SECTION 3.3: TECHNIQUES OF DIFFERENTIATION

LEARNING OBJECTIVES

• Learn how to differentiate using short cuts, including:
the Linearity Properties, the Product Rule, the Quotient Rule, and (perhaps) the Reciprocal Rule.

PART A: BASIC RULES OF DIFFERENTIATION

In Section 3.2, we discussed Rules 1 through 4 below.

Basic Short Cuts for Differentiation

Assumptions:
- c, m, b, and n are real constants.
- f and g are functions that are differentiable "where we care."

	If $h(x)=$	then $h'(x)=$	Comments
1.	c	0	The derivative of a constant is 0.
2.	$mx+b$	m	The derivative of a linear function is the slope.
3.	x^n	nx^{n-1}	Power Rule
4.	$c \cdot f(x)$	$c \cdot f'(x)$	Constant Multiple Rule (Linearity)
5.	$f(x)+g(x)$	$f'(x)+g'(x)$	Sum Rule (Linearity)
6.	$f(x)-g(x)$	$f'(x)-g'(x)$	Difference Rule (Linearity)

• **Linearity.** Because of Rules 4, 5, and 6, the differentiation operator D_x is called a linear operator. (The operations of taking limits (Ch.2) and integrating (Ch.5) are also linear.) The Sum Rule, for instance, may be thought of as "the derivative of a **sum** equals the **sum** of the derivatives, if they exist." Linearity allows us to take derivatives **term-by-term** and then to **"pop out" constant factors**.

• **Proofs.** The Limit Definition of the Derivative can be used to prove these short cuts. The Linearity Properties of **Limits** are crucial to proving the Linearity Properties of **Derivatives**. (See Footnote 1.)

Armed with these short cuts, we may now differentiate **all polynomial functions**.

Example 1 (Differentiating a Polynomial Using Short Cuts)

Let $f(x) = -4x^3 + 6x - 5$. Find $f'(x)$.

§ Solution

$$
\begin{aligned}
f'(x) &= D_x\left(-4x^3 + 6x - 5\right) \\
&= D_x\left(-4x^3\right) + D_x(6x) - D_x(5) && \text{(Sum and Difference Rules)} \\
&= -4 \cdot D_x\left(x^3\right) + D_x(6x) - D_x(5) && \text{(Constant Multiple Rule)}
\end{aligned}
$$

TIP 1: Students get used to applying the Linearity Properties, skip all of this work, and give the "answer only."

$$
\begin{aligned}
&= -4\left(3x^2\right) + 6 - 0 \\
&= -12x^2 + 6
\end{aligned}
$$

Challenge to the Reader: Observe that the "-5" term has no impact on the derivative. Why does this make sense graphically? Hint: How would the graphs of $y = -4x^3 + 6x$ and $y = -4x^3 + 6x - 5$ be different? Consider the **slopes** of corresponding tangent lines to those graphs. §

Example 2 (Equation of a Tangent Line; Revisiting Example 1)

Find an equation of the **tangent line** to the graph of $y = -4x^3 + 6x - 5$ at the point $(1, -3)$.

§ Solution

• Let $f(x) = -4x^3 + 6x - 5$, as in Example 1.

• Just to be safe, we can **verify** that the **point** $(1, -3)$ lies on the graph by verifying that $f(1) = -3$. (Remember that function values correspond to y-coordinates here.)

• Find *m*, the **slope** of the tangent line at the point where $x = 1$.
This is given by $f'(1)$, the value of the **derivative** function at $x = 1$.

$$m = f'(1)$$

From Example 1, remember that $f'(x) = -12x^2 + 6$.

$$= \left[-12x^2 + 6\right]_{x=1}$$
$$= -12(1)^2 + 6$$
$$= -6$$

• We can find a **Point-Slope Form** for the equation of the desired tangent line.

The line contains the **point**: $(x_1, y_1) = (1, -3)$.
It has **slope**: $m = -6$.

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = -6(x - 1)$$

• If we wish, we can rewrite the equation in **Slope-Intercept Form**.

$$y + 3 = -6x + 6$$
$$y = -6x + 3$$

• We can also obtain the **Slope-Intercept Form** directly.

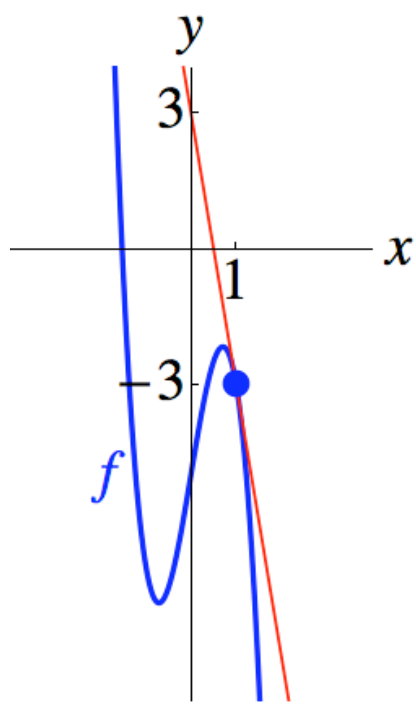
$$y = mx + b \quad \Rightarrow$$

$$(-3) = (-6)(1) + b$$

$$b = 3 \quad \Rightarrow$$

$$y = -6x + 3$$

• Observe how the **red tangent line** below is consistent with the equation above.



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Example 3 (Finding Horizontal Tangent Lines; Revisiting Example 1)

Find the *x*-coordinates of all points on the graph of $y = -4x^3 + 6x - 5$ where the **tangent line** is **horizontal**.

§ Solution

• Let $f(x) = -4x^3 + 6x - 5$, as in Example 1.

• We must find where the **slope** of the tangent line to the graph is 0.
We must solve the equation:

$$\begin{aligned} f'(x) &= 0 \\ -12x^2 + 6 &= 0 \qquad (\text{See Example 1.}) \\ -12x^2 &= -6 \\ x^2 &= \frac{1}{2} \\ x &= \pm\sqrt{\frac{1}{2}} \\ x &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

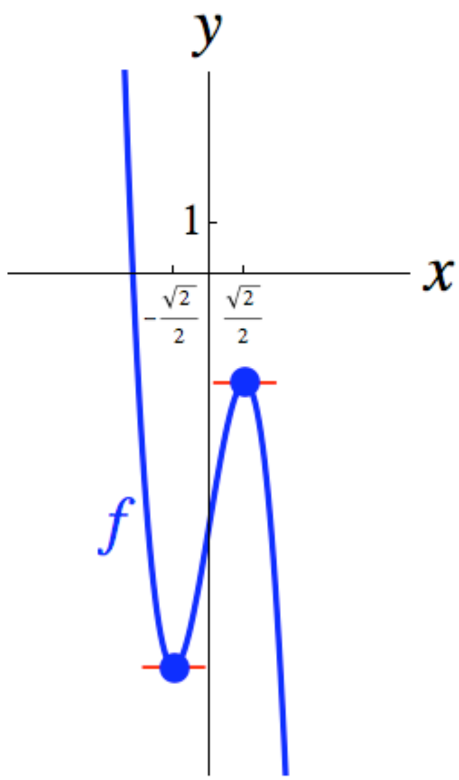
The desired *x*-coordinates are $\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$.

• The corresponding **points** on the graph are:

$\left(\frac{\sqrt{2}}{2},\ f\left(\frac{\sqrt{2}}{2}\right)\right)$, which is $\left(\frac{\sqrt{2}}{2},\ 2\sqrt{2} - 5\right)$, and

$\left(-\frac{\sqrt{2}}{2},\ f\left(-\frac{\sqrt{2}}{2}\right)\right)$, which is $\left(-\frac{\sqrt{2}}{2},\ -2\sqrt{2} - 5\right)$.

• The **red tangent lines** below are truncated.



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PART B: PRODUCT RULE OF DIFFERENTIATION

WARNING 1: The derivative of a product is typically **not** the product of the derivatives.

> Product Rule of Differentiation
>
> Assumptions:
>
> • f and g are functions that are differentiable "where we care."
>
> If $h(x) = f(x)g(x)$,
> then $h'(x) = f'(x)g(x) + f(x)g'(x)$.

• Footnote 2 uses the Limit Definition of the Derivative to prove this.

• Many sources switch terms and write: $h'(x) = f(x)g'(x) + f'(x)g(x)$, but our form is easier to extend to three or more factors.

Example 4 (Differentiating a Product)

Find $D_x\left[\left(x^4+1\right)\left(x^2+4x-5\right)\right]$.

§ Solution

TIP 2: Clearly **break** the product up into factors, as has already been done here. The **number of factors** (here, two) will equal the **number of terms** in the **derivative** when we use the Product Rule to "expand it out."

TIP 3: Pointer method. Imagine a **pointer** being moved from **factor to factor** as we write the derivative **term-by-term**. The **pointer** indicates which factor we **differentiate**, and then we **copy** the other factors to form the corresponding term in the derivative.

$\left(x^4+1\right)$	$\left(x^2+4x-5\right)$	
$\wedge\left(D_x\right)$	copy	+
copy	$\wedge\left(D_x\right)$	

$$D_x\left[\left(x^4+1\right)\left(x^2+4x-5\right)\right] = \left[D_x\left(x^4+1\right)\right]\cdot\left(x^2+4x-5\right) + \left(x^4+1\right)\cdot\left[D_x\left(x^2+4x-5\right)\right]$$

$$= \left[4x^3\right]\cdot\left(x^2+4x-5\right) + \left(x^4+1\right)\cdot\left[2x+4\right]$$

The Product Rule is especially convenient here if we do not have to simplify our result. Here, we will simplify.

$$= 6x^5 + 20x^4 - 20x^3 + 2x + 4$$

Challenge to the Reader: Find the derivative by first multiplying out the product and then differentiating term-by-term. §

The Product Rule can be extended to three or more factors.

- The Exercises include a related proof.

Example 5 (Differentiating a Product of Three Factors)

Find $\frac{d}{dt}\left[(t+4)\left(t^2+2\right)\left(\sqrt[3]{t}-t\right)\right]$. The result does not have to be simplified, and negative exponents are acceptable here. (Your instructor may object!)

§ Solution

$(t+4)$	$\left(t^2+2\right)$	$\left(t^{1/3}-t\right)$	
$\wedge (D_t)$	copy	copy	+
copy	$\wedge (D_t)$	copy	+
copy	copy	$\wedge (D_t)$	

$$\frac{d}{dt}\left[(t+4)\left(t^2+2\right)\left(\sqrt[3]{t}-t\right)\right] = \left[D_t(t+4)\right]\cdot\left(t^2+2\right)\cdot\left(\sqrt[3]{t}-t\right) +$$
$$(t+4)\cdot\left[D_t\left(t^2+2\right)\right]\cdot\left(\sqrt[3]{t}-t\right) +$$
$$(t+4)\cdot\left(t^2+2\right)\cdot\left[D_t\left(t^{1/3}-t\right)\right]$$

$$= [1]\cdot\left(t^2+2\right)\cdot\left(\sqrt[3]{t}-t\right) +$$
$$(t+4)\cdot[2t]\cdot\left(\sqrt[3]{t}-t\right) +$$
$$(t+4)\cdot\left(t^2+2\right)\cdot\left[\frac{1}{3}t^{-2/3}-1\right]$$

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TIP 4: Apply the **Constant Multiple Rule**, not the Product Rule, to something like $D_x\left(2x^3\right)$. While the Product Rule would work, it would be inefficient here.

PART C: QUOTIENT RULE (and RECIPROCAL RULE) OF DIFFERENTIATION

WARNING 2: The derivative of a quotient is typically **not** the quotient of the derivatives.

Quotient Rule of Differentiation

Assumptions:

- f and g are functions that are differentiable "where we care."
- g is nonzero "where we care."

If $h(x) = \dfrac{f(x)}{g(x)}$,

then $h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$.

- Footnote 3 proves this using the Limit Definition of the Derivative.
- Footnote 4 more elegantly proves this using the Product Rule.

TIP 5: Memorizing. The Quotient Rule can be memorized as:

$$D\left(\frac{\text{Hi}}{\text{Lo}}\right) = \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{(\text{Lo})^2 \text{, the square of what's below}}$$

Observe that the numerator and the denominator on the right-hand side **rhyme**.

- At this point, we can differentiate **all rational functions**.

Reciprocal Rule of Differentiation

If $h(x) = \dfrac{1}{g(x)}$,

then $h'(x) = -\dfrac{g'(x)}{\left[g(x)\right]^2}$.

• This is a special case of the Quotient Rule where $f(x) = 1$.

Think: $-\dfrac{\text{D}(\text{Lo})}{(\text{Lo})^2}$

TIP 6: While the **Reciprocal Rule** is useful, it is not all that necessary to memorize if the **Quotient Rule** has been memorized.

Example 6 (Differentiating a Quotient)

Find $D_x\left(\dfrac{7x-3}{3x^2+1}\right)$.

§ Solution

$$D_x\left(\frac{7x-3}{3x^2+1}\right) = \frac{\text{Lo}\cdot D(\text{Hi}) - \text{Hi}\cdot D(\text{Lo})}{(\text{Lo})^2\text{, the square of what's below}}$$

$$= \frac{(3x^2+1)\cdot\left[D_x(7x-3)\right] - (7x-3)\cdot\left[D_x(3x^2+1)\right]}{(3x^2+1)^2}$$

$$= \frac{(3x^2+1)\cdot[7] - (7x-3)\cdot[6x]}{(3x^2+1)^2}$$

$$= \frac{-21x^2+18x+7}{(3x^2+1)^2},\ \text{or}\ \frac{7-21x^2+18x}{(3x^2+1)^2},\ \text{or}\ -\frac{21x^2-18x-7}{(3x^2+1)^2}$$

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TIP 7: Rewriting. Instead of running with the first technique that comes to mind, examine the problem, **think**, and see if **rewriting or simplifying** first can help.

Example 7 (Rewriting Before Differentiating)

Let $s(w)=\dfrac{6w^2-\sqrt{w}}{3w}$. Find $s'(w)$.

§ Solution

Rewriting $s(w)$ by splitting the fraction yields a simpler solution than applying the Quotient Rule directly would have.

$$\begin{aligned} s(w) &= \frac{6w^2}{3w}-\frac{\sqrt{w}}{3w} \\ &= 2w-\frac{1}{3}w^{-1/2} \quad \Rightarrow \\ s'(w) &= 2+\frac{1}{6}w^{-3/2} \\ &= 2+\frac{1}{6w^{3/2}}, \text{ or } \frac{12w^{3/2}+1}{6w^{3/2}}, \text{ or } \frac{12w^2+\sqrt{w}}{6w^2} \end{aligned}$$

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FOOTNOTES

1. **Proof of the Sum Rule of Differentiation.** Throughout the Footnotes, we assume that f and g are functions that are differentiable "where we care." Let $p = f + g$. (We will use h for "run" in the Limit Definition of the Derivative.)

$$p'(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h} = \lim_{h\to 0} \frac{\left[f(x+h) + g(x+h)\right] - \left[f(x) + g(x)\right]}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right] + \left[g(x+h) - g(x)\right]}{h}$$
$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

(Observe that we have exploited the Sum Rule (linearity) of Limits.)

$$= f'(x) + g'(x)$$

The Difference Rule can be similarly proven, or, if we accept the Constant Multiple Rule, we can use: $f - g = f + (-g)$. Sec. 2.2, Footnote 1 extends to derivatives of linear combinations.

2. **Proof of the Product Rule of Differentiation.** Let $p = fg$.

$$p'(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h}$$
$$= \lim_{h\to 0} \frac{\left[f(x+h)g(x+h)\right] - \left[f(x)g(x)\right]}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}\right]$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)\cdot\left[g(x+h) - g(x)\right]}{h} + \lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right]\cdot g(x)}{h}$$
$$= \lim_{h\to 0} \left[f(x+h)\cdot\frac{g(x+h) - g(x)}{h}\right] + \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\cdot g(x)\right]$$
$$= \left[\lim_{h\to 0} f(x+h)\right]\cdot\left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] + \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right]\left[\lim_{h\to 0} g(x)\right]$$
$$= \left[f(x)\right]\cdot\left[g'(x)\right] + \left[f'(x)\right]\cdot\left[g(x)\right], \text{ or}$$
$$f'(x)g(x) + f(x)g'(x)$$

Note: We have: $\lim_{h\to 0} f(x+h) = f(x)$ by continuity, because differentiability implies continuity. We have something similar for g in Footnote 3.

3. **Proof of the Quotient Rule of Differentiation, I.** Let $p = f / g$, where $g(x) \neq 0$.

$$
\begin{aligned}
p'(x) &= \lim_{h\to 0} \frac{p(x+h)-p(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \left(\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \right] \cdot \frac{1}{h} \right) \\
&= \lim_{h\to 0} \left(\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right] \cdot \frac{1}{h} \right) \\
&= \lim_{h\to 0} \left[\frac{f(x+h)g(x) - f(x)g(x+h)}{h} \cdot \frac{1}{g(x+h)g(x)} \right] \\
&= \lim_{h\to 0} \left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \cdot \frac{1}{g(x+h)g(x)} \right] \\
&= \lim_{h\to 0} \left[\frac{\left[f(x+h)g(x) - f(x)g(x)\right] + \left[f(x)g(x) - f(x)g(x+h)\right]}{h} \cdot \frac{1}{g(x+h)g(x)} \right] \\
&= \lim_{h\to 0} \left[\frac{\left[f(x+h) - f(x)\right]\cdot g(x) + f(x)\cdot\left[g(x) - g(x+h)\right]}{h} \cdot \frac{1}{g(x+h)g(x)} \right] \\
&= \lim_{h\to 0} \left[\frac{\left[f(x+h) - f(x)\right]\cdot g(x) - f(x)\cdot\left[g(x+h) - g(x)\right]}{h} \cdot \frac{1}{g(x+h)g(x)} \right] \\
&= \left[\lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right]\cdot g(x)}{h} - \lim_{h\to 0} \frac{f(x)\cdot\left[g(x+h) - g(x)\right]}{h} \right] \cdot \left[\lim_{h\to 0} \frac{1}{g(x+h)g(x)} \right] \\
&= \left[\left[g(x)\right]\cdot\left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \right] - \left[f(x)\right]\cdot\left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \right] \right] \cdot \left[\frac{1}{g(x)g(x)} \right]
\end{aligned}
$$

(See Footnote 2, Note.)

$$
= \left(\left[g(x)\right]\cdot\left[f'(x)\right] - \left[f(x)\right]\cdot\left[g'(x)\right] \right) \cdot \frac{1}{\left[g(x)\right]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}
$$

4. **Proof of the Quotient Rule of Differentiation, II, using the Product Rule.**

Let $h(x)=\frac{f(x)}{g(x)}$, where $g(x)\neq 0$.

Then, $g(x)h(x)=f(x)$.

Differentiate both sides with respect to *x*. Apply the **Product Rule** to the left-hand side. We obtain: $g'(x)h(x)+g(x)h'(x)=f'(x)$. Solving for $h'(x)$, we obtain:

$h'(x)=\frac{f'(x)-g'(x)h(x)}{g(x)}$. Remember that $h(x)=\frac{f(x)}{g(x)}$. Then,

$$\begin{aligned}
h'(x)&=\frac{f'(x)-g'(x)\left[\frac{f(x)}{g(x)}\right]}{g(x)}\\
&=\frac{\left(f'(x)-g'(x)\left[\frac{f(x)}{g(x)}\right]\right)}{\left[g(x)\right]}\cdot\frac{\left[g(x)\right]}{\left[g(x)\right]}\\
&=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}
\end{aligned}$$

This approach is attributed to Marie Agnessi (1748); see *The AMATYC Review*, Fall 2002 (Vol. 24, No. 1), p.2, Letter to the Editor by Joe Browne.

• See also “Quotient Rule Quibbles” by Eugene Boman in the Fall 2001 edition (vol.23, No.1) of *The AMATYC Review*, pp.55-58. The article suggests that the **Reciprocal Rule** for $D_x\left[\frac{1}{g(x)}\right]$ can be proven directly by using the Limit Definition of the Derivative, and then the **Product Rule** can be used in conjunction with the Reciprocal Rule to differentiate $\left[f(x)\right]\left[\frac{1}{g(x)}\right]$; the Spivak and Apostol calculus texts take this approach. The article presents another proof, as well.