# CHAPTER 3:

# *Derivatives*



- Derivatives represent slopes of tangent lines and rates of change (such as velocity).
- In this chapter, we will define derivatives and derivative functions using limits.
- We will develop short cut techniques for finding derivatives.
- Tangent lines correspond to local linear approximations of functions.
- Implicit differentiation is a technique used in applied related rates problems.

# SECTION 3.1: DERIVATIVES, TANGENT LINES, AND RATES OF CHANGE

**LEARNING OBJECTIVES**

- Relate difference quotients to slopes of secant lines and average rates of change.
- Know, understand, and apply the Limit Definition of the Derivative at a Point.
- Relate derivatives to slopes of tangent lines and instantaneous rates of change.
- Relate opposite reciprocals of derivatives to slopes of normal lines.

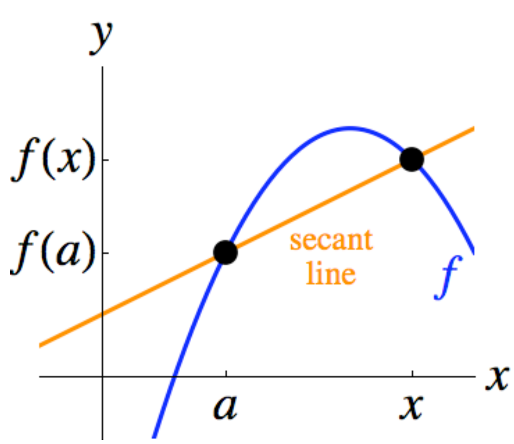
## PART A: SECANT LINES

• For now, assume that $f$ is a polynomial function of $x$. (We will relax this assumption in Part B.) Assume that $a$ is a constant.

• Temporarily fix an arbitrary real value of $x$. (By "**arbitrary**," we mean that **any** real value will do). Later, instead of thinking of $x$ as a **fixed** (or single) value, we will think of it as a "moving" or "varying" **variable** that can take on different values.

The secant line to the graph of $f$ on the interval $[a, x]$, where $a < x$, is the line that passes through the points $(a, f(a))$ and $(x, f(x))$.

• *secare* is Latin for "to cut."




The **slope** of this secant line is given by: $\dfrac{\text{rise}}{\text{run}} = \dfrac{f(x) - f(a)}{x - a}$.

• We call this a difference quotient, because it has the form: $\dfrac{\text{difference of outputs}}{\text{difference of inputs}}$.

## PART B: TANGENT LINES and DERIVATIVES

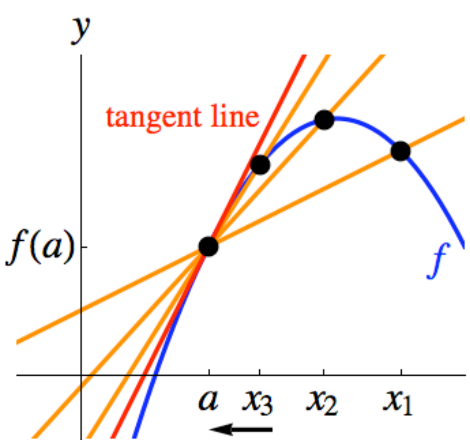
> If we now treat $x$ as a variable and let $x \to a$, the corresponding **secant lines** approach the red <u>tangent line</u> below.

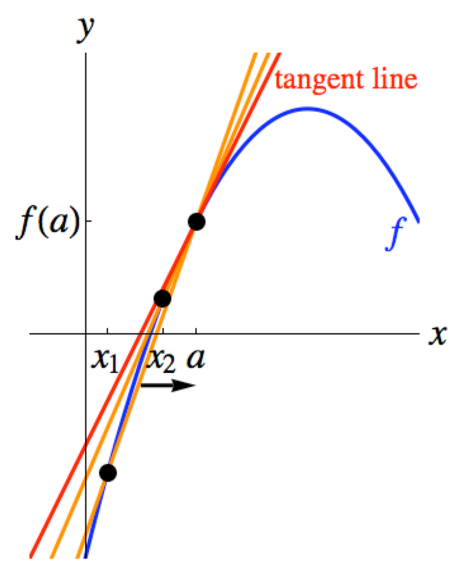
• *tangere* is Latin for "to touch." A **secant line** to the graph of $f$ must intersect it in at least two distinct points. A **tangent line** only need intersect the graph in one point, where the line might "just touch" the graph. (There could be other intersection points).

• This "limiting process" makes the tangent line a creature of **calculus**, not just precalculus.

Below, we let $x$ approach $a$ from the **right** $\left(x \to a^{+}\right)$.

Below, we let $x$ approach $a$ from the **left** $\left(x \to a^{-}\right)$.
(See Footnote 1.)




• We define the **slope** of the tangent line to be the (two-sided) **limit** of the difference quotient as $x \to a$, if that limit exists.

• We denote this slope by $f'(a)$, read as "***f*** **prime** of (or at) $a$."

> $f'(a)$, the <u>derivative</u> of $f$ at $a$, is the **slope of the tangent line** to the graph of $f$ at the point $\left(a, f(a)\right)$, if that slope exists (as a real number).

> $f$ is <u>differentiable</u> at $a$ $\Leftrightarrow$ $f'(a)$ exists.

• **Polynomial** functions are **differentiable everywhere** on $\mathbb{R}$. (See Section 3.2.)

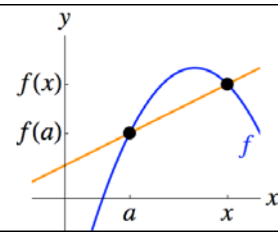
• The statements of this section apply to **any** function that is differentiable at $a$.

**Limit Definition of the Derivative at a Point $a$ (Version 1)**

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ if it exists}$$

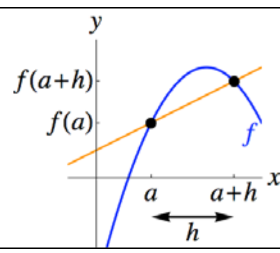
- If $f$ is continuous at $a$, we have the indeterminate Limit Form $\frac{0}{0}$.
- **Continuity** involves limits of **function values**, while **differentiability** involves limits of **difference quotients**.

Version 1: Variable endpoint ($x$)

| | | |
|---|---|---|
| Slope of secant line: $\dfrac{f(x) - f(a)}{x - a}$ |  | $a$ is constant; $x$ is variable |

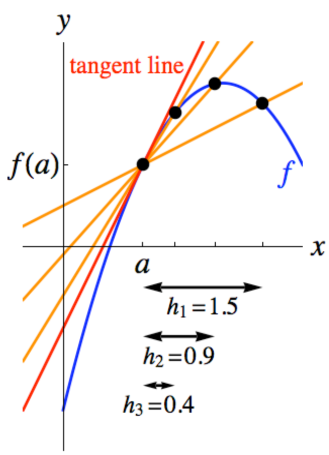
A second version, where $x$ is replaced by $a + h$, is more commonly used.

Version 2: Variable run ($h$)

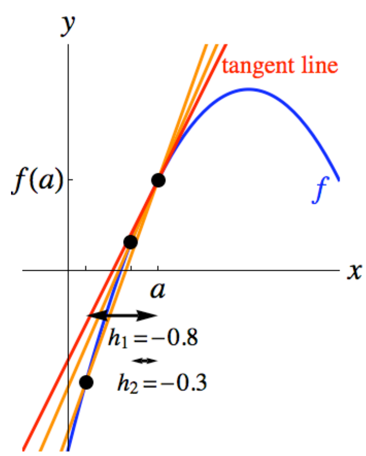
| | | |
|---|---|---|
| Slope of secant line: $\dfrac{f(a+h) - f(a)}{h}$ |  | $a$ is constant; $h$ is variable |

If we let the **run** $h \to 0$, the corresponding **secant lines** approach the red **tangent line** below.

Below, we let $h$ approach 0 from the **right** $\left(h \to 0^+\right)$.




Below, we let $h$ approach 0 from the **left** $\left(h \to 0^-\right)$. (Footnote 1.)



Limit Definition of the Derivative at a Point *a* (Version 2)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ if it exists}$$

Version 3: Two-Sided Approach

Limit Definition of the Derivative at a Point *a* (Version 3)

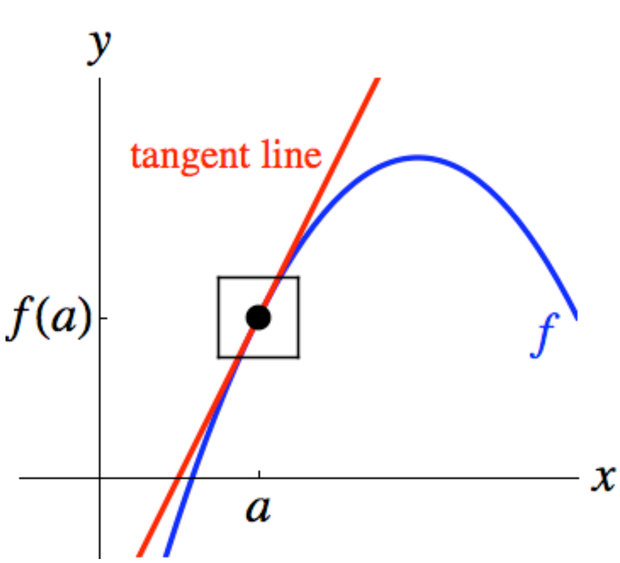
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}, \text{ if it exists}$$

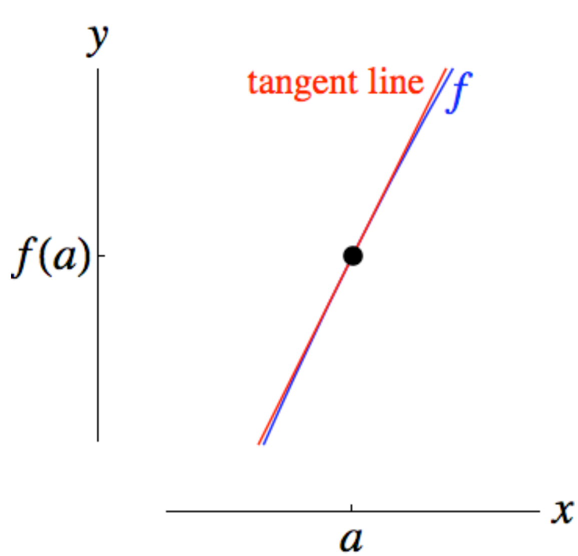
• The reader is encouraged to draw a figure to understand this approach.

Principle of Local Linearity

The **tangent line** to the graph of $f$ at the point $\left(a, f(a)\right)$, if it exists, represents the **best local linear approximation** to the function close to $a$. The graph of $f$ resembles this line if we "zoom in" on the point $\left(a, f(a)\right)$.

• The **tangent line model** linearizes the function locally around $a$.
We will expand on this in Section 3.5.




(The figure on the right is a "zoom in" on the box in the figure on the left.)

## PART C: FINDING DERIVATIVES USING THE LIMIT DEFINITIONS

*Example 1 (Finding a Derivative at a Point Using Version 1 of the Limit Definition)*

Let $f(x) = x^3$. Find $f'(1)$ using Version 1 of the Limit Definition of the Derivative at a Point.

*§ Solution*

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \qquad (\text{Here, } a = 1.)$$

$$= \lim_{x \to 1} \frac{\left[x^3\right] - \left[(1)^3\right]}{x - 1}$$

**TIP 1:** The brackets here are unnecessary, but better safe than sorry.

$$= \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \qquad \left(\text{Limit Form } \frac{0}{0}\right)$$

We will factor the numerator using the **Difference of Two Cubes** template and then simplify. **Synthetic Division** can also be used. (See Chapter 2 in the Precalculus notes).

$$= \lim_{x \to 1} \frac{\overset{(1)}{\cancel{(x-1)}}\left(x^2 + x + 1\right)}{\underset{(1)}{\cancel{(x-1)}}}$$

$$= \lim_{x \to 1} \left(x^2 + x + 1\right)$$

$$= (1)^2 + (1) + 1$$

$$= 3$$

§

*Example 2 (Finding a Derivative at a Point Using Version 2 of the Limit Definition; Revisiting Example 1)*

Let $f(x) = x^3$, as in Example 1. Find $f'(1)$ using Version 2 of the Limit Definition of the Derivative at a Point.

*§ Solution*

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad (\text{Here, } a = 1.)$$

$$= \lim_{h \to 0} \frac{\left[(1+h)^3\right] - \left[(1)^3\right]}{h}$$

We will use the **Binomial Theorem** to expand $(1+h)^3$.

(See Chapter 9 in the Precalculus notes.)

$$= \lim_{h \to 0} \frac{\left[(1)^3 + 3(1)^2(h) + 3(1)(h)^2 + (h)^3\right] - [1]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 3h + 3h^2 + h^3 \cancel{-1}}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\overset{(1)}{\cancel{h}}\left(3 + 3h + h^2\right)}{\underset{(1)}{\cancel{h}}}$$

$$= \lim_{h \to 0} \left(3 + 3h + h^2\right)$$

$$= 3 + 3(0) + (0)^2$$

$$= 3$$

We obtain the same result as in Example 1: $f'(1) = 3$. §

## PART D: FINDING EQUATIONS OF TANGENT LINES

*Example 3 (Finding Equations of Tangent Lines; Revisiting Examples 1 and 2)*

Find an equation of the **tangent line** to the graph of $y = x^3$ at the point where $x = 1$. (Review Section 0.14: Lines in the Precalculus notes.)

*§ Solution*

• Let $f(x) = x^3$, as in Examples 1 and 2.

• Find $f(1)$, the ***y*-coordinate** of the **point** of interest.

$$\begin{aligned} f(1) &= (1)^3 \\ &= 1 \end{aligned}$$

• The **point** of interest is then: $\left(1, f(1)\right) = (1, 1)$.

• Find $f'(1)$, the **slope** (*m*) of the desired tangent line.

In Part C, we showed (twice) that: $f'(1) = 3$.

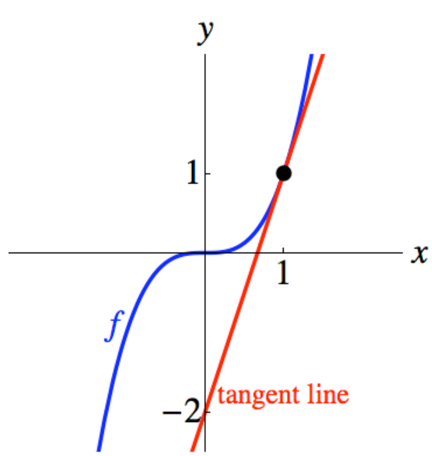
• Find a **Point-Slope Form** for the equation of the tangent line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 3(x - 1) \end{aligned}$$

• Find the **Slope-Intercept Form** for the equation of the tangent line.

$$\begin{aligned} y - 1 &= 3x - 3 \\ y &= 3x - 2 \end{aligned}$$

• Observe how the **red tangent line** below is consistent with the equation above.



• The **Slope-Intercept Form** can also be obtained directly.

Remember the **Basic Principle of Graphing**: The graph of an equation consists of all points (such as $(1,1)$ here) whose coordinates satisfy the equation.

$$y = mx + b \quad \Rightarrow$$

$$(1) = (3)(1) + b \quad \Rightarrow$$

$$(\text{Solve for } b.)$$

$$b = -2 \quad \Rightarrow$$

$$y = 3x - 2$$

§

## PART E: NORMAL LINES

Assume that $P$ is a point on a graph where a tangent line exists.

The normal line to the graph at $P$ is the line that contains $P$ and that is **perpendicular** to the **tangent line** at $P$.

*Example 4 (Finding Equations of Normal Lines; Revisiting Example 3)*

Find an equation of the **normal line** to the graph of $y = x^3$ at $P(1,1)$.

*§ Solution*

• In Examples 1 and 2, we let $f(x) = x^3$, and we found that the **slope** of the **tangent line** at $(1,1)$ was given by: $f'(1) = 3$.

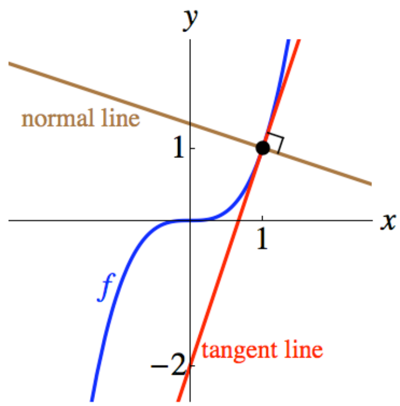
• The **slope** of the **normal line** at $(1,1)$ is then $-\frac{1}{3}$, the **opposite reciprocal** of the slope of the tangent line.

• A **Point-Slope Form** for the equation of the **normal line** is given by:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{3}(x - 1)$$

• The **Slope-Intercept Form** is given by: $y = -\frac{1}{3}x + \frac{4}{3}$.




**WARNING 1:** The Slope-Intercept Form for the equation of the **normal line** at *P* **cannot** be obtained by taking the Slope-Intercept Form for the equation of the **tangent line** at *P* and replacing the slope with its opposite reciprocal, **unless** *P* lies on the *y*-axis. In this Example, the normal line is **not** given by: $y = -\frac{1}{3}x - 2$. §

## PART F: NUMERICAL APPROXIMATION OF DERIVATIVES

The **Principle of Local Linearity** implies that the slope of the **tangent line** at the point $(a, f(a))$ can be "well approximated" by the slope of the **secant line** on a "small" interval containing *a*.

When using Version 2 of the Limit Definition of the Derivative, this implies that:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \text{ when } h \approx 0.$$

*Example 5 (Numerically Approximating a Derivative; Revisiting Example 2)*

Let $f(x) = x^3$, as in Example 2. We will find approximations of $f'(1)$. (See Example 8 in Part H.)

| $h$ | $\frac{f(1+h) - f(1)}{h}$, or $3 + 3h + h^2$ $(h \neq 0)$ (See Example 2.) |
|---|---|
| 0.1 | 3.31 |
| 0.01 | 3.0301 |
| 0.001 | 3.003001 |
| $\to 0$ | $\to 3$ |
| $-0.001$ | 2.997001 |
| $-0.01$ | 2.9701 |
| $-0.1$ | 2.71 |

• If we only have a **table of values** for a function $f$ instead of a rule for $f(x)$, we may have to resort to numerically approximating derivatives. §

## PART G: AVERAGE RATE OF CHANGE

The average rate of change of $f$ on $\left[a, b\right]$ is equal to the **slope** of the **secant line** on $\left[a, b\right]$, which is given by: $\dfrac{\text{rise}}{\text{run}} = \dfrac{f(b) - f(a)}{b - a}$. (See Footnotes 2 and 3.)

*Example 6 (Average Velocity)*

Average velocity is a common example of an average rate of change.

Let's say a car is driven due north 100 miles during a two-hour trip.
What is the average velocity of the car?

• Let $t$ = the time (in hours) elapsed since the beginning of the trip.

• Let $y = s(t)$, where $s$ is the position function for the car (in miles).
$s$ gives the **signed** distance of the car from the starting position.

•• The position ($s$) values would be **negative** if the car were **south** of the starting position.

• Let $s(0) = 0$, meaning that $y = 0$ corresponds to the starting position.
Therefore, $s(2) = 100$ (miles).

The average velocity on the time-interval $\left[a, b\right]$ is the **average rate of change of position with respect to time**. That is,

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

where $\Delta$ (uppercase delta) denotes "change in"

$$= \frac{s(b) - s(a)}{b - a}\text{, a } \textbf{difference quotient}$$

Here, the average velocity of the car on $\left[0, 2\right]$ is:

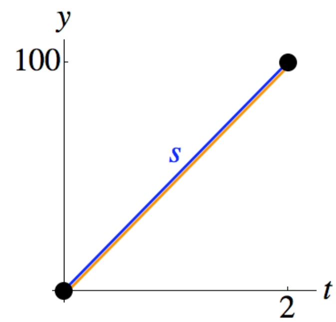
$$\frac{s(2) - s(0)}{2 - 0} = \frac{100 - 0}{2}$$

$$= 50 \ \frac{\text{miles}}{\text{hour}} \left( \text{or } \frac{\text{mi}}{\text{hr}} \text{ or mph} \right)$$

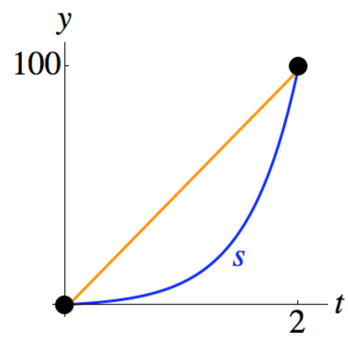
**TIP 2:** The unit of velocity is the unit of **slope** given by: $\dfrac{\text{unit of } s}{\text{unit of } t}$.

The average velocity is 50 mph on $\left[0, 2\right]$ in the three scenarios below. It is the slope of the orange secant line. (Axes are scaled differently.)

- Below, the velocity is constant (50 mph).
(We are not requiring the car to slow down to a stop at the end.)

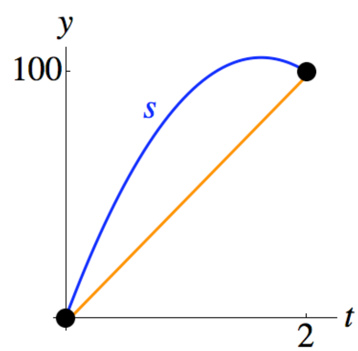



- Below, the velocity is increasing; the car is accelerating.




- Below, the car "breaks the rules," backtracks, and goes south.

**WARNING 2:** The car's velocity is **negative** in value when it is backtracking; this happens when the graph falls.




Note: The Mean Value Theorem for Derivatives in Section 4.2 will imply that the car must be going **exactly** 50 mph at some time value $t$ in $(0, 2)$. The theorem applies in all three scenarios above, because $s$ is **continuous** on $[0, 2]$ and is **differentiable** on $(0, 2)$. §

## PART H: INSTANTANEOUS RATE OF CHANGE

The instantaneous rate of change of $f$ at $a$ is equal to $f'(a)$, if it exists.

*Example 7 (Instantaneous Velocity)*

Instantaneous velocity (or simply velocity) is a common example of an instantaneous rate of change.

Let's say a car is driven due north for two hours, beginning at noon. How can we find the instantaneous velocity of the car at 1pm?

(If this is positive, this can be thought of as the **speedometer** reading at 1pm.)

- Let $t =$ the time (in hours) elapsed since noon.
- Let $y = s(t)$, where $s$ is the position function for the car (in miles).

Consider **average velocities** on **variable** time intervals of the form $[a, a+h]$, if $h > 0$, or the form $[a+h, a]$, if $h < 0$, where $h$ is a variable run. (We can let $h = \Delta t$ .)

The average velocity on the time-interval $[a, a+h]$, if $h > 0$, or $[a+h, a]$, if $h < 0$, is given by:

$$\frac{s(a+h) - s(a)}{h}$$

- This equals the **slope of the secant line** to the graph of $s$ on the interval.

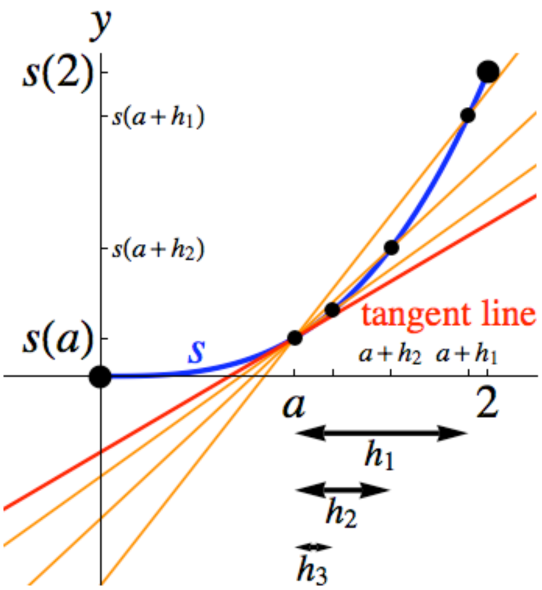
- (See Footnote 1 on the $h < 0$ case.)

Let's assume there exists a **non-vertical tangent line** to the graph of $s$ at the point $\left(a, s\left(a\right)\right)$.

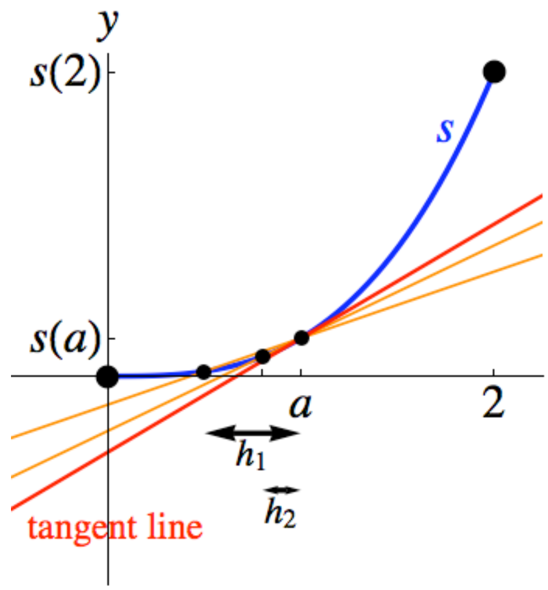
- Then, as $h \to 0$, the slopes of the **secant** lines will approach the slope of this **tangent** line, which is $s'\left(a\right)$.

- Likewise, as $h \to 0$, the **average** velocities will approach the **instantaneous** velocity at $a$.

Below, we let $h \to 0^+$.




Below, we let $h \to 0^-$.




The <u>instantaneous velocity</u> (or simply <u>velocity</u>) at $a$ is given by:

$$s'\left(a\right), \text{ or } v\left(a\right) = \lim_{h \to 0} \frac{s\left(a+h\right) - s\left(a\right)}{h}, \text{ if it exists}$$

In our Example, the instantaneous velocity of the car at 1pm is given by:

$$s'\left(1\right), \text{ or } v\left(1\right) = \lim_{h \to 0} \frac{s\left(1+h\right) - s\left(1\right)}{h}$$

Let's say $s\left(t\right) = t^3$. Example 2 then implies that $s'\left(1\right) = v\left(1\right) = 3 \text{ mph}$. §

*Example 8 (Numerically Approximating an Instantaneous Velocity; Revisiting Examples 5 and 7)*

Again, let's say the position function $s$ is defined by: $s(t) = t^3$ on $[0, 2]$.

We will approximate $v(1)$, the **instantaneous velocity** of the car at 1pm.

We will first compute **average velocities** on intervals of the form $[1, 1+h]$.

Here, we let $h \to 0^+$.

| Interval | Value of $h$ (in hours) | Average velocity, $\frac{s(1+h)-s(1)}{h}$ |
|---|---|---|
| $[1, 2]$ | 1 | $\frac{s(2)-s(1)}{1} = 7$ mph |
| $[1, 1.1]$ | 0.1 | $\frac{s(1.1)-s(1)}{0.1} = 3.31$ mph |
| $[1, 1.01]$ | 0.01 | $\frac{s(1.01)-s(1)}{0.01} = 3.0301$ mph |
| $[1, 1.001]$ | 0.001 | $\frac{s(1.001)-s(1)}{0.001} = 3.003001$ mph |
| | $\to 0^+$ | $\to 3$ mph |

• These average velocities approach 3 mph, which is $v(1)$.

• **WARNING 3: Tables can sometimes be misleading.** The table here does **not** represent a rigorous evaluation of $v(1)$. Answers are not always integer-valued.

We could also consider this approach:

| Interval | Value of $h$ | Average velocity, $\frac{s(1+h)-s(1)}{h}$ (rounded off to six significant digits) |
|---|---|---|
| $[1, 2]$ | 1 hour | 7.00000 mph |
| $\left[1, 1\frac{1}{60}\right]$ | 1 minute | 3.05028 mph |
| $\left[1, 1\frac{1}{3600}\right]$ | 1 second | 3.00083 mph |
| | $\to 0^+$ | $\to 3$ mph |

Here, we let $h \to 0^-$.

| Interval | Value of $h$ (in hours) | Average velocity, $\frac{s(1+h)-s(1)}{h}$ |
|---|---|---|
| $[0, 1]$ | $-1$ | $\frac{s(0)-s(1)}{-1} = 1$ mph |
| $[0.9, 1]$ | $-0.1$ | $\frac{s(0.9)-s(1)}{-0.1} = 2.71$ mph |
| $[0.99, 1]$ | $-0.01$ | $\frac{s(0.99)-s(1)}{-0.01} = 2.9701$ mph |
| $[0.999, 1]$ | $-0.001$ | $\frac{s(0.999)-s(1)}{-0.001} = 2.997001$ mph |
| | $\to 0^-$ | $\to 3$ mph |

• Because of the way we normally look at slopes, we may prefer to rewrite the first difference quotient $\frac{s(0)-s(1)}{-1}$ as $\frac{s(1)-s(0)}{1}$, and so forth. (See Footnote 1.) §

*Example 9 (Rate of Change of a Profit Function)*

A company sells widgets. Assume that all widgets produced are sold. $P(x)$, the profit (in dollars) if $x$ widgets are produced and sold, is modeled by: $P(x) = -x^2 + 200x - 5000$. Find the **instantaneous** rate of change of profit at 60 widgets. (In economics, this is referred to as marginal profit.)

**WARNING 4:** We will treat the **domain** of $P$ as $[0, \infty)$, even though one could argue that the domain should only consist of **integers**. Be aware of this issue with applications such as these.

*§ Solution*

We want to find $P'(60)$.

$$P'(60)$$
$$= \lim_{h \to 0} \frac{P(60+h) - P(60)}{h}$$
$$= \lim_{h \to 0} \frac{\left[ -(60+h)^2 + 200(60+h) - 5000 \right] - \left[ -(60)^2 + 200(60) - 5000 \right]}{h}$$

**WARNING 5: Grouping symbols** are essential when expanding $P(60)$ here, because we are subtracting an expression with more than one term.

$$= \lim_{h \to 0} \frac{\left[ -\left(3600 + 120h + h^2\right) + 12{,}000 + 200h - 5000 \right] - \left[ -3600 + 12{,}000 - 5000 \right]}{h}$$

**TIP 3:** Instead of simplifying within the brackets immediately, we will take advantage of **cancellations**.

$$= \lim_{h \to 0} \frac{\cancel{-3600} - 120h - h^2 \cancel{+12{,}000} + 200h \cancel{-5000} \cancel{+3600} \cancel{-12{,}000} \cancel{+5000}}{h}$$
$$= \lim_{h \to 0} \frac{80h - h^2}{h}$$

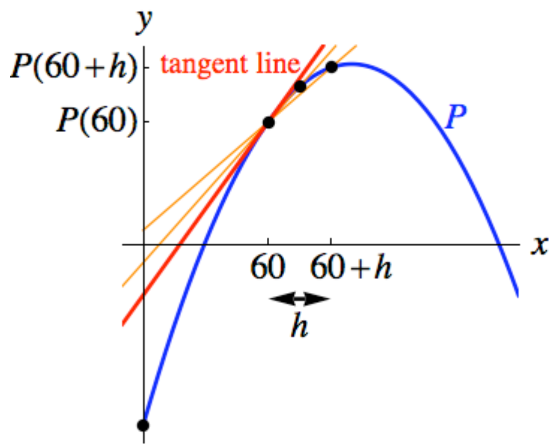
$$= \lim_{h \to 0} \frac{\overset{(1)}{\cancel{h}}(80-h)}{\underset{(1)}{\cancel{h}}}$$

$$= \lim_{h \to 0} (80-h)$$

$$= 80-(0)$$

$$= 80 \ \frac{\text{dollars}}{\text{widget}}$$

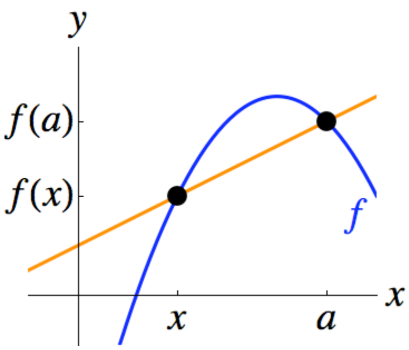
This is the slope of the **red tangent line** below.

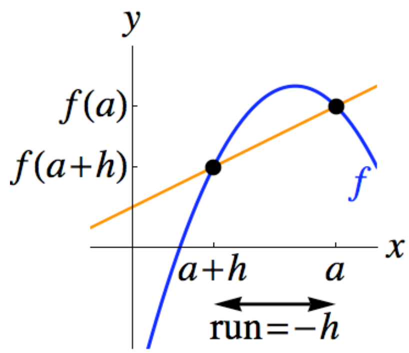



• If we produce and sell one more widget (from 60 to 61), we expect to make about $80 more in profit. What would be your business strategy if **marginal profit** is positive? §

**FOOTNOTES**

1. **Difference quotients with negative denominators.** Our forms of difference quotients allow negative denominators ("runs"), as well. They still represent slopes of secant lines.




Left figure $(x < a)$: $\text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{f(a) - f(x)}{a - x} = \dfrac{-[f(x) - f(a)]}{-(x - a)} = \dfrac{f(x) - f(a)}{x - a}$

Right figure $(h < 0)$:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(a) - f(a+h)}{a - (a+h)} = \frac{-[f(a+h) - f(a)]}{-h} = \frac{f(a+h) - f(a)}{h}$$

2. **Average rate of change and assumptions made about a function.** When defining the average rate of change of a function $f$ on an interval $[a, b]$, where $a < b$, sources typically do not state the assumptions made about $f$. The formula $\dfrac{f(b) - f(a)}{b - a}$ seems only to require the existence of $f(a)$ and $f(b)$, but we typically assume more than just that.

   • Although the slope of the secant line on $[a, b]$ can still be defined, we need more for the existence of derivatives (i.e., the differentiability of $f$) and the existence of non-vertical tangent lines.

   • We ordinarily assume that $f$ is **continuous** on $[a, b]$. Then, there are no holes, jumps, or vertical asymptotes on $[a, b]$ when $f$ is graphed. (See Section 2.8.)

   • We may also assume that $f$ is **differentiable** on $[a, b]$. Then, the graph of $f$ makes no sharp turns and does not exhibit "infinite steepness" (corresponding to vertical tangent lines). However, this assumption may lead to circular reasoning, because the ideas of secant lines and average rate of change are used to develop the ideas of derivatives, tangent lines, and instantaneous rate of change. Differentiability is defined in terms of the existence of derivatives.

   • We may also need to assume that $f'$ is continuous on $[a, b]$. (See Footnote 3.)

3. **Average rate of change of $f$ as the average value of $f'$.** Assume that $f'$ is continuous on $\left[a, b\right]$. Then, the **average rate of change of $f$** on $\left[a, b\right]$ is equal to the **average value of $f'$** on $\left[a, b\right]$. In Chapter 5, we will assume that a function (say, $g$) is continuous on $\left[a, b\right]$ and then define the **average value of $g$** on $\left[a, b\right]$ to be $\dfrac{\int_a^b g(x)\, dx}{b-a}$; the numerator is a <u>definite integral</u>, which will be defined as a limit of sums. Then, the **average value of $f'$** on $\left[a, b\right]$ is given by: $\dfrac{\int_a^b f'(x)\, dx}{b-a}$, which is equal to $\dfrac{f(b)-f(a)}{b-a}$ by the <u>Fundamental Theorem of Calculus</u>. The theorem assumes that the integrand [function], $f'$, is continuous on $\left[a, b\right]$.