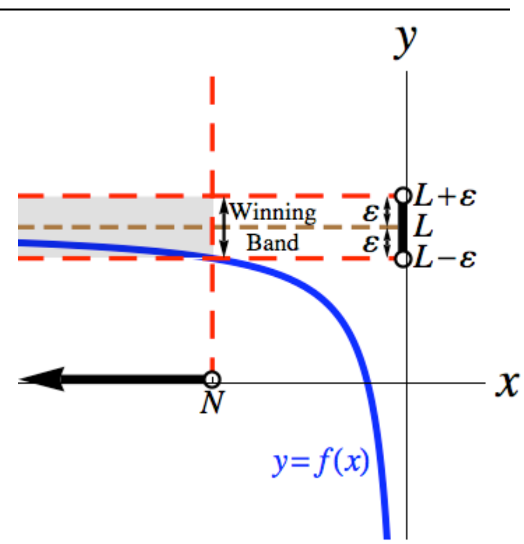
For $L \in \mathbb{R}$, if a function f is defined on some interval $(-\infty, c)$, $c \in \mathbb{R}$.

$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0,\ \exists N \in \mathbb{R} \ni$$
$$\left[x < N;\ \text{that is, } x \in (-\infty, N) \Rightarrow \left| f(x) - L \right| < \varepsilon \right].$$

$\lim_{x \to \infty} f(x) = L$; here, $f(x) = \frac{1}{x} + 2$



$\lim_{x \to -\infty} f(x) = L$; here, $f(x) = \frac{1}{x} + 2$

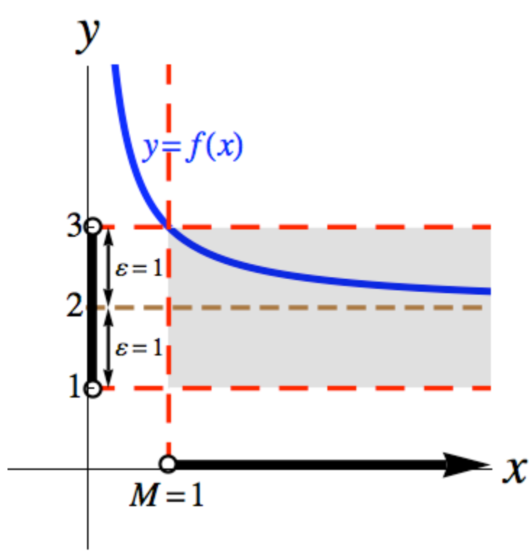


How Does the "Static" Approach to "Long-Run" Limits Relate to the "Dynamic" Approach?

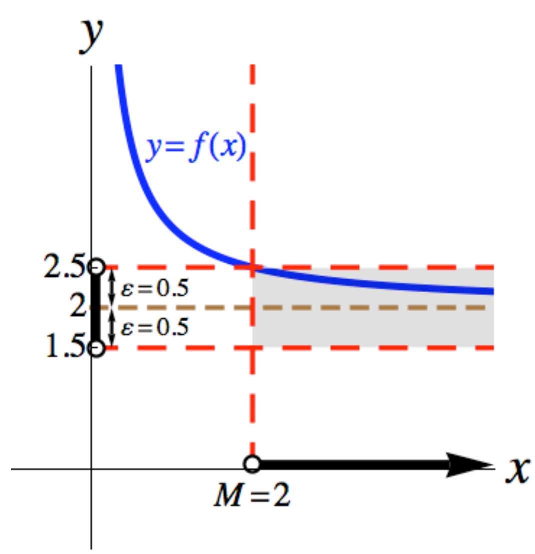
Why is $\lim\limits_{x \to \infty} \left(\frac{1}{x} + 2 \right) = 2$? Because, **regardless of how small** we make the **tolerance level** ε and how tight we make the lottery for the players, **there is** a "point of no return" M **after which all the players win**. That is, the corresponding box (see the shaded boxes in the figures below) **traps the graph** of $y = f(x)$ for all $x > M$.

As $\varepsilon \to 0^+$, we can **choose values** for M in such a way that the corresponding shaded boxes **always trap the graph** and zoom in, or collapse in, on the **HA** $y = 2$. In other words, **there are always winners as** $x \to \infty$.

If $\varepsilon = 1$, we can choose $M = 1$.



If $\varepsilon = 0.5$, we can choose $M = 2$.



For this example, if ε is **any positive real number**, we can choose $M = \frac{1}{\varepsilon}$.

PART E: DEFINING INFINITE LIMITS AT A POINT

Challenge to the reader:

Give precise "M-δ" and "N-δ" definitions of $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} f(x) = -\infty$ $(a \in \mathbb{R})$, where the function f is defined on a punctured neighborhood of *a*.

SECTION 2.8: CONTINUITY

LEARNING OBJECTIVES

• Understand and know the definitions of continuity at a point (in a one-sided and two-sided sense), on an open interval, on a closed interval, and variations thereof.
• Be able to identify discontinuities and classify them as removable, jump, or infinite.
• Know properties of continuity, and use them to analyze the continuity of rational, algebraic, and trigonometric functions and compositions thereof.
• Understand the Intermediate Value Theorem (IVT) and apply it to solutions of equations and real zeros of functions.

PART A: CONTINUITY AT A POINT

Informally, a function f with domain $\mathbb{R}$ is everywhere continuous (on $\mathbb{R}$) $\Leftrightarrow$ we can take a pencil and **trace** the graph of f between any two distinct points on the graph **without** having to lift up our pencil.

We will make this idea more precise by first defining continuity **at a point** a $(a \in \mathbb{R})$ and then continuity **on intervals**.

Continuity at a Point a

f is continuous at $x = a$ $\Leftrightarrow$

1) $f(a)$ is **defined (real)**; that is, $a \in \operatorname{Dom}(f)$,
2) $\lim_{x \to a} f(x)$ **exists (is real)**, and
3) $\lim_{x \to a} f(x) = f(a)$.

f is discontinuous at $x = a$ $\Leftrightarrow$ f is not continuous at $x = a$.

Comments

1) ensures that there is literally a **point** at $x = a$.
2) **constrains** the behavior of f **immediately around** $x = a$.
3) then ensures **"safe passage"** through the point $(a, f(a))$ on the graph of $y = f(x)$. Some sources just state 3) in the definition, since the form of 3) implies 1) and 2).

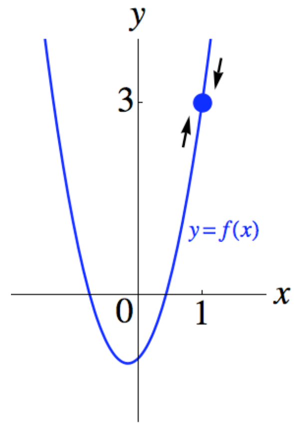
Example 1 (Continuity at a Point; Revisiting Section 2.1, Example 1)

Let $f(x) = 3x^2 + x - 1$. Show that f is continuous at $x = 1$.

§ Solution

1) $f(1) = 3$, a real number $(1 \in \text{Dom}(f))$
2) $\lim_{x \to 1} f(x) = 3$, a real number, and
3) $\lim_{x \to 1} f(x) = f(1)$.

Therefore, f is continuous at $x = 1$. The graph of $y = f(x)$ is below.



Note: The **Basic Limit Theorem for Rational Functions** in Section 2.1 basically states that a **rational** function is **continuous** at any number in its **domain**. §

Example 2 (Discontinuities at a Point; Revisiting Section 2.2, Example 2)

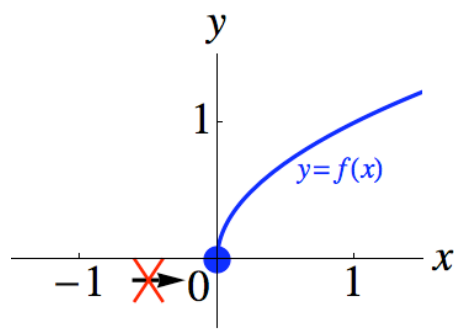
Let $f(x) = \sqrt{x}$. Explain why f is discontinuous at $x = -1$ and $x = 0$.

§ Solution

- $f(-1)$ is **not real** $(-1 \notin \text{Dom}(f))$, so f is **discontinuous** at $x = -1$.

- $f(0) = 0$, but $\lim_{x \to 0} \sqrt{x}$ **does not exist (DNE)**, so f is **discontinuous** at $x = 0$.

The graph of $y = f(x)$ is below.



Some sources do not even bother calling -1 and 0 "discontinuities" of f, since f is **not even defined** on a **punctured neighborhood** of $x = -1$ or of $x = 0$. §

PART B: CLASSIFYING DISCONTINUITIES

We now consider cases where a function f is **discontinuous** at $x = a$, even though f is defined on a punctured neighborhood of $x = a$.

We will **classify** such discontinuities as removable, jump, or infinite.
(See Footnotes 1 and 2 for another type of discontinuity.)

<ins>Removable Discontinuities</ins>

A function f has a <ins>removable discontinuity</ins> at $x = a$ $\Leftrightarrow$

1) $\lim_{x \to a} f(x)$ **exists** (call this limit L), but
2) f is still **discontinuous** at $x = a$.

• Then, the graph of $y = f(x)$ has a **hole** at the point (a, L).

Example 3 (Removable Discontinuity at a Point; Revisiting Section 2.1, Ex. 7)

Let $g(x) = x + 3,\ (x \neq 3)$. **Classify** the discontinuity at $x = 3$.

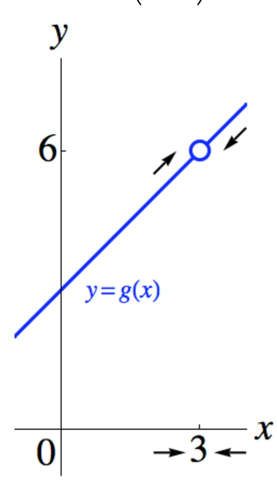
§ Solution

g has a **removable discontinuity** at $x = 3$, because:

1) $\lim_{x \to 3} g(x) = 6$, but

2) g is still discontinuous at $x = 3$; here, $g(3)$ is undefined.

The graph of $y = g(x)$ below has a **hole** at the point $(3, 6)$.



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Example 4 (Removable Discontinuity at a Point; Revisiting Section 2.1, Ex. 8)

Let $h(x)=\begin{cases} x+3, & x\neq 3 \\ 7, & x=3 \end{cases}$. **Classify** the discontinuity at $x=3$.

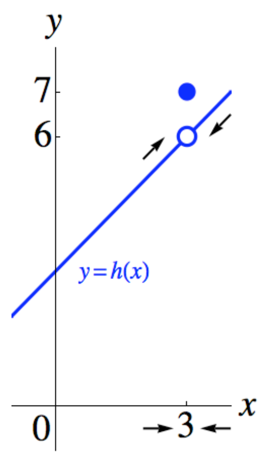
§ Solution

h has a **removable discontinuity** at $x=3$, because:

1) $\lim_{x\to 3} h(x)=6$, but

2) h is still discontinuous at $x=3$;
here, $\lim_{x\to 3} h(x)\neq h(3)$,
because $6\neq 7$.

The graph of $y=h(x)$ also has a **hole** at the point $(3, 6)$.



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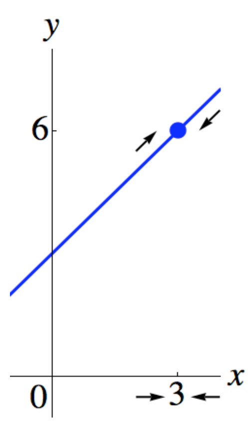
Why are These Discontinuities Called "Removable"?

The term "removable" is a bit of a misnomer here, since we have no business changing the function at hand.

The idea is that a removable discontinuity at *a* can be removed by (re)defining the function at *a*; the new function will then be continuous at *a*.

For example, if we were to **define** $g(3)=6$ in Example 3 and **redefine** $h(3)=6$ in Example 4, then we would remove the discontinuity at $x=3$ in both situations. We would obtain the graph below.



Jump Discontinuities

A function f has a jump discontinuity at $x = a \Leftrightarrow$

1) $\lim\limits_{x \to a^-} f(x)$ **exists**, and (call this limit L_1)

2) $\lim\limits_{x \to a^+} f(x)$ **exists**, but (call this limit L_2)

3) $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$. $(L_1 \neq L_2)$

• Therefore, $\lim\limits_{x \to a} f(x)$ **does not exist (DNE)**.

• It is **irrelevant** how $f(a)$ is defined, or if it is defined at all.

Example 5 (Jump Discontinuity at a Point; Revisiting Section 2.1, Example 14)

Let $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & \text{if } x > 0 \\ \dfrac{-x}{x} = -1, & \text{if } x < 0 \end{cases}$. **Classify** the discontinuity at $x = 0$.

§ Solution

f has a **jump discontinuity** at $x = 0$, because:

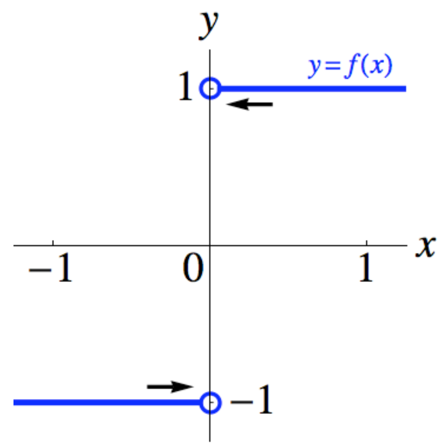
1) $\lim\limits_{x \to 0^-} f(x) = -1$, and

2) $\lim\limits_{x \to 0^+} f(x) = 1$, but

3) $\lim\limits_{x \to 0} f(x)$ does not exist (DNE), because $-1 \neq 1$.

We **cannot remove** this discontinuity by defining $f(0)$.

The graph of $y = f(x)$ is below.



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Infinite Discontinuities

A function f has an infinite discontinuity at $x = a$ $\Leftrightarrow$

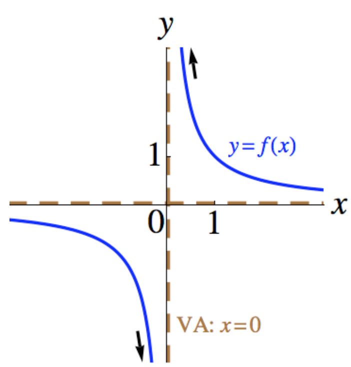
$\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ is ∞ or $-\infty$.

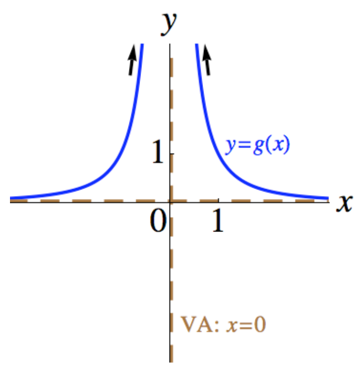
- That is, the graph of $y = f(x)$ has a **VA** at $x = a$.
- It is **irrelevant** how $f(a)$ is defined, or if it is defined at all.

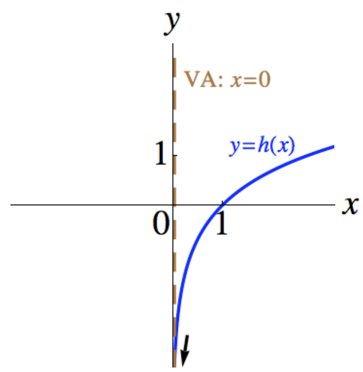
Example 6 (Infinite Discontinuities at a Point; Revisiting Section 2.4, Exs. 1 and 2)

The functions described below have **infinite discontinuities** at $x = 0$.
We will study $\ln x$ in Chapter 7 (see also the Precalculus notes, Section 3.2).

$f(x) = \frac{1}{x}$ $\qquad$ $g(x) = \frac{1}{x^2}$ $\qquad$ $h(x) = \ln x$







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PART C: CONTINUITY ON AN OPEN INTERVAL

We can extend the concept of continuity in various ways.
(For the remainder for this section, assume $a < b$.)

> Continuity on an Open Interval
>
> A function f is continuous on the **open interval** (a, b) $\Leftrightarrow$
> f is continuous at **every number (point)** in (a, b).
>
> • This extends to **unbounded** open intervals of the form (a, ∞), $(-\infty, b)$, and $(-\infty, \infty)$.

In Example 6, all three functions are continuous on the interval $(0, \infty)$.

The first two functions are also continuous on the interval $(-\infty, 0)$.

We will say that the "continuity intervals" of the first two functions are: $(-\infty, 0)$, $(0, \infty)$. However, this terminology is **not standard**.

• In Footnote 1, f has the singleton (one-element) set $\{0\}$ as a "degenerate continuity interval." See also Footnotes 2 and 3.

• Avoid using the union $(\cup)$ symbol here. In Section 2.1, Example 10, f was continuous on $(-\infty, 0]$ and $(0, 1)$, but not on $(-\infty, 1)$.

PART D: CONTINUITY ON OTHER INTERVALS; ONE-SIDED CONTINUITY

> Continuity on a Closed Interval
>
> A function f is continuous on the **closed interval** $[a, b]$ $\Leftrightarrow$
>
> 1) f is defined on $[a, b]$,
> 2) f is continuous on (a, b),
> 3) $\lim_{x \to a^+} f(x) = f(a)$, and
> 4) $\lim_{x \to b^-} f(x) = f(b)$.

3) and 4) **weaken** the continuity requirements at the **endpoints**, *a* and *b*.
Imagine taking limits as we **"push outwards"** towards the endpoints.

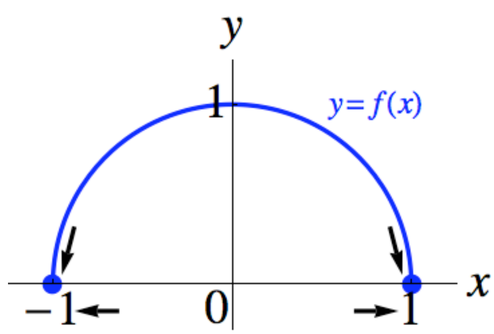
3) implies that f is continuous from the right at *a*.
4) implies that f is continuous from the left at *b*.

Example 7 (Continuity on a Closed Interval)

Let $f(x) = \sqrt{1 - x^2}$. Show that f is continuous on the closed interval $[-1, 1]$.

§ Solution

The graph of $y = f(x)$ is below.



$$y = \sqrt{1 - x^2} \quad \Leftrightarrow$$
$$y^2 = 1 - x^2 \quad (y \geq 0) \quad \Leftrightarrow$$
$$x^2 + y^2 = 1 \quad (y \geq 0)$$

The graph is the upper half of the unit circle centered at the origin, including the points $(-1, 0)$ and $(1, 0)$.

f is continuous on $[-1, 1]$, because:

1) f is defined on $[-1, 1]$,
2) f is continuous on $(-1, 1)$,
3) $\lim_{x \to -1^+} f(x) = f(-1)$, so f is continuous from the **right** at -1, and
4) $\lim_{x \to 1^-} f(x) = f(1)$, so f is continuous from the **left** at 1.

Note: $f(-1) = 0$, and $f(1) = 0$, but they need not be equal.

f has $[-1, 1]$ as its sole "**continuity interval.**" When giving "continuity intervals," we include brackets where appropriate, even though f is **not continuous** (in a two-sided sense) at -1 and at 1 (**WARNING 1**).

• Some sources would call $(-1, 1)$ the continuity set of f; it is the set of all real numbers at which f is continuous. (See Footnotes 2 and 3.) §

Challenge to the Reader: Draw a graph where f is defined on $[a, b]$, and f is continuous on (a, b), but f is **not** continuous on the closed interval $[a, b]$.

Continuity on Half-Open, Half-Closed Intervals

f is continuous on an interval of the form $[a, b)$ or $[a, \infty)$ $\Leftrightarrow$ f is continuous on (a, b) or (a, ∞), respectively, and it is continuous **from the right** at a.

f is continuous on an interval of the form $(a, b]$ or $(-\infty, b]$ $\Leftrightarrow$ f is continuous on (a, b) or $(-\infty, b)$, respectively, and it is continuous **from the left** at b.

Example 8 (Continuity from the Right; Revisiting Example 2)

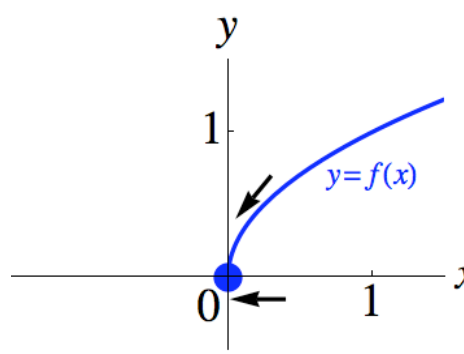
Let $f(x) = \sqrt{x}$.

f is continuous on $(0, \infty)$.

$\lim_{x \to 0^+} \sqrt{x} = 0 = f(0)$, so f is continuous **from the right** at 0.

The sole **"continuity interval"** of f is $[0, \infty)$.

The graph of $y = f(x)$ is below.



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Example 9 (Continuity from the Left)

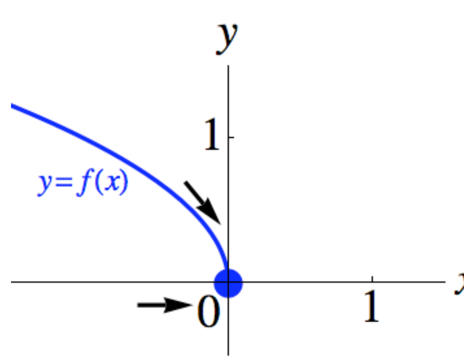
Let $f(x) = \sqrt{-x}$.

f is continuous on $(-\infty, 0)$.

$\lim_{x \to 0^-} \sqrt{-x} = 0 = f(0)$, so f is continuous **from the left** at 0.

The sole **"continuity interval"** of f is $(-\infty, 0]$.

The graph of $y = f(x)$ is below.



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PART E: CONTINUITY THEOREMS

Properties of Continuity / Algebra of Continuity Theorems

If f and g are functions that are continuous at $x = a$, then so are the functions:

- $f + g$, $f - g$, and fg.
- $\dfrac{f}{g}$, if $g(a) \neq 0$.
- f^n, if n is a positive integer exponent $\left(n \in \mathbb{Z}^+\right)$.
- $\sqrt[n]{f}$, if:
  - (n is an **odd** positive integer), or
  - (n is an **even** positive integer, **and** $f(a) > 0$).

In Section 2.2, we showed how similar properties of **limits** justified the Basic Limit Theorem for Rational Functions. Similarly, the properties above, together with the fact that **constant** functions and the **identity** function (represented by $f(x) = x$) are **everywhere continuous** on $\mathbb{R}$, justify the following:

Continuity of Rational Functions

A rational function is continuous on its domain.

- That is, the "continuity interval(s)" of a rational function f **are** its domain interval(s).

In particular, **polynomial** functions are **everywhere continuous** (on $\mathbb{R}$).

Although this is typically true for **algebraic** functions in general, there are counterexamples (see Footnote 4).

Example 10 (Continuity of a Rational Function; Revisiting Example 6)

If $f(x) = \dfrac{1}{x}$, then $\text{Dom}(f) = (-\infty, 0) \cup (0, \infty)$.

f is **rational**, so the "**continuity intervals**" of f are: $(-\infty, 0)$, $(0, \infty)$. §

When analyzing the continuity of functions that are **not rational**, we may need to check for **one-sided** continuity at **endpoints** of domain intervals.

Example 11 (Continuity of an Algebraic Function; Revisiting Chapter 1, Ex. 6)

Let $h(x) = \dfrac{\sqrt{x+3}}{x-10}$. What are the **"continuity intervals"** of h?

§ Solution

In Chapter 1, we found that $\mathrm{Dom}(h) = [-3, 10) \cup (10, \infty)$.

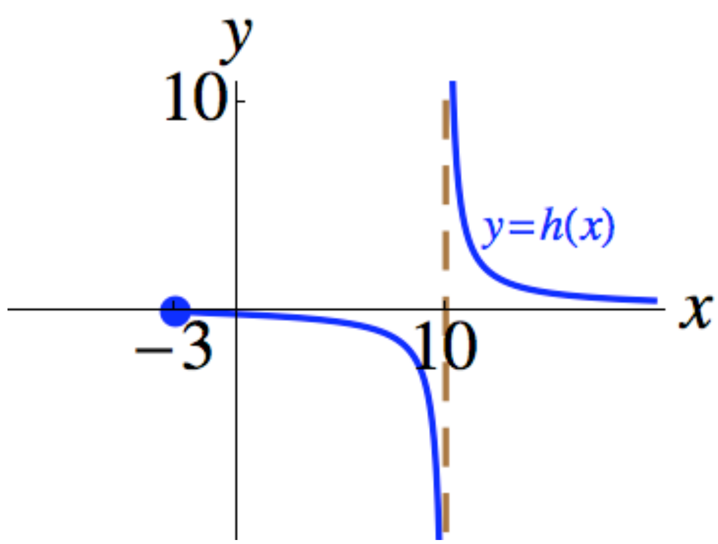
We will show that the **"continuity intervals"** are, in fact, the **domain intervals**, $[-3, 10)$ and $(10, \infty)$.

By the **Algebra of Continuity Theorems**, we find that h is continuous on $(-3, 10)$ and $(10, \infty)$.

Now, $\lim\limits_{x \to -3^+} h(x) = 0 = h(-3)$, because $\left(\text{Limit Form } \dfrac{\sqrt{0^+}}{-13} \right) \Rightarrow 0$.

Therefore, h is continuous **from the right** at $x = -3$, and its **"continuity intervals"** are: $[-3, 10)$ and $(10, \infty)$.

The graph of $y = h(x)$ is below.



§

<u>Continuity of Composite Functions</u>

If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

(See Footnote 5.)

<u>Continuity of Basic Trigonometric Functions</u>

The six basic **trigonometric** functions (sine, cosine, tangent, cosecant, secant, and cotangent) are **continuous** on their **domain intervals**.

Example 12 (Continuity of a Composite Function)

Let $h(x) = \sec\left(\dfrac{1}{x}\right)$. Where is h continuous?

§ Solution

Observe that $h(x) = (f \circ g)(x) = f(g(x))$, where:

the **"inside"** function is given by $g(x) = \dfrac{1}{x}$, and

the **"outside"** function f is given by $f(\theta) = \sec\theta$, where $\theta = \dfrac{1}{x}$.

g is continuous at all real numbers **except** 0 $(x \neq 0)$.
f is continuous on its domain intervals.

$$\begin{aligned}
\sec\theta \text{ is real} \quad &\Leftrightarrow \quad \cos\theta \neq 0, \text{ and } x \neq 0 \\
&\Leftrightarrow \quad \theta \neq \frac{\pi}{2} + \pi n \ \ (n \in \mathbb{Z}), \text{ and } x \neq 0 \\
&\Leftrightarrow \quad \frac{1}{x} \neq \frac{\pi}{2} + \pi n \ \ (n \in \mathbb{Z}), \text{ and } x \neq 0
\end{aligned}$$

We can replace both sides of the inequation with their **reciprocals**, because we exclude the case $x = 0$, and both sides are never 0.

$$\Leftrightarrow \quad x \neq \frac{1}{\dfrac{\pi}{2} + \pi n} \ \ (n \in \mathbb{Z}), \text{ and } x \neq 0$$

$$\Leftrightarrow \quad x \neq \frac{1}{\left(\frac{\pi}{2} + \pi n\right)} \cdot \frac{2}{2} \quad (n \in \mathbb{Z}), \text{ and } x \neq 0$$

$$\Leftrightarrow \quad x \neq \frac{2}{\pi + 2\pi n} \quad (n \in \mathbb{Z}), \text{ and } x \neq 0$$

h is continuous on:

$$\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{2}{\pi + 2\pi n} \ (n \in \mathbb{Z}), \text{ and } x \neq 0 \right\}, \text{ or}$$

$$\left\{ x \in \mathbb{R} \,\middle|\, x \neq \frac{2}{\pi(2n+1)} \ (n \in \mathbb{Z}), \text{ and } x \neq 0 \right\}.$$

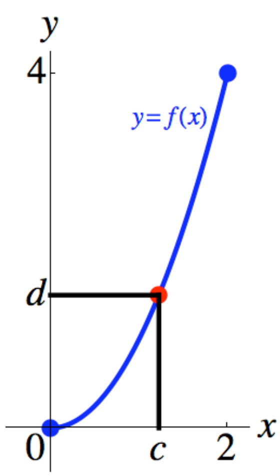
§

<u>PART F: THE INTERMEDIATE VALUE THEOREM (IVT)</u>

Continuity of a function **constrains** its behavior in important (and useful) ways. Continuity is central to some key theorems in calculus. We will see the <u>Extreme Value Theorem (EVT)</u> in Chapter 4 and <u>Mean Value Theorems (MVTs)</u> in Chapters 4 and 5. We now discuss the <u>Intermediate Value Theorem (IVT)</u>, which directly relates to the **meaning** of continuity. We will motivate it before stating it.

<u>Example 13 (Motivating the IVT)</u>

Let $f(x) = x^2$ on the x-interval $[0, 2]$. The graph of $y = f(x)$ is below.



f is **continuous** on $[0, 2]$,
$f(0) = 0$, and
$f(2) = 4$.

The **IVT** guarantees that **every** real number (d) **between** 0 and 4 is a value of (is taken on by) f at **some** x-value (c) in $[0, 2]$. §

The Intermediate Value Theorem (IVT): Informal Statement

If a function f is **continuous** on the **closed** interval $[a, b]$,
then f takes on **every** real number **between** $f(a)$ and $f(b)$ on $[a, b]$.

The Intermediate Value Theorem (IVT): Precise Statement

Let $\min(f(a), f(b))$ be the **lesser** of $f(a)$ and $f(b)$;
if they are equal, then we take their common value.

Let $\max(f(a), f(b))$ be the **greater** of $f(a)$ and $f(b)$;
if they are equal, then we take their common value.

A function f is **continuous** on $[a, b]$ $\Rightarrow$
$\forall d \in [\min(f(a), f(b)),\ \max(f(a), f(b))],\ \exists c \in [a, b] \ni f(c) = d$.

Example 14 (Applying the IVT to Solutions of Equations)

Prove that $x^2 = 3$ has a **solution** in $[0, 2]$.

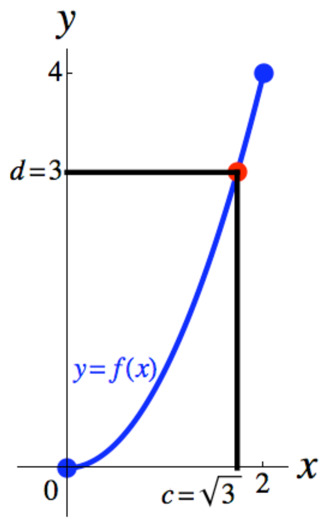
§ Solution

Let $f(x) = x^2$. (We also let the desired **function value**, $d = 3$.)

f is **continuous** on $[0, 2]$,
$f(0) = 0$,
$f(2) = 4$, **and**
$3 \in [0, 4]$.

Therefore, by the **IVT**, $\exists c \in [0, 2] \ni$ (such that) $f(c) = 3$.
That is, $x^2 = 3$ has a **solution** (c) in $[0, 2]$.
Q.E.D. §

In Example 14, $c = \sqrt{3}$ was our solution to $x^2 = 3$ in $[0,2]$; $d = 3$ here.



To **verify the conclusion of the IVT** in general, we can give a **formula** for *c* given **any** real number *d* in $[0, 4]$, where $c \in [0,2]$ and $f(c) = d$.

Example 15 (Verifying the Conclusion of the IVT; Revisiting Examples 13 and 14)

Verify the conclusion of the IVT for $f(x) = x^2$ on the *x*-interval $[0,2]$.

§ Solution

f is **continuous** on $[0, 2]$, so the IVT applies. $f(0) = 0$, and $f(2) = 4$.
Let $d \in [0, 4]$, and let $c = \sqrt{d}$.

- The following justifies our **formula** for c:

$$\begin{aligned} f(c) = d \text{ and } c \in [0, 2] \quad &\Leftrightarrow \\ c^2 = d \text{ and } c \in [0, 2] \quad &\Leftrightarrow \\ c = \sqrt{d}, \text{ a real number in } [0, 2] & \end{aligned}$$

WARNING 2: We do not write $c = \pm\sqrt{d}$, because either $d = 0$, or a value for *c* would fall outside of $[0, 2]$.

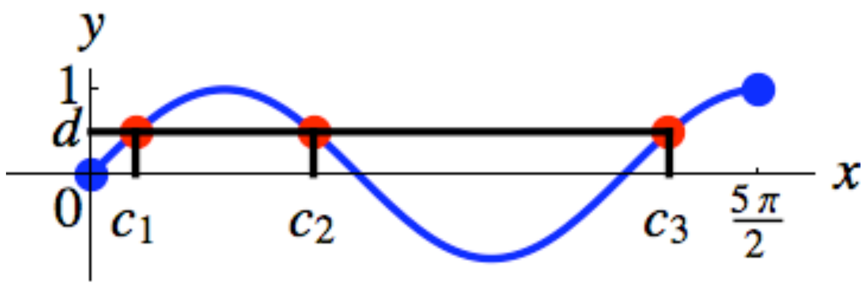
Observe: $0 \le d \le 4 \Rightarrow 0 \le \sqrt{d} \le 2$.

Then, $c \in [0, 2]$, and $f(c) = c^2 = \left(\sqrt{d}\right)^2 = d$.

Therefore, $\forall d \in [0, 4], \exists c \in [0, 2] \ni f(c) = d$. §

Example 16 (c Might Not Be Unique)

Let $f(x) = \sin x$ on the x-interval $\left[0, \frac{5\pi}{2}\right]$. The graph of $y = f(x)$ is below.



$f(0) = 0$, and $f\left(\frac{5\pi}{2}\right) = 1$. Because f is **continuous** on $\left[0, \frac{5\pi}{2}\right]$, the IVT guarantees that **every** real number d **between** 0 and 1 is taken on by f at **some** x-value c in $\left[0, \frac{5\pi}{2}\right]$.

WARNING 3: Given an appropriate value for d, there **might be more than one** appropriate choice for c. The IVT does not forbid that.

WARNING 4: Also, there are real numbers **outside of** $[0, 1]$ that are taken on by f on the x-interval $\left[0, \frac{5\pi}{2}\right]$. The IVT does not forbid that, either. §

PART G: THE BISECTION METHOD FOR APPROXIMATING A ZERO OF A FUNCTION

Our ability to **solve equations** is equivalent to our ability to **find zeros** of functions. For example, $f(x) = g(x) \Leftrightarrow f(x) - g(x) = 0$; we can solve the first equation by finding the zeros of $h(x)$, where $h(x) = f(x) - g(x)$.

We may have to use computer algorithms to **approximate zeros** of functions if we can't find them exactly.

- While we do have (nastier) analogs of the Quadratic Formula for 3^{rd}- and 4^{th}-degree polynomial functions, it has actually been proven that there is **no similar formula** for higher-degree polynomial functions.

The Bisection Method, which is the basis for some of these algorithms, uses the **IVT** to produce a **sequence of smaller and smaller intervals** that are guaranteed to contain a **zero** of a given function.

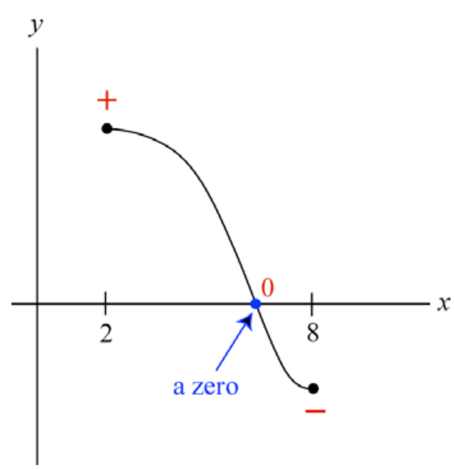
The Bisection Method for Approximating a Zero of a Continuous Function f

Let's say we want to **approximate a zero** of a function f.

Find x-values a_1 and b_1 $(a_1 < b_1)$ such that $f(a_1)$ and $f(b_1)$ have **opposite signs** and f is **continuous** on $[a_1, b_1]$. (The method fails if such x-values cannot be found.)

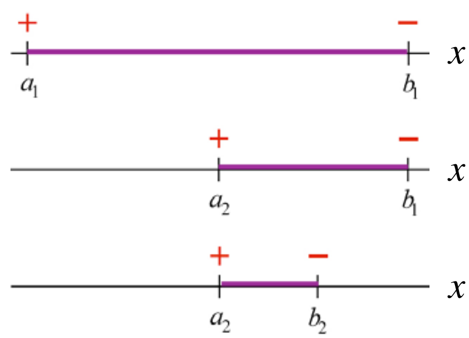
According to the **IVT**, there must be a **zero** of f in $[a_1, b_1]$, which we call our **"search interval."**

For example, consider the graph of $y = f(x)$ below. Our search interval is apparently $[2, 8]$.



If $f(a_1)$ or $f(b_1)$ were 0, then we would have found a **zero** of f, and we could either stop or try to approximate another zero.

If neither is 0, then we take the **midpoint** of the search interval and determine the sign of $f(x)$ there (in red below). We can then **shrink the search interval** (in purple below) and **repeat** the process. We call the Bisection Method an iterative method because of this repetition.



We stop when we **find a zero**, or until the search interval is **small enough** so that we are satisfied with taking its **midpoint** as our approximation.

A key drawback to numerical methods such as the Bisection Method is that, unless we manage to find n distinct real zeros of an n^{th}-degree polynomial $f(x)$, we may need other techniques to be sure that we have found **all** of the real zeros, if we are looking for all of them. §

Example 17 (Applying the Bisection Method; Revisiting Example 14)

We can approximate $\sqrt{3}$ by approximating the positive real **solution** of $x^2 = 3$, or the positive real **zero** of $h(x)$, where $h(x) = x^2 - 3$.

Search interval $[a, b]$	Sign of $h(a)$	Sign of $h(b)$	Midpoint	Sign of h there
$[0, 2]$	−	+	1	−
$[1, 2]$	−	+	1.5	−
$[1.5, 2]$	−	+	1.75	+
$[1.5, 1.75]$	−	+	1.625	−

etc. §

In Section 4.8, we will use Newton's Method for approximating zeros of a function, which tends to be more efficient. However, Newton's Method requires **differentiability** of a function, an idea we will develop in Chapter 3.

FOOTNOTES

1. **A function with domain $\mathbb{R}$ that is only continuous at 0.** (Revisiting Footnote 1 in Section 2.1.) Let $f(x)=\begin{cases} 0, & \text{if } x \text{ is a rational number } \left(x\in\mathbb{Q}\right) \\ x, & \text{if } x \text{ is an irrational number } \left(x\notin\mathbb{Q};\ \text{really, } x\in\mathbb{R}\setminus\mathbb{Q}\right)\end{cases}$.
f is continuous at $x=0$, because $f(0)=0$, and we can use the Squeeze (Sandwich) Theorem to prove that $\lim\limits_{x\to 0} f(x)=0$, also. The discontinuities at the nonzero real numbers are not categorized as removable, jump, or infinite.

2. **Continuity sets and a nowhere continuous function.** See *Cardinality of the Set of Real Functions With a Given Continuity Set* by Jiaming Chen and Sam Smith. The 19th-century German mathematician Dirichlet came up with a nowhere continuous function, *D*:
$$D(x)=\begin{cases} 0, & \text{if } x \text{ is a rational number } \left(x\in\mathbb{Q}\right) \\ 1, & \text{if } x \text{ is an irrational number } \left(x\notin\mathbb{Q};\ \text{really, } x\in\mathbb{R}\setminus\mathbb{Q}\right)\end{cases}$$

3. **Continuity on a set.** This is tricky to define! See "Continuity on a Set" by R. Bruce Crofoot, *The College Mathematics Journal*, Vol. 26, No. 1 (Jan. 1995) by the Mathematical Association of America (MAA). Also see Louis A. Talman, *The Teacher's Guide to Calculus* (web). Talman suggests:
Let *S* be a subset of $\text{Dom}(f)$; that is, $S\subseteq \text{Dom}(f)$. f is continuous on *S* $\Leftrightarrow$
$\forall a\in S,\ \forall \varepsilon>0,\ \exists \delta>0 \ni \left[\left(x\in S \text{ and } \left|x-a\right|<\delta\right) \Rightarrow \left|f(x)-f(a)\right|<\varepsilon\right]$.
• The definition essentially states that, for every number *a* in the set of interest, its function value is arbitrarily close to the function values of nearby *x*-values in the set. Note that we use $f(a)$ instead of *L*, which we used to represent $\lim\limits_{x\to a} f(x)$, because we need $\lim\limits_{x\to a} f(x)=f(a)$ (or possibly some one-sided variation) in order to have continuity on *S*.
• This definition covers / subsumes our definitions of continuity on open intervals; closed intervals; half-open, half-closed intervals; and unions (collections) thereof.
• One possible criticism against this definition is that it implies that the functions described in Footnote 4 are, in fact, continuous on the singleton set {0}. This conflicts with our definition of continuity at a point in Part A because of the issue of nonexistent limits. Perhaps we should require that f be defined on some interval of the form $\left[a, c\right)$ with $c>a$ or the form $\left(c, a\right]$ with $c<a$.
• Crofoot argues for the following definition: f is continuous on *S* if the restriction of f to *S* is continuous at each number in *S*. He acknowledges the use of one-sided continuity when dealing with closed intervals.

4. **An algebraic function that is not continuous on its domain.** Let $f(x)=\sqrt{x}+\sqrt{-x}$.
$\text{Dom}(f)=\left\{0\right\}$, a singleton (a set consisting of a single element), but f is not continuous at 0 (by Part A), because $\lim\limits_{x\to 0} f(x)$ does not exist (DNE). The same is true for $f(x)=\sqrt{-x^2}$.

5. **Continuity and the limit properties in Section 2.2, Part A.** Let $a, K \in \mathbb{R}$.
If $\lim_{x \to a} g(x) = K$, and f is continuous at K, then:
$\lim_{x \to a} (f \circ g)(x) = \lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(K)$. Basically, continuity allows f to commute with a limit operator: $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$. Think: "The limit of a (blank) is the (blank) of the limit." This relates to Property 5) on the limit of a power, Property 6) on the limit of a constant multiple, and Property 7) on the limit of a root in Section 2.2.
For example, f could represent the squaring function.

6. **A function that is continuous at every irrational point and discontinuous at every rational point.** See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.27. Also see Tom Vogel, http://www.math.tamu.edu/~tvogel/gallery/node6.html (web). If x is rational, where $x = \frac{a}{b}$ $(a, b \in \mathbb{Z})$, $b > 0$, and the fraction is simplified, then let $f(x) = \frac{1}{b}$. If x is irrational, let $f(x) = 0$. Vogel calls this the "ruler function," appealing to the image of markings on a ruler. However, there does not exist a function that is continuous at every rational point and discontinuous at every irrational point.

7. **An everywhere continuous function that is nowhere monotonic (either increasing or decreasing).** See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.29. There is no open interval on which the function described there is either increasing or decreasing.