SECTION 3.6: CHAIN RULE

LEARNING OBJECTIVES

- Understand the Chain Rule and use it to differentiate composite functions.
- Know when and how to apply the Generalized Power Rule and the Generalized Trigonometric Rules, which are based on the Chain Rule.

PART A: THE IDEA OF THE CHAIN RULE

Yul, Uma, and Xavier run in a race. Let y, u, and x represent their positions (in miles), respectively.

- Assume that Yul always runs **twice** as fast as Uma. That is, $\frac{dy}{du} = 2$.

(If Uma runs Δu miles, then Yul runs Δy miles, where $\Delta y = 2\,\Delta u$.)

- Assume that Uma always runs **three** times as fast as Xavier. That is, $\frac{du}{dx} = 3$.

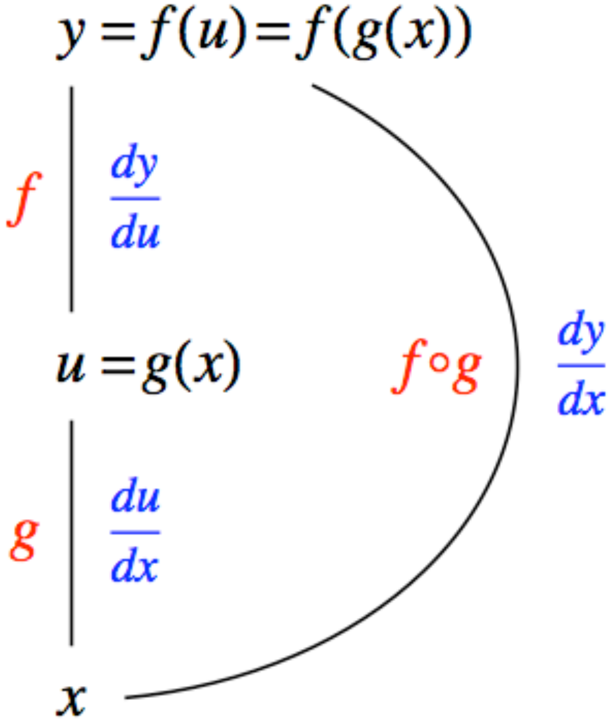
- Therefore, Yul always runs **six** times as fast as Xavier. That is, $\frac{dy}{dx} = 6$.

This is an example of the Chain Rule, which states that: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Here, $6 = 2 \cdot 3$.

WARNING 1: The Chain Rule is a calculus rule, not an algebraic rule, in that the "*du*"s should **not** be thought of as "canceling."

We can think of *y* as a function of *u*, which, in turn, is a function of *x*.
Call these functions f and *g*, respectively.
Then, *y* is a **composite function** of *x*; this function is denoted by $f \circ g$.



- **In multivariable calculus**, you will see bushier trees and more complicated forms of the Chain Rule where you add products of derivatives along paths, extending what we have done above.

TIP 1: The Chain Rule is used to differentiate **composite functions** such as $f \circ g$. The derivative of a **product** of functions is **not** necessarily the product of the derivatives (see Section 3.3 on the Product Rule), but the derivative of a **composition** of functions **is** the product of the derivatives. (Composite functions were reviewed in Chapter 1.)

PART B: FORMS OF THE CHAIN RULE

Chain Rule

Let $y = f(u)$ and $u = g(x)$, where f and g are differentiable "where we care." Then,

Form 1) $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Form 2) $(f \circ g)'(x) = [f'(u)][g'(x)]$

Form 3) $y' = [f'(u)][u']$

- Essentially, the derivative of a **composite function** is obtained by taking the derivative of the "outer function" at u **times** the derivative of the "inner function" at x.

- Following *How to Ace Calculus* by Adams, Thompson, and Hass (Times, 1998), we will refer to the derivative of the **inner** function as the "**tail**." In the forms above, the tail is denoted by $\dfrac{du}{dx}$, $g'(x)$, and u'.

WARNING 2: Forgetting the "tail" is a very common error students make when applying the Chain Rule.

- See Footnote 1 for a partial proof.

- See Footnote 2 on a controversial form.

Many differentiation rules, such as the Generalized Power Rule and the Generalized Trigonometric Rules we will introduce in this section, are **based on the Chain Rule**.

PART C: GENERALIZED POWER RULE

The **Power Rule of Differentiation**, which we introduced in Part B of Section 3.2, can be used to find $D_x\left(x^7\right)$. However, it **cannot** be used to find $D_x\left[\left(3x^2+4\right)^7\right]$ without expanding the indicated seventh power, something we would rather not do.

Example 1 (Using the Chain Rule to Motivate the Generalized Power Rule)

Use the **Chain Rule** to find $D_x\left[\left(3x^2+4\right)^7\right]$.

§ Solution

Let $y=\left(3x^2+4\right)^7$. We will treat y as a **composite function** of x.
$y=(f\circ g)(x)=f\left(g(x)\right)$, where:

$$\begin{aligned} u &= g(x)=3x^2+4 && (g \text{ is the “inner function”}) \\ y &= f(u)=u^7 && (f \text{ is the “outer function”}) \end{aligned}$$

Observe that $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ can be readily found using **basic rules**.

We can then find $\dfrac{dy}{dx}$ using the **Chain Rule**.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= \left[D_u\left(u^7\right)\right]\left[D_x\left(3x^2+4\right)\right] \\ &= \left[7u^6\right]\left[6x\right] \end{aligned}$$

$(\text{Since } u \text{ was our creation, we must express } u \text{ in terms of } x.)$

$$\begin{aligned} &= \left[7\left(3x^2+4\right)^6\right]\left[6x\right] \quad (\text{See Example 2 for a short cut.}) \\ &= 42x\left(3x^2+4\right)^6 \end{aligned}$$

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Example 1 suggests the following short cut.

Generalized Power Rule

Let u be a function of x that is differentiable "where we care."
Let n be a real constant.

$$D_x\left(u^n\right)=\left(nu^{n-1}\right)\underbrace{\left(D_x u\right)}_{\text{"tail"}}$$

- **Rationale.** If $y=u^n$, then, by the Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \quad \Rightarrow$$

$$D_x\left(u^n\right)=\left[D_u\left(u^n\right)\right]\cdot\left[D_x u\right]$$

$$=\left[nu^{n-1}\right]\cdot\left[D_x u\right]$$

Example 2 (Using the Generalized Power Rule; Revisiting Example 1)

Use the **Generalized Power Rule** to find $D_x\left[\left(3x^2+4\right)^7\right]$.

§ Solution

$$D_x\left[\left(3x^2+4\right)^7\right]=\left[7\left(3x^2+4\right)^6\right]\left[D_x\left(3x^2+4\right)\right] \quad \text{(See Warning 3.)}$$

$$=\left[7\left(3x^2+4\right)^6\right]\left[6x\right]$$

$$=42x\left(3x^2+4\right)^6$$

WARNING 3: Copy the base. The base u, which is $\left(3x^2+4\right)$ here, is **copied** under the exponent. Do **not** differentiate it until you get to the "**tail**."

WARNING 4: Remember the exponent. Many students forget to write the exponent, 6, because the base can take some time to write. You may want to write the exponent **before** writing out the base.

WARNING 5: Identifying "tails." The D_x notation helps us keep track of "how far" to unravel tails. **A tail may have a tail of its own.** If we forget tails, we're **not going far enough**. If we attach inappropriate tails (such as an additional "6" after the "6*x*" above), we're going **too far**. §

Example 3 (Using the Generalized Power Rule in Conjunction with the Quotient Rule or the Product Rule)

Find $D_x\left(\frac{x^3}{\sqrt{2x-1}}\right)$.

§ Solution Method 1 (Using the Quotient Rule)

$$D_x\left(\frac{x^3}{\sqrt{2x-1}}\right)=\frac{\text{Lo}\cdot D(\text{Hi})-\text{Hi}\cdot D(\text{Lo})}{(\text{Lo})^2\text{, the square of what's below}}$$

$$=\frac{\left(\sqrt{2x-1}\right)\cdot\left[D_x\left(x^3\right)\right]\;-\;\left(x^3\right)\cdot\left(D_x\left[(2x-1)^{1/2}\right]\right)}{\left(\sqrt{2x-1}\right)^2}$$

$$=\frac{\left(\sqrt{2x-1}\right)\cdot\left[3x^2\right]\;-\;\left(x^3\right)\cdot\left[\frac{1}{2}(2x-1)^{-1/2}\right]\cdot\left[D_x(2x-1)\right]}{2x-1}$$

$$=\frac{\left(\sqrt{2x-1}\right)\cdot\left[3x^2\right]\;-\;\left(x^3\right)\cdot\left[\cancel{\frac{1}{2}}(2x-1)^{-1/2}\right]\cdot\left[\cancel{2}\right]}{2x-1}$$

(We could factor the numerator at this point.)

$$=\frac{\left(\sqrt{2x-1}\right)\cdot\left[3x^2\right]\;-\;\frac{x^3}{\sqrt{2x-1}}}{2x-1}$$

$$=\frac{\left[\left(\sqrt{2x-1}\right)\cdot\left[3x^2\right]\;-\;\frac{x^3}{\sqrt{2x-1}}\right]}{2x-1}\cdot\frac{\sqrt{2x-1}}{\sqrt{2x-1}}$$

WARNING 6: Distribute before canceling. Do **not** cancel in the numerators until we have distributed $\sqrt{2x-1}$ through the first numerator.

$$=\frac{(2x-1)\cdot\left[3x^2\right]-x^3}{(2x-1)^{3/2}}$$

$$=\frac{6x^3-3x^2-x^3}{(2x-1)^{3/2}}$$

$$=\frac{5x^3-3x^2}{(2x-1)^{3/2}},\quad\text{or}\quad\frac{x^2(5x-3)}{(2x-1)^{3/2}}$$

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§ Solution Method 2 (Using the Product Rule)

If we had forgotten the Quotient Rule, we could have rewritten:

$D_x\left(\dfrac{x^3}{\sqrt{2x-1}}\right) = D_x\left[x^3(2x-1)^{-1/2}\right]$ and applied the **Product Rule**.

We would then use the **Generalized Power Rule** to find $D_x\left[(2x-1)^{-1/2}\right]$.

The key **drawback** here is that we obtain two terms, and students may find it difficult to combine them into a **single, simplified fraction**. Observe:

$$D_x\left(\frac{x^3}{\sqrt{2x-1}}\right) = D_x\left[x^3(2x-1)^{-1/2}\right] \quad \text{(Rewriting)}$$

$$= \left[D_x\left(x^3\right)\right]\cdot\left[(2x-1)^{-1/2}\right] + \left(x^3\right)\cdot\left(D_x\left[(2x-1)^{-1/2}\right]\right)$$

(by the Product Rule)

$$= \left[3x^2\right]\cdot\left[(2x-1)^{-1/2}\right] + \left(x^3\right)\cdot\left(\left[-\frac{1}{2}(2x-1)^{-3/2}\right]\cdot\left[D_x(2x-1)\right]\right)$$

(by the Generalized Power Rule)

$$= \left[3x^2\right]\cdot\left[(2x-1)^{-1/2}\right] + \left(x^3\right)\cdot\left(\left[-\frac{\cancel{1}}{\cancel{2}}(2x-1)^{-3/2}\right]\cdot\left[\cancel{2}\right]\right)$$

$$= \left[3x^2\right]\cdot\left[(2x-1)^{-1/2}\right] + \left(x^3\right)\cdot\left(\left[-(2x-1)^{-3/2}\right]\right)$$

$$= \frac{3x^2}{(2x-1)^{1/2}} - \frac{x^3}{(2x-1)^{3/2}}$$

$$= \frac{3x^2}{(2x-1)^{1/2}}\cdot\frac{(2x-1)}{(2x-1)} - \frac{x^3}{(2x-1)^{3/2}}$$

Build up the first fraction to obtain the LCD, $(2x-1)^{3/2}$.

$$= \frac{3x^2(2x-1)-x^3}{(2x-1)^{3/2}}$$

$$= \frac{6x^3-3x^2-x^3}{(2x-1)^{3/2}}$$

$$= \frac{5x^3-3x^2}{(2x-1)^{3/2}}, \text{ or } \frac{x^2(5x-3)}{(2x-1)^{3/2}} \quad \text{(as in Method 1)}$$

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Example 4 (Using the Generalized Power Rule to Differentiate a Power of a Trigonometric Function)

Let $f(\theta) = \sec^5 \theta$. Find $f'(\theta)$.

§ Solution

First, rewrite $f(\theta)$:

$$\begin{aligned} f(\theta) &= \sec^5 \theta \\ &= (\sec\theta)^5 \end{aligned}$$

WARNING 7: Rewriting before differentiating. When differentiating a **power of a trigonometric function**, rewrite the power in this way. Students get very confused otherwise. Also, do **not** write $\sec\theta^5$ here; that is equivalent to $\sec\left(\theta^5\right)$, not $(\sec\theta)^5$.

$$f'(\theta) = \left[5(\sec\theta)^4\right] \cdot \left[D_\theta(\sec\theta)\right]$$

(by the Generalized Power Rule)

$$\begin{aligned} &= \left[5(\sec\theta)^4\right] \cdot \left[\sec\theta\tan\theta\right] \\ &= 5(\sec\theta)^5 \tan\theta \\ &= 5\sec^5\theta\tan\theta \end{aligned}$$

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Example 5 (Using the Generalized Power Rule to Prove the Reciprocal Rule)

Prove the **Reciprocal Rule** from Section 3.3: $D_x\left[\dfrac{1}{g(x)}\right] = -\dfrac{g'(x)}{\left[g(x)\right]^2}$.

§ Solution

$$D_x\left[\frac{1}{g(x)}\right] = D_x\left(\left[g(x)\right]^{-1}\right)$$

WARNING 8: “-1” here denotes a reciprocal, **not** a function inverse.

$$= \left(-\left[g(x)\right]^{-2}\right) \cdot \left[g'(x)\right] \quad \text{(by the Generalized Power Rule)}$$

$$= -\frac{g'(x)}{\left[g(x)\right]^2}$$

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PART D: GENERALIZED TRIGONOMETRIC RULES

The **Basic Trigonometric Rules of Differentiation**, which we introduced in Section 3.4, can be used to find $D_x(\sin x)$. However, they **cannot** be used to find $D_x\left[\sin\left(x^2\right)\right]$.

Example 6 (Using the Chain Rule to Motivate the Generalized Trigonometric Rules)

Use the **Chain Rule** to find $D_x\left[\sin\left(x^2\right)\right]$.

§ Solution

Let $y = \sin\left(x^2\right)$. We will treat *y* as a **composite function** of *x*.

$y = (f \circ g)(x) = f\left(g(x)\right)$, where:

$$u = g(x) = x^2 \qquad \text{(g is the “inner function”)}$$

$$y = f(u) = \sin u \qquad \text{(f is the “outer function”)}$$

Observe that $\frac{dy}{du}$ and $\frac{du}{dx}$ can be readily found using **basic rules**.

We can then find $\frac{dy}{dx}$ using the **Chain Rule**.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left[D_u(\sin u)\right]\left[D_x\left(x^2\right)\right] \\ &= [\cos u][2x]\end{aligned}$$

$$\left(\text{Since } u \text{ was our creation, we must express } u \text{ in terms of } x.\right)$$

$$\begin{aligned}&= \left[\cos\left(x^2\right)\right][2x] \qquad \left(\text{See Example 7 for a short cut.}\right) \\ &= 2x\cos\left(x^2\right)\end{aligned}$$

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Example 6 suggests the following short cuts.

Generalized Trigonometric Rules

Let u be a function of x that is differentiable "where we care" (see Footnote 4).

$$D_x(\sin u) = (\cos u)\underbrace{(D_x u)}_{\text{"tail"}} \qquad D_x(\cos u) = (-\sin u)\underbrace{(D_x u)}_{\text{"tail"}}$$

$$D_x(\tan u) = (\sec^2 u)\underbrace{(D_x u)}_{\text{"tail"}} \qquad D_x(\cot u) = (-\csc^2 u)\underbrace{(D_x u)}_{\text{"tail"}}$$

$$D_x(\sec u) = (\sec u \tan u)\underbrace{(D_x u)}_{\text{"tail"}} \qquad D_x(\csc u) = (-\csc u \cot u)\underbrace{(D_x u)}_{\text{"tail"}}$$

WARNING 9: In the **bottom two rules**, the "**tail**" is still written only **once**. The "tail" is the derivative of the common argument u.

- **Radians.** See Footnote 3 on how these rules encourage us to use radians (as opposed to degrees) when differentiating trigonometric functions.

Example 7 (Using the Generalized Trigonometric Rules; Revisiting Example 6)

Use the **Generalized Trigonometric Rules** to find $D_x\left[\sin\left(x^2\right)\right]$.

§ Solution

$$\begin{aligned} D_x\left[\sin\left(x^2\right)\right] &= \left[\cos\left(x^2\right)\right]\left[D_x\left(x^2\right)\right] \quad \text{(See Warning 10.)} \\ &= \left[\cos\left(x^2\right)\right][2x] \\ &= 2x\cos\left(x^2\right) \end{aligned}$$

WARNING 10: Copy the argument. The sine function's argument u, which is $\left(x^2\right)$ here, is **copied** as the cosine function's argument. Do **not** differentiate it until you get to the "**tail**." §

TIP 2: Consistency with the Basic Trigonometric Rules. Observe:

$$\begin{aligned} D_x(\sin x) &= [\cos x][D_x(x)] \\ &= [\cos x][1] \\ &= \cos x \end{aligned}$$

The "tail" is simply 1 when the argument (x here) is just the variable of differentiation, so we can ignore the tail in the Basic Trigonometric Rules.

Example 8 (Using the Generalized Trigonometric Rules)

Let $g(\theta)=\sec\left(9\theta^2-1\right)$. Find $g'(\theta)$.

§ Solution

$$g'(\theta)=D_\theta\left[\sec\left(9\theta^2-1\right)\right]$$

$$=\left[\sec\left(9\theta^2-1\right)\tan\left(9\theta^2-1\right)\right]\cdot\left[D_\theta\left(9\theta^2-1\right)\right]$$

(See Warning 9.)

$$=\left[\sec\left(9\theta^2-1\right)\tan\left(9\theta^2-1\right)\right]\cdot[18\theta]$$

$$=18\theta\sec\left(9\theta^2-1\right)\tan\left(9\theta^2-1\right)$$

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Example 9 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)

Let $f(x)=\cos^5(7x)$. Find $f'(x)$.

§ Solution

First, rewrite $f(x)$:

$$f(x)=\cos^5(7x)$$

$$=\left[\cos(7x)\right]^5 \quad \text{(See Warning 7.)}$$

Overall, we are differentiating a **power**, so we will first apply the **Generalized Power Rule**.

$$f'(x)=\left(5\left[\cos(7x)\right]^4\right)\cdot\left(D_x\left[\cos(7x)\right]\right) \quad \text{(See Warning 5 and Tip 3.)}$$

(by the Generalized Power Rule)

$$=\left(5\left[\cos(7x)\right]^4\right)\cdot\left[-\sin(7x)\right]\cdot\left[D_x(7x)\right]$$

(by the Generalized Trigonometric Rules)

$$=\left(5\left[\cos(7x)\right]^4\right)\cdot\left[-\sin(7x)\right]\cdot[7]$$

$$=-35\cos^4(7x)\sin(7x)$$

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TIP 3: Linear arguments. If a is a real constant, then $D_x\left[\sin(ax)\right]=a\cos(ax)$, $D_x\left[\cos(ax)\right]=-a\sin(ax)$, $D_x\left[\sec(ax)\right]=a\sec(ax)\tan(ax)$, etc. In Example 9, we saw that: $D_x\left[\cos(7x)\right]=-7\sin(7x)$. These can be very useful short cuts.

- More generally, $D_x\left[\sin(ax+b)\right]=a\cos(ax+b)$, and so forth; the "tail" is still the **coefficient of *x*** in the linear argument.

Example 10 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)

Show that $D_x\left[\tan^4(\pi x)\right]=4\pi\tan^3(\pi x)\sec^2(\pi x)$.
(Left to the reader.) The solution is similar to that in Example 9.
Hint: First rewrite: $D_x\left[\tan^4(\pi x)\right]=D_x\left(\left[\tan(\pi x)\right]^4\right)$. §

Example 11 (Using the Generalized Trigonometric Rules, Followed by the Generalized Power Rule)

Find $D_x\left(\csc\left[\left(x^2+1\right)^3\right]\right)$.

§ Solution

Overall, we are differentiating a **trigonometric function**, so we will first apply the **Generalized Trigonometric Rules**.

$$D_x\left(\csc\left[\left(x^2+1\right)^3\right]\right)=\left(-\csc\left[\left(x^2+1\right)^3\right]\cot\left[\left(x^2+1\right)^3\right]\right)\cdot\left(D_x\left[\left(x^2+1\right)^3\right]\right)$$

$$\left(\text{by the Generalized Trigonometric Rules}\right)$$

$$=\left(-\csc\left[\left(x^2+1\right)^3\right]\cot\left[\left(x^2+1\right)^3\right]\right)\cdot\left[3\left(x^2+1\right)^2\right]\cdot\left[D_x\left(x^2+1\right)\right]$$

$$\left(\text{by the Generalized Power Rule}\right)$$

$$=\left(-\csc\left[\left(x^2+1\right)^3\right]\cot\left[\left(x^2+1\right)^3\right]\right)\cdot\left[3\left(x^2+1\right)^2\right]\cdot[2x]$$

$$=-6x\left(x^2+1\right)^2\csc\left[\left(x^2+1\right)^3\right]\cot\left[\left(x^2+1\right)^3\right]$$

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PART E: EXAMPLES WITH TANGENT LINES

Example 12 (Finding Horizontal Tangent Lines to a Polynomial Graph)

Let $f(x)=\left(x^2-9\right)^7$. Find the *x*-coordinates of all points on the graph of $y=f(x)$ where the **tangent line** is **horizontal**.

§ Solution

- We must find where the **slope** of the tangent line to the graph is 0. We must solve the equation:

$$\begin{aligned} f'(x) &= 0 \\ D_x\left[\left(x^2-9\right)^7\right] &= 0 \\ \left[7\left(x^2-9\right)^6\right]\cdot\left[D_x\left(x^2-9\right)\right] &= 0 \quad \text{(by the Generalized Power Rule)} \\ \left[7\left(x^2-9\right)^6\right]\cdot[2x] &= 0 \\ 14x\left(x^2-9\right)^6 &= 0 \end{aligned}$$

- The **Generalized Power Rule** is a great help here. The alternative? We could have expanded $\left(x^2-9\right)^7$ by the Binomial Theorem, differentiated the result term-by-term, and then factored the result as $14x\left(x^2-9\right)^6$ or as $14x(x+3)^6(x-3)^6$ … after quite a bit of work!

- Instead of factoring further, we will apply the **Zero Factor Property** directly:

$$\begin{aligned} x=0 \quad \text{or} \quad x^2-9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The desired *x*-coordinates are: -3, 0, and 3.

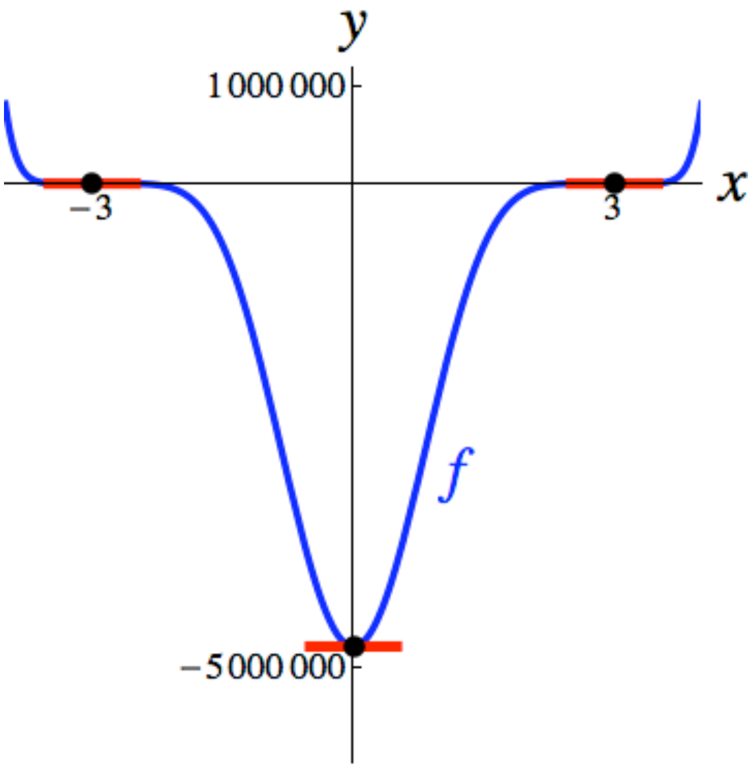
- Why does the graph of $y = \left(x^2 - 9\right)^7$ below make sense?

  - • Observe that f is an **even** function.

  - • $f(x) = \left(x^2 - 9\right)^7 = (x+3)^7 (x-3)^7$, which means that -3 and 3 are zeros of f of multiplicity 7 (see Chapter 2 of the Precalculus notes). As a result, the graph has ***x*-intercepts** at $(-3, 0)$ and $(3, 0)$, and the higher multiplicity indicates greater **flatness** around those points. Because the multiplicities are odd, the graph **"cuts through" the *x*-axis** at the *x*-intercepts, instead of "bouncing off" of the *x*-axis there.

  - • The ***y*-intercept** is extremely low, because $f(0) = (-9)^7 = -4{,}782{,}969$.

- The **red tangent lines** below are truncated.



(Axes are scaled differently.)

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Example 13 (Finding Horizontal Tangent Lines to a Trigonometric Graph)

Let $f(x) = x + \cos(2x)$. Find the x-coordinates of all points on the graph of $y = f(x)$ where the **tangent line** is **horizontal**.

§ Solution

- We must find where the **slope** of the tangent line to the graph is 0.

We must solve the equation:

$$\begin{aligned}
f'(x) &= 0 \\
D_x\left[x + \cos(2x)\right] &= 0 \\
1 + \left[-\sin(2x)\right] \cdot \left[D_x(2x)\right] &= 0 \quad \text{(by Generalized Trigonometric Rules)} \\
1 + \left[-\sin(2x)\right] \cdot [2] &= 0 \\
1 - 2\sin(2x) &= 0 \quad \text{(See Tip 3 for a short cut.)} \\
-2\sin(2x) &= -1 \\
\sin\underbrace{(2x)}_{\theta} &= \frac{1}{2}
\end{aligned}$$

Use the **substitution** $\theta = 2x$.

$$\sin\theta = \frac{1}{2}$$

Our solutions for θ are:

$$\theta = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad \theta = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z})$$

To find our solutions for x, replace θ with $2x$, and solve for x.

$$\begin{aligned}
2x &= \frac{\pi}{6} + 2\pi n & \text{or} \quad 2x &= \frac{5\pi}{6} + 2\pi n & (n \in \mathbb{Z}) \\
x &= \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \pi n & \text{or} \quad x &= \left(\frac{1}{2}\right)\left(\frac{5\pi}{6}\right) + \pi n & (n \in \mathbb{Z}) \\
x &= \frac{\pi}{12} + \pi n & \text{or} \quad x &= \frac{5\pi}{12} + \pi n & (n \in \mathbb{Z})
\end{aligned}$$

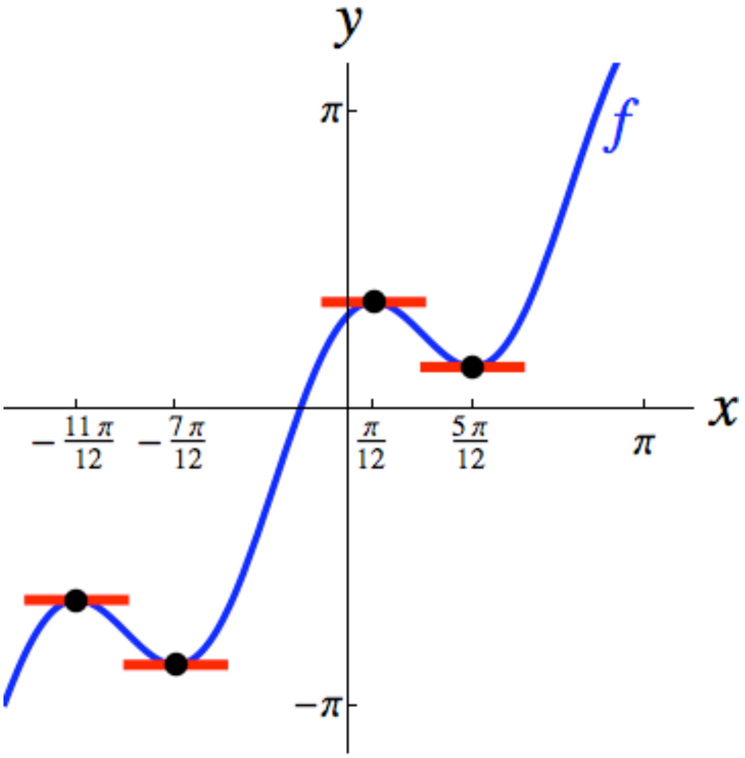
The desired *x*-coordinates are given by:

$$\left\{ x \in \mathbb{R} \,\middle|\, x = \frac{\pi}{12} + \pi n, \quad \text{or} \quad x = \frac{5\pi}{12} + \pi n, \ (n \in \mathbb{Z}) \right\}.$$

- Observe that there are **infinitely many** points on the graph where the tangent line is horizontal.

- Why does the graph of $y = x + \cos(2x)$ below make sense? The "x" term leads to upward drift; the graph oscillates about the line $y = x$.

- The **red tangent lines** below are truncated.



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FOOTNOTES

1. **Partial proof of the Chain Rule.** Assume that g is differentiable at a, and f is differentiable at $g(a)$. Let $b = g(a)$. More generally, let $u = g(x)$. As an optional step, we can let $p = f \circ g$. Then, $p(x) = (f \circ g)(x) = f(g(x))$. We will show that p, or $f \circ g$, is differentiable at a, with $p'(a) = (f \circ g)'(a) = [f'(b)][g'(a)]$.

$$\begin{aligned} p'(a) &= \lim_{x \to a} \frac{p(x) - p(a)}{x - a} \\ &= \lim_{x \to a} \left[\frac{p(x) - p(a)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right], \quad [g(x) \neq g(a)] \quad \text{(See Note 1 below.)} \\ &= \lim_{x \to a} \left[\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right] \\ &= \left[\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \right] \cdot \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right] \\ &= \left[\lim_{u \to b} \frac{f(u) - f(b)}{u - b} \right] \cdot \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right] \quad \text{(See Note 2 below.)} \\ &= [f'(b)] \cdot [g'(a)] \end{aligned}$$

Note 1: We have a problem if $g(x) = g(a)$ "near" $x = a$; that is, the partial proof fails if $g(x) = g(a)$ somewhere on every "punctured" open interval about $x = a$. The function in Footnote 4 exhibits this problem, where $a = 0$. Larson gives a more general proof in Appendix A of his calculus text (9th ed., p.A8). It is not for the faint of heart!

Note 2: We assume that g is differentiable (and thus **continuous**) at a. Therefore, as $x \to a$, then $u \to b$, since $\lim_{x \to a} u = \lim_{x \to a} g(x) = g(a) = b$.

2. **A controversial form of the Chain Rule.** Some sources give the Chain Rule as: $(f \circ g)'(x) = [f'(g(x))][g'(x)]$. However, some object to the use of the notation $f'(g(x))$.

3. **Radians.** The proofs in Section 3.4 showing that $D_x(\sin x) = \cos x$ and $D_x(\cos x) = -\sin x$ utilized the limit statement $\lim_{h \to 0} \frac{\sin h}{h} = 1$, which was proven in Footnote 1 of Section 3.4 under the assumption that h was measured in radians (or as "pure" real numbers).

- Define the "sind" and "cosd" functions as follows:

$\text{sind}(x) =$ the sine of x degrees, and $\text{cosd}(x) =$ the cosine of x degrees.

Now, $x \text{ degrees} = \left(x \ \cancel{\text{degrees}}\right)\left(\frac{\pi \text{ [radians]}}{180 \ \cancel{\text{degrees}}}\right) = \frac{\pi}{180} x \text{ [radians]}$.

Therefore, $\text{sind}(x) = \sin\left(\frac{\pi}{180} x\right)$, and $\text{cosd}(x) = \cos\left(\frac{\pi}{180} x\right)$.

- Unfortunately, $D_x\left[\operatorname{sind}(x)\right]$ is **not** simply $\operatorname{cosd}(x)$, as demonstrated below:

$$\begin{aligned} D_x\left[\operatorname{sind}(x)\right] &= D_x\left[\sin\left(\frac{\pi}{180}x\right)\right] \\ &= \left[\cos\left(\frac{\pi}{180}x\right)\right]\cdot\left[D_x\left(\frac{\pi}{180}x\right)\right] \quad \text{(by Generalized Trigonometric Rules)} \\ &= \left[\cos\left(\frac{\pi}{180}x\right)\right]\cdot\left[\frac{\pi}{180}\right] \\ &= \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\right) \\ &= \frac{\pi}{180}\operatorname{cosd}(x) \end{aligned}$$

Therefore, we prefer the use of our original sine and cosine functions, together with radian measure.

- See *The Math Forum @ Drexel* on the web: http://mathforum.org/, particularly http://mathforum.org/library/drmath/view/53779.html with Dr. Peterson.

4. **Applicability of the Chain Rule and short cuts.** In Section 3.2, Footnote 7, we defined a piecewise-defined function f as follows: $f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$. It turns out that $f'(x)=\begin{cases} 2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$. The Product, Power, and Generalized Trigonometric Rules give us the top rule for $f'(x)$ when $x\neq 0$. However, these rules do **not** apply when $x=0$, since it is not true that $f(x)=x^2\sin\left(\frac{1}{x}\right)$ when $x=0$; in fact, we would have had a problem using these methods at $x=0$ if there were no open interval containing $x=0$ throughout which the rule applied. Nevertheless, $f'(0)$ does exist! In Section 3.2, Footnote 7, we showed that $f'(0)=0$ using the Limit Definition of the Derivative.