SECTION 3.6: CHAIN RULE

LEARNING OBJECTIVES

• Understand the Chain Rule and use it to differentiate composite functions.
• Know when and how to apply the Generalized Power Rule and the Generalized Trigonometric Rules, which are based on the Chain Rule.

PART A: THE IDEA OF THE CHAIN RULE

Yul, Uma, and Xavier run in a race. Let $y$, $u$, and $x$ represent their positions (in miles), respectively.

• Assume that Yul always runs **twice** as fast as Uma. That is, $\frac{dy}{du} = 2$.
  (If Uma runs $\Delta u$ miles, then Yul runs $\Delta y$ miles, where $\Delta y = 2 \Delta u$.)

• Assume that Uma always runs **three** times as fast as Xavier. That is, $\frac{du}{dx} = 3$.

• Therefore, Yul always runs **six** times as fast as Xavier. That is, $\frac{dy}{dx} = 6$.

This is an example of the **Chain Rule**, which states that: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Here, $6 = 2 \cdot 3$.

**WARNING 1:** The Chain Rule is a calculus rule, not an algebraic rule, in that the “$du$”s should **not** be thought of as “canceling.”
We can think of $y$ as a function of $u$, which, in turn, is a function of $x$. Call these functions $f$ and $g$, respectively. Then, $y$ is a **composite function** of $x$; this function is denoted by $f \circ g$.

- **In multivariable calculus**, you will see bushier trees and more complicated forms of the Chain Rule where you add products of derivatives along paths, extending what we have done above.

**TIP 1:** The Chain Rule is used to differentiate **composite functions** such as $f \circ g$. The derivative of a **product** of functions is not necessarily the product of the derivatives (see Section 3.3 on the Product Rule), but the derivative of a **composition** of functions is the product of the derivatives. (Composite functions were reviewed in Chapter 1.)
PART B: FORMS OF THE CHAIN RULE

**Chain Rule**

Let \( y = f(u) \) and \( u = g(x) \), where \( f \) and \( g \) are differentiable “where we care.” Then,

Form 1) \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)

Form 2) \( (f \circ g)'(x) = [f'(u)] \cdot g'(x) \)

Form 3) \( y' = [f'(u)][u'] \)

- Essentially, the derivative of a composite function is obtained by taking the derivative of the “outer function” at \( u \) times the derivative of the “inner function” at \( x \).

- Following *How to Ace Calculus* by Adams, Thompson, and Hass (Times, 1998), we will refer to the derivative of the inner function as the “tail.” In the forms above, the tail is denoted by \( \frac{du}{dx} \), \( g'(x) \), and \( u' \).

**WARNING 2:** Forgetting the “tail” is a very common error students make when applying the Chain Rule.

- See Footnote 1 for a partial proof.
- See Footnote 2 on a controversial form.

Many differentiation rules, such as the Generalized Power Rule and the Generalized Trigonometric Rules we will introduce in this section, are based on the Chain Rule.
PART C: GENERALIZED POWER RULE

The Power Rule of Differentiation, which we introduced in Part B of Section 3.2, can be used to find $D_x \left( x^7 \right)$. However, it cannot be used to find $D_x \left[ \left( 3x^2 + 4 \right)^7 \right]$ without expanding the indicated seventh power, something we would rather not do.

Example 1 (Using the Chain Rule to Motivate the Generalized Power Rule)

Use the Chain Rule to find $D_x \left[ \left( 3x^2 + 4 \right)^7 \right]$.

§ Solution

Let $y = \left( 3x^2 + 4 \right)^7$. We will treat $y$ as a composite function of $x$.

$y = (f \circ g)(x) = f\left(g(x)\right)$, where:

$u = g(x) = 3x^2 + 4 \quad (g \text{ is the “inner function”})$

$y = f(u) = u^7 \quad (f \text{ is the “outer function”})$

Observe that $\frac{dy}{du}$ and $\frac{du}{dx}$ can be readily found using basic rules.

We can then find $\frac{dy}{dx}$ using the Chain Rule.

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[ D_u \left( u^7 \right) \right] \left[ D_x \left( 3x^2 + 4 \right) \right]
\]

\[
= \left[ 7u^6 \right] [6x]
\]

(Since $u$ was our creation, we must express $u$ in terms of $x$.)

\[
= \left[ 7 \left( 3x^2 + 4 \right)^6 \right] [6x] \quad \text{(See Example 2 for a short cut.)}
\]

$= 42x \left( 3x^2 + 4 \right)^6$
Example 1 suggests the following short cut.

**Generalized Power Rule**

Let $u$ be a function of $x$ that is differentiable “where we care.”
Let $n$ be a real constant.

$$D_x \left( u^n \right) = \left( nu^{n-1} \right) \left( D_x u \right)$$

• **Rationale.** If $y = u^n$, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow$$

$$D_x \left( u^n \right) = \left[ D_u \left( u^n \right) \right] \cdot [D_x u]$$

$$= \left[ nu^{n-1} \right] \cdot [D_x u]$$

**Example 2 (Using the Generalized Power Rule; Revisiting Example 1)**

Use the **Generalized Power Rule** to find $D_x \left[ (3x^2 + 4)^7 \right]$.

**Solution**

$$D_x \left[ (3x^2 + 4)^7 \right] = \left[ 7(3x^2 + 4)^6 \right] \left[ D_x (3x^2 + 4) \right] \quad (\text{See Warning 3.})$$

$$= \left[ 7(3x^2 + 4)^6 \right] [6x]$$

$$= 42x (3x^2 + 4)^6$$

**WARNING 3:** Copy the base. The base $u$, which is $(3x^2 + 4)$ here, is copied under the exponent. Do not differentiate it until you get to the “tail.”

**WARNING 4:** Remember the exponent. Many students forget to write the exponent, 6, because the base can take some time to write. You may want to write the exponent before writing out the base.

**WARNING 5:** Identifying “tails.” The $D_x$ notation helps us keep track of “how far” to unravel tails. A tail may have a tail of its own. If we forget tails, we’re not going far enough. If we attach inappropriate tails (such as an additional “6” after the “6x” above), we’re going too far. §
Example 3 (Using the Generalized Power Rule in Conjunction with the Quotient Rule or the Product Rule)

Find \( D_x \left( \frac{x^3}{\sqrt{2x - 1}} \right) \).

§ Solution Method 1 (Using the Quotient Rule)

\[
D_x \left( \frac{x^3}{\sqrt{2x - 1}} \right) = \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{(\text{Lo})^2}, \text{ the square of what's below}
\]

\[
= \frac{(\sqrt{2x - 1}) \cdot [D_x (x^3)] - (x^3) \cdot (D_x [(2x - 1)^{1/2}])}{(\sqrt{2x - 1})^2}
\]

\[
= \frac{(\sqrt{2x - 1}) \cdot [3x^2] - (x^3) \cdot \left[ \frac{1}{2} (2x - 1)^{-1/2} \right] \cdot [D_x (2x - 1)]}{2x - 1}
\]

\[
= \frac{(\sqrt{2x - 1}) \cdot [3x^2] - (x^3) \cdot \left[ \frac{1}{2} (2x - 1)^{-1/2} \right] \cdot [\neq]}{2x - 1}
\]

(We could factor the numerator at this point.)

\[
= \frac{(\sqrt{2x - 1}) \cdot [3x^2] - \frac{x^3}{\sqrt{2x - 1}}}{2x - 1}
\]

\[
= \frac{\left[ (\sqrt{2x - 1}) \cdot [3x^2] - \frac{x^3}{\sqrt{2x - 1}} \right]}{2x - 1} \cdot \frac{\sqrt{2x - 1}}{\sqrt{2x - 1}}
\]

WARNING 6: Distribute before canceling. Do not cancel in the numerators until we have distributed \( \sqrt{2x - 1} \) through the first numerator.

\[
= \frac{(2x - 1) \cdot [3x^2] - x^3}{(2x - 1)^{3/2}}
\]

\[
= \frac{6x^3 - 3x^2 - x^3}{(2x - 1)^{3/2}}
\]

\[
= \frac{5x^3 - 3x^2}{(2x - 1)^{3/2}}, \text{ or } \frac{x^2 (5x - 3)}{(2x - 1)^{3/2}}
\]
§ Solution Method 2 (Using the Product Rule)

If we had forgotten the Quotient Rule, we could have rewritten:

\[ D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) = D_x \left[ x^3 (2x-1)^{-1/2} \right] \text{ and applied the Product Rule.} \]

We would then use the Generalized Power Rule to find \( D_x \left[ (2x-1)^{-1/2} \right] \).

The key drawback here is that we obtain two terms, and students may find it difficult to combine them into a single, simplified fraction. Observe:

\[ D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) = D_x \left[ x^3 (2x-1)^{-1/2} \right] \quad \text{(Rewriting)} \]

\[ = \left[ D_x (x^3) \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left[ D_x \left[ (2x-1)^{-1/2} \right] \right] \quad \text{(by the Product Rule)} \]

\[ = \left[ 3x^2 \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left[ \left[ -\frac{1}{2} (2x-1)^{-3/2} \right] \cdot \left[ D_x (2x-1) \right] \right] \quad \text{(by the Generalized Power Rule)} \]

\[ = \left[ 3x^2 \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left[ \left[ -\frac{1}{2} (2x-1)^{-3/2} \right] \cdot \left[ 2 \right] \right] \]

\[ = \frac{3x^2}{(2x-1)^{1/2}} - \frac{x^3}{(2x-1)^{3/2}} \]

\[ = \frac{3x^2}{(2x-1)^{1/2}} \cdot \left[ \frac{1}{2} (2x-1) \right] - \frac{x^3}{(2x-1)^{3/2}} \]

Build up the first fraction to obtain the LCD, \( (2x-1)^{3/2} \).

\[ = \frac{3x^2 (2x-1) - x^3}{(2x-1)^{3/2}} \]

\[ = \frac{6x^3 - 3x^2 - x^3}{(2x-1)^{3/2}} \]

\[ = \frac{5x^3 - 3x^2}{(2x-1)^{3/2}}, \text{ or } \frac{x^2 (5x - 3)}{(2x-1)^{3/2}} \quad \text{(as in Method 1)} \]
Example 4 (Using the Generalized Power Rule to Differentiate a Power of a Trigonometric Function)

Let \( f(\theta) = \sec^5 \theta \). Find \( f'(\theta) \).

**Solution**

First, rewrite \( f(\theta) \):

\[
 f(\theta) = \sec^5 \theta \\
= (\sec \theta)^5
\]

**WARNING 7:** Rewriting before differentiating. When differentiating a power of a trigonometric function, rewrite the power in this way. Students get very confused otherwise. Also, do **not** write \( \sec \theta^5 \) here; that is equivalent to \( \sec(\theta^5) \), not \( (\sec \theta)^5 \).

\[
f'(\theta) = \left[ 5(\sec \theta)^4 \right] \cdot \left[ D_\theta (\sec \theta) \right]
= \left[ 5(\sec \theta)^4 \right] \cdot [\sec \theta \tan \theta]
= 5(\sec \theta)^5 \tan \theta
= 5 \sec^5 \theta \tan \theta
\]

Example 5 (Using the Generalized Power Rule to Prove the Reciprocal Rule)

Prove the Reciprocal Rule from Section 3.3: \( D_x \left[ \frac{1}{g(x)} \right] = -\frac{g'(x)}{[g(x)]^2} \).

**Solution**

\[
 D_x \left[ \frac{1}{g(x)} \right] = D_x \left[ (g(x))^{-1} \right]
= \left( -[g(x)]^{-2} \right) \cdot [g'(x)] \quad \text{(by the Generalized Power Rule)}
= -\frac{g'(x)}{[g(x)]^2}
\]

**WARNING 8:** “\(-1\)” here denotes a reciprocal, not a function inverse.
PART D: GENERALIZED TRIGONOMETRIC RULES

The Basic Trigonometric Rules of Differentiation, which we introduced in Section 3.4, can be used to find $D_x (\sin x)$. However, they cannot be used to find $D_x \left[ \sin \left( x^2 \right) \right]$.

Example 6 (Using the Chain Rule to Motivate the Generalized Trigonometric Rules)

Use the Chain Rule to find $D_x \left[ \sin \left( x^2 \right) \right]$.

§ Solution

Let $y = \sin \left( x^2 \right)$. We will treat $y$ as a composite function of $x$.

$y = (f \circ g)(x) = f(g(x))$, where:

\[
\begin{align*}
    u &= g(x) = x^2 \quad (g \text{ is the “inner function”}) \\
    y &= f(u) = \sin u \quad (f \text{ is the “outer function”})
\end{align*}
\]

Observe that $\frac{dy}{du}$ and $\frac{du}{dx}$ can be readily found using basic rules.

We can then find $\frac{dy}{dx}$ using the Chain Rule.

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[ D_u (\sin u) \right] \left[ D_x \left( x^2 \right) \right]
\]

\[
= [\cos u] [2x]
\]

(Since $u$ was our creation, we must express $u$ in terms of $x$.)

\[
= [\cos \left( x^2 \right)] [2x] \quad \text{(See Example 7 for a short cut.)}
\]

\[
= 2x \cos \left( x^2 \right)
\]
Example 6 suggests the following short cuts.

**Generalized Trigonometric Rules**

Let \( u \) be a function of \( x \) that is differentiable “where we care” (see Footnote 4).

\[
D_x (\sin u) = (\cos u) \left( D_x u \right) \\
D_x (\cos u) = (\sin u) \left( D_x u \right) \\
D_x (\tan u) = (\sec^2 u) \left( D_x u \right) \\
D_x (\cot u) = (\csc^2 u) \left( D_x u \right) \\
D_x (\sec u) = (\sec u \tan u) \left( D_x u \right) \\
D_x (\csc u) = (\csc u \cot u) \left( D_x u \right)
\]

**WARNING 9:** In the bottom two rules, the “tail” is still written only once. The “tail” is the derivative of the common argument \( u \).

- **Radians.** See Footnote 3 on how these rules encourage us to use radians (as opposed to degrees) when differentiating trigonometric functions.

**Example 7 (Using the Generalized Trigonometric Rules; Revisiting Example 6)**

Use the **Generalized Trigonometric Rules** to find \( D_x [\sin (x^2)] \).

**Solution**

\[
D_x [\sin (x^2)] = [\cos (x^2)] \left[ D_x (x^2) \right] \\
\quad = [\cos (x^2)] [2x] \\
\quad = 2x \cos (x^2)
\]

**WARNING 10:** Copy the argument. The sine function’s argument \( u \), which is \((x^2)\) here, is copied as the cosine function’s argument. Do not differentiate it until you get to the “tail.”

**TIP 2:** Consistency with the Basic Trigonometric Rules. Observe:

\[
D_x (\sin x) = [\cos x] [D_x (x)] \\
\quad = [\cos x] [1] \\
\quad = \cos x
\]

The “tail” is simply 1 when the argument \((x\) here) is just the variable of differentiation, so we can ignore the tail in the Basic Trigonometric Rules.
Example 8 (Using the Generalized Trigonometric Rules)

Let $g(\theta) = \sec(9\theta^2 - \theta)$. Find $g'(\theta)$.

§ Solution

$$g'(\theta) = D_\theta\left[\sec(9\theta^2 - \theta)\right]$$
$$= \left[\sec(9\theta^2 - \theta)\tan(9\theta^2 - \theta)\right] \cdot D_\theta(9\theta^2 - \theta)$$

(See Warning 9.)
$$= \left[\sec(9\theta^2 - \theta)\tan(9\theta^2 - \theta)\right] \cdot [18\theta - 1]$$
$$= (18\theta - 1)\sec(9\theta^2 - \theta)\tan(9\theta^2 - \theta)$$

§

Example 9 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)

Let $f(x) = \cos^5(7x)$. Find $f'(x)$.

§ Solution

First, rewrite $f(x)$:
$$f(x) = \cos^5(7x)$$
$$= \left[\cos(7x)\right]^5 \quad (\text{See Warning 7.})$$

Overall, we are differentiating a power, so we will first apply the Generalized Power Rule.

$$f'(x) = \left[5\left[\cos(7x)\right]^4\right] \cdot \left(D_x\left[\cos(7x)\right]\right) \quad (\text{See Warning 5 and Tip 3.})$$

(by the Generalized Power Rule)
$$= \left[5\left[\cos(7x)\right]^4\right] \cdot [-\sin(7x)] \cdot \left[D_x(7x)\right]$$

(by the Generalized Trigonometric Rules)
$$= \left[5\left[\cos(7x)\right]^4\right] \cdot [-\sin(7x)] \cdot [7]$$
$$= -35 \cos^4(7x)\sin(7x)$$

§
**TIP 3: Linear arguments.** If \( a \) is a real constant, then 
\[
D_x \left[ \sin(ax) \right] = a \cos(ax), \\
D_x \left[ \cos(ax) \right] = -a \sin(ax), \\
D_x \left[ \sec(ax) \right] = a \sec(ax) \tan(ax), 
\]
and so forth; the “tail” is still the coefficient of \( x \) in the linear argument.

In Example 9, we saw that: 
\[
D_x \left[ \cos(7x) \right] = -7 \sin(7x). 
\]
These can be very useful short cuts.

• More generally, 
\[
D_x \left[ \sin(ax + b) \right] = a \cos(ax + b), 
\]
and so forth; the “tail” is still the coefficient of \( x \) in the linear argument.

**Example 10 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)**

Show that 
\[
D_x \left[ \tan^4(\pi x) \right] = 4\pi \tan^3(\pi x)\sec^2(\pi x). 
\]
(Left to the reader.) The solution is similar to that in Example 9. 

Hint: First rewrite: 
\[
D_x \left[ \tan^4(\pi x) \right] = D_x \left[ \left[ \tan(\pi x) \right]^4 \right]. 
\]

**Example 11 (Using the Generalized Trigonometric Rules, Followed by the Generalized Power Rule)**

Find 
\[
D_x \left( \csc \left[ (x^2 + 1)^3 \right] \right). 
\]

**Solution**

Overall, we are differentiating a **trigonometric function**, so we will first apply the **Generalized Trigonometric Rules**.

\[
D_x \left( \csc \left[ (x^2 + 1)^3 \right] \right) = \left( -\csc \left[ (x^2 + 1)^3 \right] \cot \left[ (x^2 + 1)^3 \right] \right) \cdot \left( D_x \left[ (x^2 + 1)^3 \right] \right) \]  \quad \text{(by the Generalized Trigonometric Rules)}
\]

\[
= \left( -\csc \left[ (x^2 + 1)^3 \right] \cot \left[ (x^2 + 1)^3 \right] \right) \cdot 3(x^2 + 1)^2 \cdot D_x (x^2 + 1) \]  \quad \text{(by the Generalized Power Rule)}
\]

\[
= \left( -\csc \left[ (x^2 + 1)^3 \right] \cot \left[ (x^2 + 1)^3 \right] \right) \cdot 3(x^2 + 1)^2 \cdot 2x
\]

\[
= -6x(x^2 + 1)^2 \csc \left[ (x^2 + 1)^3 \right] \cot \left[ (x^2 + 1)^3 \right]
\]
PART E: EXAMPLES WITH TANGENT LINES

Example 12 (Finding Horizontal Tangent Lines to a Polynomial Graph)

Let \( f(x) = (x^2 - 9)^7 \). Find the \( x \)-coordinates of all points on the graph of \( y = f(x) \) where the tangent line is horizontal.

\[ \text{Solution} \]

- We must find where the slope of the tangent line to the graph is 0. We must solve the equation:

\[
\frac{d}{dx} f(x) = 0
\]

\[
D_x \left[ (x^2 - 9)^7 \right] = 0
\]

\[
7(x^2 - 9)^6 \cdot D_x (x^2 - 9) = 0 \quad \text{(by the Generalized Power Rule)}
\]

\[
7(x^2 - 9)^6 \cdot 2x = 0
\]

\[
14x(x^2 - 9)^6 = 0
\]

- The Generalized Power Rule is a great help here. The alternative? We could have expanded \( (x^2 - 9)^7 \) by the Binomial Theorem, differentiated the result term-by-term, and then factored the result as \( 14x(x^2 - 9)^6 \) or as \( 14x(x + 3)^6(x - 3)^6 \) … after quite a bit of work!

- Instead of factoring further, we will apply the Zero Factor Property directly:

\[
x = 0 \quad \text{or} \quad x^2 - 9 = 0
\]

\[
x^2 = 9
\]

\[
x = \pm 3
\]

The desired \( x \)-coordinates are: \(-3\), 0, and 3.
• Why does the graph of \( y = \left( x^2 - 9 \right)^7 \) below make sense?

  • Observe that \( f \) is an **even** function.

  • \( f(x) = \left( x^2 - 9 \right)^7 = (x + 3)^7 (x - 3)^7 \), which means that \(-3\) and \(3\) are zeros of \( f \) of multiplicity 7 (see Chapter 2 of the Precalculus notes). As a result, the graph has **x-intercepts** at \((-3,0)\) and \((3,0)\), and the higher multiplicity indicates greater **flatness** around those points. Because the multiplicities are odd, the graph “**cuts through**” the **x-axis** at the x-intercepts, instead of “bouncing off” of the x-axis there.

  • The **y-intercept** is extremely low, because 
  \[
  f(0) = (-9)^7 = -4,782,969.
  \]

• The **red tangent lines** below are truncated.

(Axes are scaled differently.)
Example 13 (Finding Horizontal Tangent Lines to a Trigonometric Graph)

Let \( f(x) = x + \cos(2x) \). Find the \( x \)-coordinates of all points on the graph of \( y = f(x) \) where the tangent line is horizontal.

§ Solution

• We must find where the slope of the tangent line to the graph is 0. We must solve the equation:

\[
 f'(x) = 0 \\
 D_x \left[ x + \cos(2x) \right] = 0 \\
 1 + \left[ -\sin(2x) \right] \cdot \left[ D_x(2x) \right] = 0 \quad \text{(by Generalized Trigonometric Rules)}
\]

\[ 1 - 2\sin(2x) = 0 \quad \text{(See Tip 3 for a short cut.)} \]

\[ -2\sin(2x) = -1 \]

\[ \sin \left( \frac{2x}{\theta} \right) = \frac{1}{2} \]

Use the substitution \( \theta = 2x \).

\[ \sin \theta = \frac{1}{2} \]

Our solutions for \( \theta \) are:

\[ \theta = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad \theta = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z}) \]

To find our solutions for \( x \), replace \( \theta \) with \( 2x \), and solve for \( x \).

\[ 2x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z}) \]

\[ x = \left( \frac{1}{2} \right) \left( \frac{\pi}{6} \right) + \pi n \quad \text{or} \quad x = \left( \frac{1}{2} \right) \left( \frac{5\pi}{6} \right) + \pi n \quad (n \in \mathbb{Z}) \]

\[ x = \frac{\pi}{12} + \pi n \quad \text{or} \quad x = \frac{5\pi}{12} + \pi n \quad (n \in \mathbb{Z}) \]
The desired $x$-coordinates are given by:

$$\left\{ x \in \mathbb{R} \middle| x = \frac{\pi}{12} + \pi n, \text{ or } x = \frac{5\pi}{12} + \pi n, \ (n \in \mathbb{Z}) \right\}.$$ 

- Observe that there are **infinitely many** points on the graph where the tangent line is horizontal.

- Why does the graph of $y = x + \cos(2x)$ below make sense? The “$x$” term leads to upward drift; the graph oscillates about the line $y = x$.

- The **red tangent lines** below are truncated.
FOOTNOTES

1. **Partial proof of the Chain Rule.** Assume that \( g \) is differentiable at \( a \), and \( f \) is differentiable at \( g(a) \). Let \( b = g(a) \). More generally, let \( u = g(x) \). As an optional step, we can let\[ p = f \circ g. \]Then, \( p(x) = (f \circ g)(x) = f(g(x)) \). We will show that \( p \), or \( f \circ g \), is differentiable at \( a \), with\[ p'(a) = \lim_{x \to a} \frac{p(x) - p(a)}{x - a} = \lim_{x \to a} \left[ \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right] \]

\[ = \lim_{x \to a} \left[ \frac{f'(g(x))}{g'(x)} \right] \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \]

\[ = \left[ \lim_{u \to b} \frac{f(u) - f(b)}{u - b} \right] \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \]

(See Note 1 below.)

**Note 1:** We have a problem if \( g(x) = g(a) \) “near” \( x = a \); that is, the partial proof fails if \( g(x) = g(a) \) somewhere on every “punctured” open interval about \( x = a \). The function in Footnote 4 exhibits this problem, where \( a = 0 \). Larson gives a more general proof in Appendix A of his calculus text (9th ed., p.A8). It is not for the faint of heart!

**Note 2:** We assume that \( g \) is differentiable (and thus continuous) at \( a \). Therefore, as \( x \to a \), then \( u \to b \), since \( \lim_{x \to a} u = \lim_{x \to a} g(x) = g(a) = b \).

2. **A controversial form of the Chain Rule.** Some sources give the Chain Rule as:\[ (f \circ g)'(x) = \left[ f'(g(x)) \right] g'(x). \]However, some object to the use of the notation \( f'(g(x)) \).

3. **Radians.** The proofs in Section 3.4 showing that \( D_x (\sin x) = \cos x \) and \( D_x (\cos x) = -\sin x \) utilized the limit statement \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \), which was proven in Footnote 1 of Section 3.4 under the assumption that \( h \) was measured in radians (or as “pure” real numbers).

- Define the “sind” and “cosd” functions as follows:
  - \( \text{sind}(x) = \) the sine of \( x \) degrees, and \( \text{cosd}(x) = \) the cosine of \( x \) degrees.

Now, \( x \) degrees = \( x \) degrees \( \frac{\pi \text{ [radians]}}{180 \text{ [degrees]}} = \frac{\pi}{180}x \text{ [radians]} \).

Therefore, \( \text{sind}(x) = \sin \left( \frac{\pi}{180}x \right) \), and \( \text{cosd}(x) = \cos \left( \frac{\pi}{180}x \right) \).
• Unfortunately, \( D_x \left[ \text{sind}(x) \right] \) is not simply \( \text{cosd}(x) \), as demonstrated below:

\[
D_x \left[ \text{sind}(x) \right] = D_x \left[ \sin \left( \frac{\pi}{180} x \right) \right]
\]
\[
= \left[ \cos \left( \frac{\pi}{180} x \right) \right] \cdot \left[ D_x \left( \frac{\pi}{180} x \right) \right] \quad \text{(by Generalized Trigonometric Rules)}
\]
\[
= \left[ \cos \left( \frac{\pi}{180} x \right) \right] \cdot \left[ \frac{\pi}{180} \right]
\]
\[
= \frac{\pi}{180} \cos \left( \frac{\pi}{180} x \right)
\]
\[
= \frac{\pi}{180} \text{cosd}(x)
\]

Therefore, we prefer the use of our original sine and cosine functions, together with radian measure.


4. **Applicability of the Chain Rule and short cuts.** In Section 3.2, Footnote 7, we defined a piecewise-defined function \( f \) as follows: 

\[
f(x) = \begin{cases} 
   x^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\
   0, & x = 0
\end{cases}
\]

It turns out that \( f'(x) = \begin{cases} 
   2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right), & x \neq 0 \\
   0, & x = 0
\end{cases} \). The Product, Power, and Generalized Trigonometric Rules give us the top rule for \( f'(x) \) when \( x \neq 0 \). However, these rules do not apply when \( x = 0 \), since it is not true that \( f(x) = x^2 \sin \left( \frac{1}{x} \right) \) when \( x = 0 \); in fact, we would have had a problem using these methods at \( x = 0 \) if there were no open interval containing \( x = 0 \) throughout which the rule applied. Nevertheless, \( f'(0) \) does exist! In Section 3.2, Footnote 7, we showed that \( f'(0) = 0 \) using the Limit Definition of the Derivative.