SECTION 3.7: IMPLICIT DIFFERENTIATION

LEARNING OBJECTIVES

- Understand that an equation can "determine" many implicit functions.
- Perform Implicit Differentiation and obtain templates of differentiation rules built on basic rules such as the Chain Rule.
- Relate derivatives obtained from Implicit Differentiation to slopes of tangent lines to a graph, even if the graph fails the Vertical Line Test (VLT).

PART A: EXPLICIT vs. IMPLICIT DEFINITIONS OF FUNCTIONS

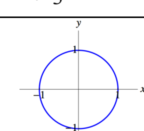
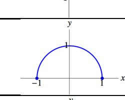
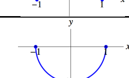
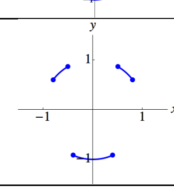
• The equation $y = x + 1$ defines y explicitly as a function of x.
If $f(x) = x + 1$, then f is the corresponding explicit function.

• The equation $y - x = 1$ defines y implicitly as a function of x.
$y - x = 1$ is equivalent to our first equation, $y = x + 1$.
However, it is **not** solved for y. y is "buried in" the equation.
If $f(x) = x + 1$, then f is the corresponding implicit function.
(Technically, if we restrict the domain of f, we get other implicit functions.)

• The equation $x^2 + y^2 = 1$ "determines" (a questionable term, but used by some sources) **many** implicit functions f: if $y = f(x)$, then the equation is satisfied.

•• The graph of $x^2 + y^2 = 1$ **fails** the Vertical Line Test (VLT), and solving for y yields $y = \pm\sqrt{1 - x^2}$, in which y is **not** a well-defined function of x.

•• However, any non-empty **part** of the graph that **passes** the VLT corresponds to an **implicit function** – examples are f_1, f_2, and f_3 below.

Graph of $x^2 + y^2 = 1$	
Graph of $y = f_1(x)$, where $f_1(x) = \sqrt{1 - x^2}$	
Graph of $y = f_2(x)$, where $f_2(x) = -\sqrt{1 - x^2}$	
Graph of $y = f_3(x)$, where $f_3(x) = \begin{cases} \sqrt{1 - x^2}, & 0.5 \le \lvert x \rvert \le 0.8 \\ -\sqrt{1 - x^2}, & \lvert x \rvert \le 0.4 \end{cases}$	

• $x^2 + y^2 = -1$, whose graph is **empty**, determines **no** implicit functions.

PART B: IMPLICIT DIFFERENTIATION

• We assume that the equations in this section determine at least one implicit function f that is **differentiable** "where needed." Here, $y = f(x)$, but we can also have $h = s(t)$, etc.

• **Notation.** We assume that $y' = \dfrac{dy}{dx} = D_x y$.

WARNING 1: y'. The y' notation can be ambiguous if we have many variables, or if we are dealing with $\dfrac{dy}{dt}$, say, instead of $\dfrac{dy}{dx}$. In Section 3.8, we will often prefer Leibniz notation such as $\dfrac{dy}{dt}$.

• **Templates (patterns).** Implicit Differentiation can apply prior rules such as the Chain Rule to build new differentiation rules and templates.

Example 1 (Using Implicit Differentiation for a "Constant Multiple" Template)

$$D_x(7y) = 7 \cdot D_x(y), \text{ or } 7\left(\frac{dy}{dx}\right), \text{ or } 7y'.$$

y could be $\sin x$, $x^2 + 1$, etc. y is a kind of placeholder. (Imagine a kernel popping.) The derivative of 7 times "it" is 7 times the derivative of "it."

Example Set 2 (Implicit Differentiation)

#	Example	Comments
1	$D_x(7y) = 7y'$	See Example 1.
2	$D_x(y^2) = 2yy'$	by Generalized Power Rule
3	$D_x(y^3) = 3y^2y'$	by Generalized Power Rule
4	$D_x(x^3y^2) = \left[D_x(x^3)\right]\cdot\left[y^2\right] + \left[x^3\right]\cdot\left[D_x(y^2)\right]$ $= \left[3x^2\right]\cdot\left[y^2\right] + \left[x^3\right]\cdot\left[2yy'\right]$ $= 3x^2y^2 + 2x^3yy'$	by Product, Gen. Power Rules
5	$D_x\left[(x+y)^4\right] = \left[4(x+y)^3\right]\cdot\left[D_x(x+y)\right]$ $= 4(x+y)^3\left[1+y'\right]$	by Generalized Power Rule

§

Example 3 (Implicit Differentiation)

$$D_x\left[\cos^3(xy)\right] = D_x\left(\left[\cos(xy)\right]^3\right) \quad \text{(Rewriting)}$$

$$= \left(3\left[\cos(xy)\right]^2\right)\cdot\left(D_x\left[\cos(xy)\right]\right) \quad \text{(by Gen. Power Rule)}$$

$$= \left(3\left[\cos(xy)\right]^2\right)\cdot\left[-\sin(xy)\right]\cdot\left[D_x(xy)\right]$$

$$\text{(by Gen. Trig Rules)}$$

$$= \left(3\left[\cos(xy)\right]^2\right)\cdot\left[-\sin(xy)\right]\cdot\left(\left[D_x(x)\right]\cdot y + x\cdot\left[D_x(y)\right]\right)$$

$$\text{(by Product Rule)}$$

$$= \left(3\left[\cos(xy)\right]^2\right)\cdot\left[-\sin(xy)\right]\cdot\left(1\cdot y + x\cdot y'\right)$$

$$= -3\left(y+xy'\right)\left[\cos^2(xy)\right]\sin(xy)$$

§

Example 4 (Using Implicit Differentiation to Find y')

Consider the given equation $x^2-2x+y^2+6y=15$. Assume that it "determines" implicit differentiable functions f such that $y=f(x)$.

Find a general formula for y', or $\dfrac{dy}{dx}$.

§ Solution

• If we solve for y by using Completing the Square (CTS), we obtain: $y=-3\pm\sqrt{25-(x-1)^2}$. (See Example 7.)

• Instead, Implicit Differentiation may be easier.

Step 1. Implicitly differentiate **both** sides of the given equation with respect to x. We expect the result to include y'.

$$x^2-2x+y^2+6y=15 \quad \Rightarrow$$
$$D_x\left(x^2-2x+y^2+6y\right)=D_x(15)$$

WARNING 2: Write this last step. Otherwise, you are in danger of copying "15" instead of differentiating it! Very common error…

$$2x-2+2yy'+6y'=0$$

Simplify by dividing both sides by 2.

$$x-1+yy'+3y'=0$$

Step 2. Isolate terms with y' on one side.

$$yy'+3y'=1-x$$

Step 3. Factor out y' on that side.

$$y'(y+3)=1-x$$

Step 4. Solve for y' by dividing both sides by the other factor.

$$y'=\frac{1-x}{y+3}$$

Note 1: If we had not divided by 2 earlier, we would have had $y'=\dfrac{2-2x}{2y+6}$, which must be **simplified** to, say, $y'=\dfrac{1-x}{y+3}$.

Note 2: Our formula for y' **includes *y***, itself! This allows us to analyze points on the graph of $x^2-2x+y^2+6y=15$ with the same *x*-coordinate but **different *y*-coordinates** (like "tiebreakers"). §

Example 5 (Evaluating y'; Revisiting Example 4)

Find the **slope of the tangent line** to the graph of $x^2-2x+y^2+6y=15$ at the point $(4,1)$ in the usual *xy*-plane.

§ Solution

Although it shouldn't be necessary, we could **check** to ensure that the point $(4,1)$ **lies on the graph**. We can plug in (substitute) $x=4$ and $y=1$ into the equation to see if $(4,1)$ is a solution point:

$$(4)^2-2(4)+(1)^2+6(1)\stackrel{?}{=}15$$

$$15=15 \quad \text{(Checks)}$$

Therefore, $(4,1)$ lies on the graph. (Let's assume it's not an endpoint.)

Our formula for y' from Example 4 will be **evaluated** at $(4, 1)$ to find the **slope of the tangent line** at that point. Plug in (substitute) $x = 4$ and $y = 1$.

$$\left[y'\right]_{(4,\,1)} = \left[\frac{1-x}{y+3}\right]_{(4,\,1)} = \frac{1-(4)}{(1)+3} = -\frac{3}{4}$$

The desired slope is $-\dfrac{3}{4}$.

WARNING 3: Know the difference. Substituting into the **original equation** is a check to see if the point is on the graph. Substituting into the $\boldsymbol{y'}$ **formula** gives us the slope.

Note 1: If we had a point, such as $(0, 0)$, that were **not on the graph**, then the corresponding derivative would be **undefined (DNE)**: $\left[y'\right]_{(0,\,0)}$ DNE.

Note 2: If we just want a **single** derivative value, then it is **not** necessary to have a general formula for y', such as $y' = \dfrac{1-x}{y+3}$. In Example 4, we could have **plugged in (substituted)** $x = 4$ and $y = 1$ soon after we implicitly **differentiated** both sides of the given equation.

$$\begin{aligned}
x^2 - 2x + y^2 + 6y = 15 \quad &\Rightarrow \\
D_x\left(x^2 - 2x + y^2 + 6y\right) = D_x(15)& \\
2x - 2 + 2yy' + 6y' = 0& \\
x - 1 + yy' + 3y' = 0 \quad &\Rightarrow \quad (\text{at } (4, 1)) \\
(4) - 1 + (1)y' + 3y' = 0& \\
3 + 4y' = 0& \\
4y' = -3& \\
\left[y'\right]_{(4,\,1)} = -\frac{3}{4}&
\end{aligned}$$

Note 3: Check for yourself that:

$$y'' = D_x\left(y'\right) = D_x\left(\frac{1-x}{y+3}\right) = -\frac{(y+3)+(1-x)y'}{(y+3)^2} = -\frac{(y+3)^2+(1-x)^2}{(y+3)^3}.$$

In the last step, we substituted $y' = \dfrac{1-x}{y+3}$ and simplified. §

Example 6 (Evaluating y'; Revisiting Examples 4 and 5)

Repeat Example 5 for the points $(4,-7)$ and $(1,2)$, which lie on the graph.

§ Solution

Point $(4,-7)$: $\left[y'\right]_{(4,-7)} = \left[\dfrac{1-x}{y+3}\right]_{(4,-7)} = \dfrac{1-(4)}{(-7)+3} = \dfrac{3}{4}$.

Point $(1,2)$: $\left[y'\right]_{(1,2)} = \left[\dfrac{1-x}{y+3}\right]_{(1,2)} = \dfrac{1-(1)}{(2)+3} = 0$. §

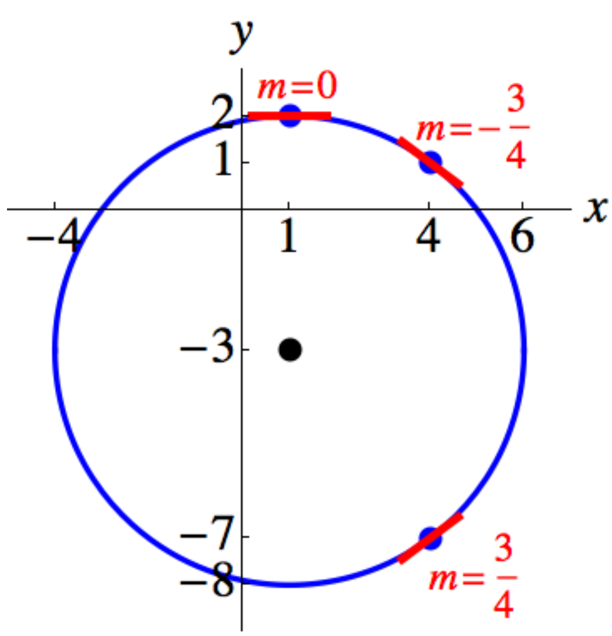
Example 7 (The Big Picture; Revisiting Examples 4-6)

The graph of $x^2-2x+y^2+6y=15$ is a **circle**.
To see this, **Complete the Square (CTS)** to obtain the standard form $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r $(r>0)$ is the radius.

$$x^2-2x+y^2+6y=15$$
$$\left(x^2-2x+1\right)+\left(y^2+6y+9\right)=15+1+9$$
$$(x-1)^2+(y+3)^2=25$$

The graph is a circle with center $(h,k)=(1,-3)$ and radius $r=\sqrt{25}=5$.

The desired tangent-line slopes we obtained in Examples 5 and 6 are labeled.



• **Slopes** represented by y', or $\dfrac{dy}{dx}$, are "read" **left-to-right**.
Don't think about revolving along the entirety of the circle.

• What points on the circle have ***y*-coordinate -3**? What is y' there? What is true of the tangent lines there?

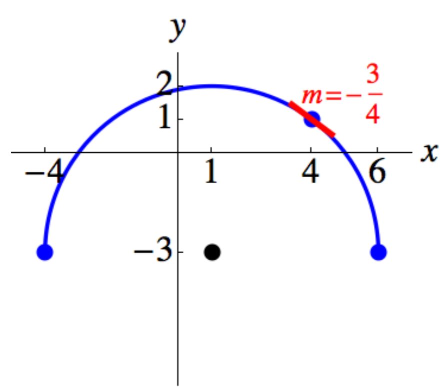
• How does the formula $y' = \dfrac{1-x}{y+3}$ hint at the **center** of the circle?

• Two **implicit functions** "determined" by the equation $x^2 - 2x + y^2 + 6y = 15$ are f_1 and f_2, where:

$f_1(x) = -3 + \sqrt{25 - (x-1)^2}$, and

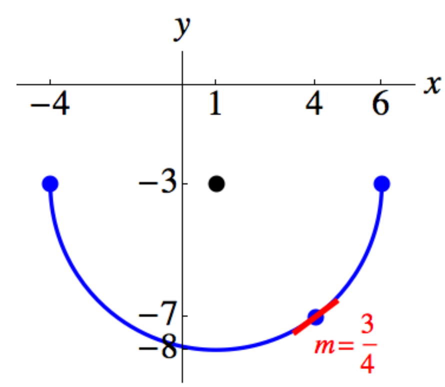
$f_2(x) = -3 - \sqrt{25 - (x-1)^2}$. (See Examples 4 and 7.)

Graph of $y = f_1(x)$



$$f_1'(4) = -\frac{3}{4}$$

Graph of $y = f_2(x)$



$$f_2'(4) = \frac{3}{4}$$

Our usual Lagrange notation can be used here. The *y*-coordinates need not be specified. §