

SECTION 3.8: RELATED RATES**LEARNING OBJECTIVES**

- Perform Implicit Differentiation with respect to different variables.
- See how Related Rates equations describe relationships between variables and/or their derivatives (rates of change) with respect to other variables.
- Be able to model and solve “word problems” that apply these ideas, use relevant constraints, and see why our results make sense.

PART A: REVIEWING IMPLICIT DIFFERENTIATION (SECTION 3.7)*Example 1 (Reviewing Implicit Differentiation)*

Consider the given equation $x^2 + y^2 = 25$.

Assume that it “determines” implicit differentiable functions f such that $y = f(x)$ and $f(3) = 4$. (Assumptions like these will go without saying.)

Find $\left[\frac{dy}{dx} \right]_{(3,4)}$. That is, find $\frac{dy}{dx}$, or y' , when $x = 3$ and $y = 4$.

WARNING 1: Avoid the y' notation in this section. We will also see $\frac{dy}{dt}$.

§ Solution

- Although it shouldn't be necessary, we could **check** that $(3, 4)$ is a solution point for $x^2 + y^2 = 25$:

$$\begin{aligned} (3)^2 + (4)^2 & \stackrel{?}{=} 25 \\ 25 &= 25 \quad (\text{Checks}) \end{aligned}$$

- Since we want a value for $\frac{dy}{dx}$, we will **implicitly differentiate** both sides

of $x^2 + y^2 = 25$ **with respect to x** . We expect the result to include $\frac{dy}{dx}$.

$$\begin{aligned}x^2 + y^2 &= 25 \Rightarrow \\D_x(x^2 + y^2) &= D_x(25) \\2x + 2y\left(\frac{dy}{dx}\right) &= 0 \\x + y\left(\frac{dy}{dx}\right) &= 0 \Rightarrow \text{(at } (3, 4)\text{)}\end{aligned}$$

- Plug in (substitute) $x = 3$ and $y = 4$; see Section 3.7, Example 5, Note 2.

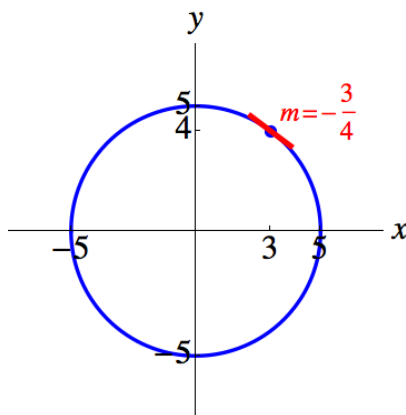
WARNING 2: Do this **after** differentiating.

$$(3) + (4)\left(\frac{dy}{dx}\right) = 0$$

- Solve for $\frac{dy}{dx}$.

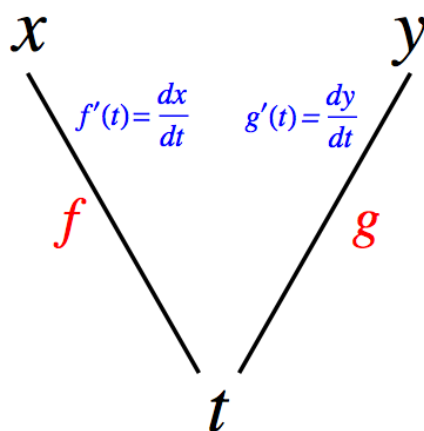
$$\begin{aligned}4\left(\frac{dy}{dx}\right) &= -3 \\ \frac{dy}{dx} &= -\frac{3}{4}\end{aligned}$$

- More precisely, $\left[\frac{dy}{dx}\right]_{(3,4)} = -\frac{3}{4}$. This is the **slope of the tangent line** to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$.



PART B: IMPLICIT DIFFERENTIATION WITH RESPECT TO DIFFERENT VARIABLES*Example 2 (Implicit Differentiation With Respect to t)*

Assume that f and g are implicit differentiable functions of t such that $x = f(t)$ and $y = g(t)$.



Find $D_t(4x^3 + xy - 3y^2)$.

§ Solution

Think of x and y as popping kernels. They could be $(\sin t)$, $(t^2 + 1)$, etc. We use the Generalized Power Rule and the Product Rule.

$$D_t(4x^3 + xy - 3y^2) = 12x^2\left(\frac{dx}{dt}\right) + \left(\frac{dx}{dt}\right)y + x\left(\frac{dy}{dt}\right) - 6y\left(\frac{dy}{dt}\right)$$

This may be rewritten as: $(12x^2 + y)\left(\frac{dx}{dt}\right) + (x - 6y)\left(\frac{dy}{dt}\right)$. §

PART C: RELATED VELOCITIES AS RELATED RATES*Example 3 (Related Rates, Including Related Velocities; Revisiting Example 1)*

A circular street is modeled by $x^2 + y^2 = 25$ in the usual xy -plane, where x and y are measured in miles and the positive y -axis points north. A car at the point $(3, 4)$ has 20 mph (miles per hour) as its **horizontal** component of velocity. Find the car's **vertical** component of velocity at $(3, 4)$... that is, when $x = 3$ miles and $y = 4$ miles. Interpret the result.

§ Solution

- t is **time** measured in hours.
- $\frac{dx}{dt}$ is the car's **horizontal** component of velocity.

Imagine lights projecting the car's shadow onto a screen along the **x -axis**.

$\frac{dx}{dt}$ is the velocity of the shadow along the screen.

- $\frac{dy}{dt}$ is the car's **vertical** component of velocity.

Imagine lights projecting the car's shadow onto a screen along the **y -axis**.

$\frac{dy}{dt}$ is the velocity of the shadow along the screen.

- The problem can now be **restated**:

If $x^2 + y^2 = 25$, find $\frac{dy}{dt}$ when $x = 3$ and $y = 4$ if $\frac{dx}{dt} = 20$ mph then, where x and y are measured in miles, and t is measured in hours.

WARNING 3: We avoid the notation $\left[\frac{dy}{dt} \right]_{(3,4)}$, because that may imply $t = 3$ instead of $x = 3$.

- We want a value for $\frac{dy}{dt}$, so we will **implicitly differentiate** both sides of

$x^2 + y^2 = 25$ **with respect to t** . We expect the result to include $\frac{dy}{dt}$.

The last two equations here are related rates equations, because they indicate a **relationship between derivatives** that are rates of change. The equations relate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\begin{aligned} x^2 + y^2 &= 25 \Rightarrow \\ D_t(x^2 + y^2) &= D_t(25) \\ 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) &= 0 \quad (\text{Related rates equation}) \\ x\left(\frac{dx}{dt}\right) + y\left(\frac{dy}{dt}\right) &= 0 \quad (\text{Related rates equation}) \Rightarrow (\text{at } (3, 4)) \end{aligned}$$

- Plug in (substitute) $x = 3$ and $y = 4$, and also $\frac{dx}{dt} = 20$.

WARNING 4: Units. Miles and (miles per hour) are **compatible** units here, so we will ignore units until we get the answers. Your physics and chemistry professors may prefer to see more units! If we had **incompatible** units such as miles and (feet per hour), then we would need to **convert** units.

$$(3)(20) + (4)\left(\frac{dy}{dt}\right) = 0$$

- Solve for $\frac{dy}{dt}$.

$$\begin{aligned} 60 + 4\left(\frac{dy}{dt}\right) &= 0 \\ 4\left(\frac{dy}{dt}\right) &= -60 \\ \frac{dy}{dt} &= -15 \text{ mph} \end{aligned}$$

- The car's **vertical** component of velocity at the point $(3, 4)$ is -15 mph.

Interpretation: At the point $(3, 4)$, the car's vertical component of velocity is 15 mph directed **south**.

WARNING 5: Don't indicate and interpret a negative sign at the same time. Here, do not write -15 mph directed south.

- Is the car moving clockwise or counterclockwise along the street? §

§ *Alternative Solution (Using Example 1)*

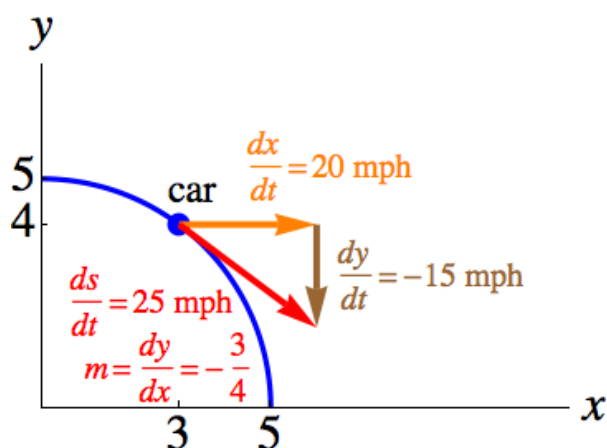
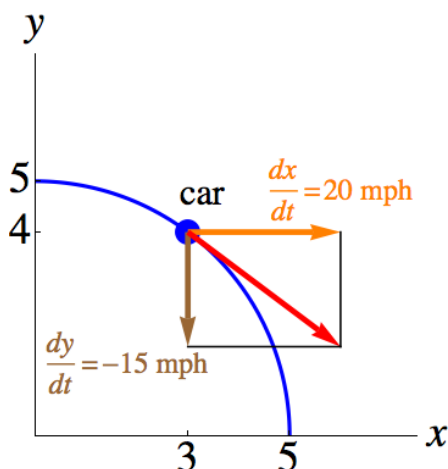
Use the **Chain Rule** and our result from Example 1, $\left[\frac{dy}{dx} \right]_{(3,4)} = -\frac{3}{4}$.

When $x = 3$ miles and $y = 4$ miles,

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \left(-\frac{3}{4} \right) (20) = -15 \text{ mph}$$

The interpretation is the same as before.

- The **velocity vectors** below represent these rates of change. The diagonal **resultant vector**, which is the sum of the orange and brown component vectors, is the **velocity vector** for the car at the point $(3, 4)$.



- $\frac{ds}{dt}$ is the car's velocity along the circle (more precisely, in the **direction of the tangent line**). It is given by the car's **speedometer**. s represents **arc length**. $\frac{ds}{dt}$ is the rate of change of the distance traveled by the car along the

arc (here, the circle). By the Pythagorean Theorem, $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

It is **never negative**. We will revisit this in Chapter 6.

- If the street were **elliptical** instead of circular, we may have a **different slope** $\left[\frac{dy}{dx}\right]_{(3,4)}$ and thus a **different proportion** between $\frac{dy}{dt}$ and $\frac{dx}{dt}$.
- What if the street were **horizontal**? What would $\frac{dy}{dt}$ be if $\frac{dx}{dt}$ were 20 mph, 30 mph, or 40 mph? §

PART D: A CONE PROBLEM

Example 4 (Related Rates Involving a Cone)

A pile of waste expands as a right circular cone at a rate of $20 \frac{\text{ft}^3}{\text{day}}$.

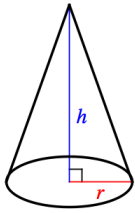
If the radius grows at a rate of $36 \frac{\text{in}}{\text{day}}$, how fast does the height change at

the instant that the radius is 10 feet and the height is $\frac{6}{\pi}$ feet?

§ Solution

Our first four steps may be reordered or done simultaneously. For example, Steps 3 and 4 may help identify the **variables** of interest in Step 2.

Step 1: Read the problem.**Step 2: Define variables, maybe through a general diagram.**

<p>Let t = time in days. Let V = the volume of the cone. Let r = the base radius of the cone. Let h = the height of the cone.</p>	
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The diagram should be that of a **generic** right circular cone – **generic** in the sense that the diagram could apply **at any time** during the cone’s expansion. We will say that the diagram applies “**throughout the process.**”

WARNING 6: Keep it generic! Do not include anything specific to the “instant of interest.” In particular, do **not** use $r = 10$ ft or $h = \frac{6}{\pi}$ ft yet.

Step 3: Identify what we are **GIVEN throughout the process and what we have to **FIND at the instant of interest.****

Here, we match numerical measures with **variables** and **derivatives**.

- For example, $20 \frac{\text{ft}^3}{\text{day}}$ is the rate of change of V (the cone’s **volume** in ft^3)

with respect to t (**time** in days). Therefore, $\frac{dV}{dt} = 20 \frac{\text{ft}^3}{\text{day}}$; this is true “throughout the process.”

- **TIP 1: Use units as clues to identify the relevant derivative.** To help see that $\frac{dV}{dt} = 20 \frac{\text{ft}^3}{\text{day}}$, observe that ft^3 is a measure of **volume** (V), and a day is a measure of **time** (t).

- **WARNING 7: Convert units if they are incompatible.** For example, $\frac{dr}{dt} = 36 \frac{\text{in}}{\text{day}}$, and the **(inches per day)** unit is incompatible with (feet) and (cubic feet). We will convert from **(inches per day)** to **(feet per day)** by multiplying by 1 in the form $\frac{1 \text{ ft}}{12 \text{ in}}$. We want the **(inches)** units to cancel, and we want to introduce **(feet)**.

$$\frac{dr}{dt} = 36 \frac{\text{in}}{\text{day}} = \left(36 \frac{\cancel{\text{in}}}{\text{day}} \right) \left(\frac{1 \text{ ft}}{12 \cancel{\text{in}}} \right) = 3 \frac{\text{ft}}{\text{day}}$$

- **TIP 2: “Linear” conversions are easier.** It is harder to convert from **(cubic feet per day)** to **(cubic inches per day)**. For example, many students would forget to **cube the 12 (WARNING 8)** here:

$$\frac{dV}{dt} = 20 \frac{\text{ft}^3}{\text{day}} = \left(20 \frac{\text{ft}^3}{\text{day}} \right) \left(\frac{12 \text{ in}}{1 \text{ ft}} \right)^3 = \left(20 \frac{\cancel{\text{ft}^3}}{\text{day}} \right) \left(\frac{1728 \text{ in}^3}{1 \cancel{\text{ft}^3}} \right) = 34,560 \frac{\text{in}^3}{\text{day}}$$

- **WARNING 9: Some numbers might not be used in calculations.** If it is stated that the cone grows in the year 2014, we still wouldn't use the number 2014 in calculations.

<u>GIVEN</u> <u>“throughout the process”:</u>	<u>FIND</u> <u>at the instant of interest:</u>
$\frac{dV}{dt} = 20 \frac{\text{ft}^3}{\text{day}}$ $\frac{dr}{dt} = 36 \frac{\text{in}}{\text{day}} = \left(36 \frac{\cancel{\text{in}}}{\text{day}} \right) \left(\frac{1 \text{ ft}}{12 \cancel{\text{in}}} \right) = 3 \frac{\text{ft}}{\text{day}}$	$\frac{dh}{dt} \text{ when:}$ $r = 10 \text{ ft}$ $h = \frac{6}{\pi} \text{ ft}$

Step 4: Key Formula

The classic **volume** formula for a right circular **cone** relates V , r , and h .

$$V = \frac{1}{3} \pi r^2 h$$

- This is $\frac{1}{3}$ the volume of the circumscribing cylinder, the right circular cylinder in which the cone fits snugly. The volume of the cylinder is given by $\pi r^2 h$, or the area of the circular base times the height. See Parts H, I.

Step 5: Perform Implicit Differentiation on both sides of the Key Formula from Step 4.

$\frac{dV}{dt}$, $\frac{dr}{dt}$, and $\frac{dh}{dt}$ are derivatives taken **with respect to t** , so we will

implicitly differentiate **both** sides of $V = \frac{1}{3}\pi r^2 h$ **with respect to t** .

• **TIP 3: Plug in (substitute) constants.** If $r = 10$ ft were a constant “throughout the process,” then it is easier to plug in (substitute) $r = 10$ now, before we differentiate. However, we cannot assume r is constant here.

$$\begin{aligned}
 V &= \frac{1}{3}\pi r^2 h \Rightarrow \\
 D_t(V) &= D_t\left(\frac{1}{3}\pi r^2 h\right) \\
 \frac{dV}{dt} &= \frac{\pi}{3} \cdot D_t(r^2 h) \quad (\text{by the Constant Multiple Rule}) \\
 \frac{dV}{dt} &= \frac{\pi}{3} \cdot \left(\left[D_t(r^2) \right] \cdot [h] + [r^2] \cdot \left[D_t(h) \right] \right) \quad (\text{by the Product Rule}) \\
 \frac{dV}{dt} &= \frac{\pi}{3} \cdot \left(\left[2r \left(\frac{dr}{dt} \right) \right] \cdot [h] + [r^2] \cdot \left[\frac{dh}{dt} \right] \right) \quad (\text{by the Gen. Power Rule}) \\
 \frac{dV}{dt} &= \frac{\pi}{3} \left[2rh \left(\frac{dr}{dt} \right) + r^2 \left(\frac{dh}{dt} \right) \right]
 \end{aligned}$$

Step 6: Plug in (substitute) values from Step 3.

WARNING 10: We can safely do this **after** we differentiate.

$$20 = \frac{\pi}{3} \left[2(10) \left(\frac{6}{\pi} \right) (3) + (10)^2 \left(\frac{dh}{dt} \right) \right]$$

Step 7: Solve for the derivative we want to find.

Here, we solve for $\frac{dh}{dt}$.

$$20 = \frac{\pi}{3} \left[\frac{360}{\pi} + 100 \left(\frac{dh}{dt} \right) \right]$$

$$20 = 120 + \frac{100\pi}{3} \left(\frac{dh}{dt} \right)$$

$$-100 = \frac{100\pi}{3} \left(\frac{dh}{dt} \right)$$

$$-100 \left(\frac{3}{100\pi} \right) = \left(\frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = -\frac{3}{\pi} \frac{\text{ft}}{\text{day}} \approx -0.9549 \frac{\text{ft}}{\text{day}}$$

• **TIP 4: Units.** The unit here is $\frac{\text{ft}}{\text{day}}$, because the unit for **height** (h) is

(**feet**), and the unit for **time** (t) is (**days**). If units had been converted, use the units **after the conversions**, since we plug in (substitute) numbers in Step 6 after the conversions.

Step 8: Conclusion

Write out our interpretation of the desired value:

At the instant of interest (when the radius is 10 feet and the height is $\frac{6}{\pi}$ feet), the height of the cone shrinks at a rate of $\frac{3}{\pi} \frac{\text{ft}}{\text{day}}$, which is about $0.9549 \frac{\text{ft}}{\text{day}}$.

Step 9: Does the answer make sense?

Should the negative value of $\frac{dh}{dt}$ bother us? Not necessarily. Although the cone's volume is increasing, **so is its radius**. The height may well shrink. Perhaps the waste pile is slippery! §

PART E: CONSTRAINT EQUATIONS*Example 5 (Related Rates Involving a Cone and a Constraint Equation)*

A water tank is in the shape of a right circular cone standing on its vertex. The tank has radius 6 feet and height 12 feet.

Water is pumped into the tank at the rate of $100 \frac{\text{ft}^3}{\text{hr}}$.

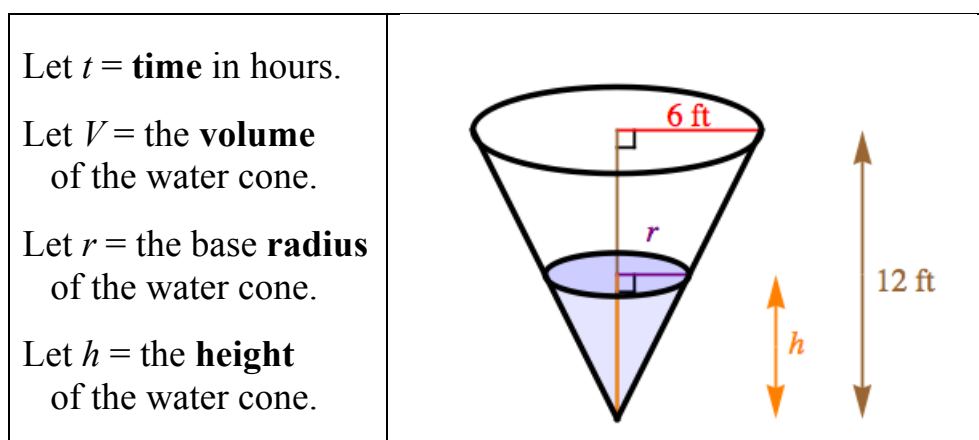
To four significant digits, approximate the rate at which the water level rises in the tank when the water is 2 feet deep.

§ Solution

Step 1: Read the problem.

Step 2: Define variables, maybe through a general diagram.

We assume that the tank is rigid, and the water in the tank forms a “water cone” that is **similar** to the “tank cone.”



The radius and height of the “tank cone” stay **constant** “throughout the process,” so we can label them. However, do **not** use 2 feet as a label yet.

Step 3: Identify what we are GIVEN throughout the process and what we have to FIND at the instant of interest.

<u>GIVEN</u> “throughout” the process:	<u>FIND</u> at the instant of interest:
$\frac{dV}{dt} = 100 \frac{\text{ft}^3}{\text{hr}}$	$\frac{dh}{dt} \text{ when:}$ $h = 2 \text{ ft}$

Step 4: Key Formula

The classic **volume** formula for a right circular **cone** relates V , r , and h .

$$V = \frac{1}{3}\pi r^2 h$$

• **TIP 5: Now, incorporate constraints that apply “throughout the process.”** It is often easier to do this now rather than later.

Here by properties of **similar triangles**, $\frac{r}{h} = \frac{6 \text{ ft}}{12 \text{ ft}} \Rightarrow \frac{r}{h} = \frac{1}{2} \Rightarrow r = \frac{h}{2}$.

The **radius** is always **half the height** for the “water cone,” as is the case for the “tank cone.” We now incorporate this constraint (or secondary equation) into our **key formula** through **substitution**.

Our **revised Key Formula** is:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h^2}{4}\right) h$$

$$V = \frac{\pi h^3}{12}$$

V is now expressed as a function of h alone.

• **TIP 6: Choose the substitution ... and your independent variable ... wisely.** We could have expressed V as a function of r alone.

Since $r = \frac{h}{2} \Rightarrow h = 2r$, we could have substituted $h = 2r$ and obtained

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (2r) = \frac{2\pi r^3}{3}.$$

This isn't wrong, but it leads to a less efficient solution, since our answer is a value of $\frac{dh}{dt}$, not $\frac{dr}{dt}$.

Step 5: Perform Implicit Differentiation on both sides of the Key Formula from Step 4.

$\frac{dV}{dt}$ and $\frac{dh}{dt}$ are derivatives taken **with respect to t** , so we will implicitly

differentiate **both** sides of $V = \frac{\pi h^3}{12}$ **with respect to t** .

$$\begin{aligned}
 V &= \frac{1}{12}\pi h^3 \quad \Rightarrow \\
 D_t(V) &= D_t\left(\frac{1}{12}\pi h^3\right) \\
 \frac{dV}{dt} &= \frac{\pi}{12} \cdot D_t(h^3) \quad (\text{by the Constant Multiple Rule}) \\
 \frac{dV}{dt} &= \frac{\pi}{12} \cdot \left[3h^2\left(\frac{dh}{dt}\right)\right] \quad (\text{by the Gen. Power Rule}) \\
 \frac{dV}{dt} &= \frac{\pi h^2}{4} \left(\frac{dh}{dt}\right)
 \end{aligned}$$

- Had we not used our **revised Key Formula** $V = \frac{\pi h^3}{12}$, we would have used our **old Key Formula** $V = \frac{1}{3}\pi r^2 h$. The Product Rule would have given us:

$$\begin{aligned}
 \frac{dV}{dt} &= D_t\left(\frac{1}{3}\pi r^2 h\right) \\
 \frac{dV}{dt} &= \frac{\pi}{3} \left[2rh\left(\frac{dr}{dt}\right) + r^2\left(\frac{dh}{dt}\right)\right]
 \end{aligned}$$

(We saw this in Example 4.) Then, we would have used the substitution $r = \frac{1}{2}h$, as well as the resulting **related rates** substitution:

$$\begin{aligned}
 D_t(r) &= D_t\left(\frac{1}{2}h\right) \\
 \frac{dr}{dt} &= \frac{1}{2}\left(\frac{dh}{dt}\right)
 \end{aligned}$$

Step 6: Plug in (substitute) values from Step 3.

$$100 = \frac{\pi(2)^2}{4} \left(\frac{dh}{dt} \right)$$

Step 7: Solve for the derivative we want to find.

We solve for $\frac{dh}{dt}$.

$$100 = \frac{\pi(2)^2}{4} \left(\frac{dh}{dt} \right)$$

$$100 = \pi \left(\frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = \frac{100}{\pi} \frac{\text{ft}}{\text{hr}} \approx 31.83 \frac{\text{ft}}{\text{hr}}$$

Step 8: Conclusion

At the instant of interest (when the water level is 2 feet deep), the water level in the tank **ris**es at a rate of $\frac{100}{\pi} \frac{\text{ft}}{\text{hr}}$, which is about $31.83 \frac{\text{ft}}{\text{hr}}$.

Step 9: Does the answer make sense?

The value of $\frac{dh}{dt}$ better be **positive**, and it is.

(What would a negative value imply?)

- How fast does the water level rise when the water is 3 feet deep?
(Answer below.) How do the calculations change? Should the water level be rising at a faster rate or at a slower rate than when the water is 2 feet deep?

- Answer: When the water is 3 feet deep, the water level rises at a rate of $\frac{400}{9\pi} \frac{\text{ft}}{\text{hr}} \approx 14.15 \frac{\text{ft}}{\text{hr}}$.

WARNING 11: Performing “double divisions” on calculators.

When approximating $\frac{400}{9\pi}$, do **not** enter: $400 \div 9 \times \pi =$; that gives us

the value of $\frac{400\pi}{9}$. This works: $400 \div 9 \div \pi =$. §

PART F: “TWO-CAR” PROBLEMS

Example 6 is meant to motivate Example 7, which directly applies **related rates**.

Example 6 (Preliminary Example Involving Motion of Two Objects)

Car A and Car B leave Point P at noon.

Car A moves east at 40 mph.

Car B moves north at 50 mph.

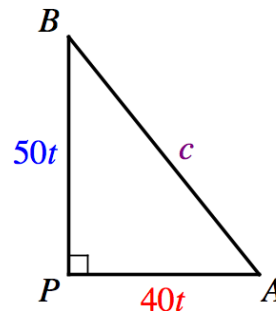
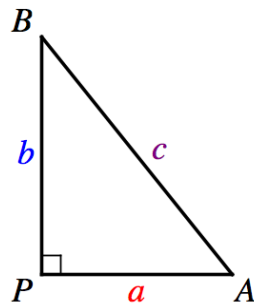
At 3pm, how fast are they moving apart?

§ Solution

Let t = the number of hours after noon ($t \geq 0$). Think: **time**.

Let a , b , and c be measured in miles.

A and B represent the positions of Cars A and B, respectively.



“Constraints”:

$$a = \left(40 \frac{\text{mi}}{\text{hr}} \right) \left(t \cancel{\text{hr}} \right) = 40t \text{ mi}$$

$$b = \left(50 \frac{\text{mi}}{\text{hr}} \right) \left(t \cancel{\text{hr}} \right) = 50t \text{ mi}$$

- We are breaking the convention in Trigonometry that assigns, for example, the label A to the angle facing opposite the side (or side length) labeled a .

- Although we could do a related-rates approach, we will instead express c in terms of t and find $\left[\frac{dc}{dt} \right]_{t=3}$.

- The **Key Formula** is the **Pythagorean Theorem**, which **relates side lengths** of a right triangle.

$$c^2 = a^2 + b^2$$

- We combine the Key Formula and the “**constraints.**”

$$\begin{aligned} c^2 &= (40t)^2 + (50t)^2 \\ &= 4100t^2 \end{aligned}$$

$c \geq 0$, so:

$$\begin{aligned} c &= \sqrt{4100t^2} \\ c &= (10\sqrt{41})t \Rightarrow \\ \frac{dc}{dt} &= 10\sqrt{41} \text{ mph} \approx 64.03 \text{ mph} \quad (t > 0) \Rightarrow \\ \left[\frac{dc}{dt} \right]_{t=3} &= 10\sqrt{41} \text{ mph} \approx 64.03 \text{ mph} \end{aligned}$$

- **Interpreting** the last two lines:

For $t > 0$, the two cars are moving apart at about 64.03 mph.

(We ignore issues such as the curvature of the Earth.)

In particular, this is true at 3pm. §

Example 7 (Variation on Example 6 Using Related Rates)

Car A leaves Point P at noon. Car B leaves Point P at 1pm.

Car A moves east at 40 mph.

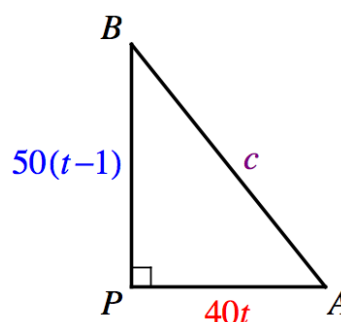
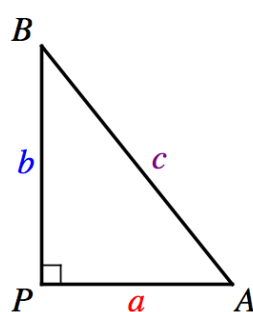
Car B moves north at 50 mph.

At 4pm, how fast are they moving apart (to four significant digits)?

§ Solution 1 (Using the Method from Example 6)

Let t = the number of hours after noon ($t \geq 1$). Think: **time**.

Let a , b , and c be measured in miles.



“Constraints”:

$$a = \left(40 \frac{\text{mi}}{\text{hr}} \right) (t \cancel{\text{hr}}) = 40t \text{ mi}$$

$$b = \left(50 \frac{\text{mi}}{\text{hr}} \right) [(t-1) \cancel{\text{hr}}] = 50(t-1) \text{ mi}$$

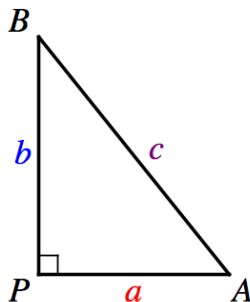
Using the method from Example 6, we obtain a more complicated

expression for c in terms of t . It is harder to find $\left[\frac{dc}{dt} \right]_{t=4}$.

In summary, for $t > 1$:

$$\begin{aligned} c &= \sqrt{(40t)^2 + [50(t-1)]^2} \Rightarrow \\ \frac{dc}{dt} &= \frac{4100t - 2500}{\sqrt{4100t^2 - 5000t + 2500}} = \frac{10(41t - 25)}{\sqrt{41t^2 - 50t + 25}} \Rightarrow \\ \left[\frac{dc}{dt} \right]_{t=4} &\approx 63.38 \text{ mph} \end{aligned}$$

At 4pm, the cars are moving at about 63.38 mph away from each other. §

§ Solution 2 (Using Related Rates)**Step 1: Read the problem.****Step 2: Define variables, maybe through a general diagram.**Let t = the number of hours after noon ($t \geq 1$). Think: **time**.Let a , b , and c be measured in miles.**Step 3: Identify what we are GIVEN throughout the process and what we have to FIND at the instant of interest.**

<u>GIVEN</u> <u>“throughout” the process:</u>	<u>FIND</u> <u>at the instant of interest:</u>
$\frac{da}{dt} = 40 \text{ mph}$ $\frac{db}{dt} = 50 \text{ mph}$	$\frac{dc}{dt} \text{ when:}$ $t = 4 \text{ hr}$

Step 4: Key FormulaAgain, we use the **Pythagorean Theorem**.

$$c^2 = a^2 + b^2$$

- Instead of incorporating the constraints from Solution 1, we will compute particular values at the **instant of interest** later.

Step 5: Perform Implicit Differentiation on both sides of the Key Formula from Step 4.

$\frac{da}{dt}$, $\frac{db}{dt}$, and $\frac{dc}{dt}$ are derivatives taken **with respect to t** , so we will implicitly differentiate **both** sides of $c^2 = a^2 + b^2$ **with respect to t** .

$$\begin{aligned}
 c^2 &= a^2 + b^2 \quad \Rightarrow \\
 D_t(c^2) &= D_t(a^2 + b^2) \\
 2c \left(\frac{dc}{dt} \right) &= 2a \left(\frac{da}{dt} \right) + 2b \left(\frac{db}{dt} \right) \quad (\text{by the Gen. Power Rule}) \\
 c \left(\frac{dc}{dt} \right) &= a \left(\frac{da}{dt} \right) + b \left(\frac{db}{dt} \right)
 \end{aligned}$$

Step 6: Plug in (substitute) values from Step 3.

$$c \left(\frac{dc}{dt} \right) = a(40) + b(50)$$

To obtain the values of a , b , and c at the **instant of interest** (4pm), use $t = 4$ hr.

- Car A has been moving for 4 hours, so $a = \left(40 \frac{\text{mi}}{\text{hr}} \right) (4 \text{ hr}) = 160$ mi.
- Car B has been moving for 3 hours, so $b = \left(50 \frac{\text{mi}}{\text{hr}} \right) (3 \text{ hr}) = 150$ mi.

• **TIP 7: Reuse the Key Formula.** We use the Pythagorean Theorem to find c at 4pm.

$$\begin{aligned}
 c^2 &= a^2 + b^2 \quad \Rightarrow \\
 c^2 &= (160)^2 + (150)^2 \\
 c^2 &= 48,100 \quad (\text{and } c > 0) \quad \Rightarrow \\
 c &= \sqrt{48,100} \text{ mi} = 10\sqrt{481} \text{ mi}
 \end{aligned}$$

We will not give an exact, simplified answer, anyway, so we can use $\sqrt{48,100}$ mi.

- **Plug in (substitute)** the values of a , b , and c at 4pm.

$$\left(10\sqrt{481}\right)\left(\frac{dc}{dt}\right) = (160)(40) + (150)(50)$$

Step 7: Solve for the derivative we want to find.

We solve for $\frac{dc}{dt}$.

$$\left(10\sqrt{481}\right)\left(\frac{dc}{dt}\right) = (160)(40) + (150)(50)$$

$$\left(10\sqrt{481}\right)\left(\frac{dc}{dt}\right) = 13,900$$

$$\frac{dc}{dt} = \frac{13,900}{10\sqrt{481}} = \frac{1390}{\sqrt{481}} = \frac{1390\sqrt{481}}{481} \text{ mph} \approx 63.38 \text{ mph}$$

Step 8: Conclusion

At 4pm, the cars move apart at a rate of about 63.38 mph.

Step 9: Does the answer make sense?

The value of $\frac{dc}{dt}$ better be **positive**, since the cars move further and further away from each other, and it is. §

- In Example 6, $\frac{dc}{dt}$ was constant. That is not the case in Example 7.

For example, at 5pm, $\frac{dc}{dt}$ is about 63.64 mph. What happens as $t \rightarrow \infty$?

- In Example 7, Solution 1, we found that $\frac{dc}{dt} = \frac{10(41t-25)}{\sqrt{41t^2-50t+25}}$ for $t > 1$.

Explain why it makes sense that $\lim_{t \rightarrow \infty} \frac{10(41t-25)}{\sqrt{41t^2-50t+25}} = 10\sqrt{41}$ mph, the answer

to Example 6. Hint: How are the two setups different? What are the effects of this difference in the long run?

PART G: A MISSILE PROBLEM*Example 8 (Related Rates Involving a Trigonometric Function)*

An underground missile silo fires a missile vertically. An observer is 10,000 feet away from the firing point and at the same elevation. Find the velocity of the missile when the angle of elevation from the observer to the missile is 60° , at which time the angle of elevation changes at $1.5 \frac{\text{deg}}{\text{sec}}$. The velocity must be expressed to four significant digits in miles per hour.

§ Solution

Step 1: Read the problem.

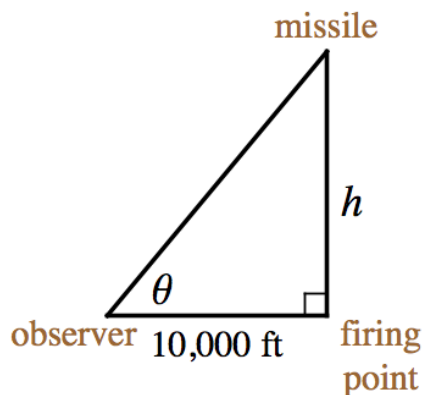
Step 2: Define variables, maybe through a general diagram.

Let t = the number of seconds after the missile is fired ($t \geq 0$). Think: **time**.

Let h = the **height** of the missile (from the firing point) in miles.

Let θ = the **angle of elevation** from the observer to the missile in **radians**.

WARNING 12: Radians. Radian measure is particularly important when derivatives involving angles are used. We will convert units in Step 3.



- See **WARNING 6: Keep it generic!** We do not use $\theta = 60^\circ$ or

$$\frac{d\theta}{dt} = 1.5 \frac{\text{deg}}{\text{sec}} \text{ yet.}$$

- See **TIP 3: Plug in (substitute) constants.** The length labeled 10,000 ft stays constant “**throughout the process.**”

Step 3: Identify what we are GIVEN throughout the process and what we have to FIND at the instant of interest.

<p><u>GIVEN</u> <u>“throughout” the process:</u></p> <p>The side length labeled “10,000 ft” in the diagram in Step 2.</p>	<p><u>FIND</u> <u>at the instant of interest:</u></p> <p>$\frac{dh}{dt}$ when:</p> <p>$\theta = 60^\circ = \frac{\pi}{3}$</p> <p>$\frac{d\theta}{dt} = \left(1.5 \frac{\text{deg}}{\text{sec}}\right) = \left(\frac{3 \cancel{\text{deg}}}{2 \text{ sec}}\right) \left(\frac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = \frac{\pi}{120} \frac{\text{rad}}{\text{sec}}$</p>
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Step 4: Key Formula

“SOH-CAH-TOA” gives us a **trigonometric** function that **relates side lengths and angles**.

$$\tan \theta = \frac{h}{10,000}$$

We want to find a value for $\frac{dh}{dt}$, so let’s solve for h .

$$h = 10,000 \tan \theta$$

Step 5: Perform Implicit Differentiation on both sides of the Key Formula from Step 4.

$\frac{dh}{dt}$ and $\frac{d\theta}{dt}$ are derivatives taken **with respect to t** , so we will implicitly differentiate **both** sides of $h = 10,000 \tan \theta$ **with respect to t** .

$$\begin{aligned}
 h &= 10,000 \tan \theta \Rightarrow \\
 D_t(h) &= D_t(10,000 \tan \theta) \\
 \frac{dh}{dt} &= \left(10,000 \sec^2 \theta\right) \left(\frac{d\theta}{dt}\right) \quad (\text{by the Gen. Trig Rules})
 \end{aligned}$$

Step 6: Plug in (substitute) values from Step 3, and

Step 7: Solve for the derivative we want to find.

$$\begin{aligned}\frac{dh}{dt} &= \left[10,000 \sec^2 \left(\frac{\pi}{3} \right) \right] \left(\frac{\pi}{120} \right) \\ &= 10,000 [2]^2 \left(\frac{\pi}{120} \right) \\ &= \frac{1000\pi}{3} \frac{\text{ft}}{\text{sec}}\end{aligned}$$

- We could have written $\sec^2(60^\circ)$ instead of $\sec^2\left(\frac{\pi}{3}\right)$, since we can evaluate a trigonometric function at a degree measure or a radian measure.

It turns out that the **conversion** $\theta = 60^\circ = \frac{\pi}{3}$ was **unnecessary**.

- If $\frac{dh}{dt}$ were something like $(\theta^2) \left(\frac{d\theta}{dt} \right)$, say, then the **conversion** would have been **necessary**.

- The **conversion** $\frac{d\theta}{dt} = 1.5 \frac{\text{deg}}{\text{sec}} = \frac{\pi}{120} \frac{\text{rad}}{\text{sec}}$ was **necessary**.

Step 7.5: Convert units in our answer.

This particular example required an answer in miles per hour.

$$\frac{dh}{dt} = \left(\frac{1000\pi}{3} \frac{\cancel{\text{ft}}}{\cancel{\text{sec}}} \right) \left(\frac{1 \text{ mi}}{5280 \cancel{\text{ft}}} \right) \left(\frac{3600 \cancel{\text{sec}}}{1 \text{ hr}} \right) = \frac{2500\pi}{11} \frac{\text{mi}}{\text{hr}} \approx 714.0 \text{ mph}$$

Step 8: Conclusion

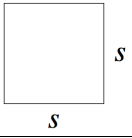
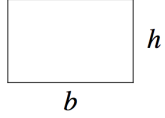
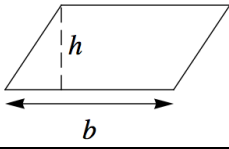
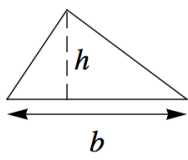
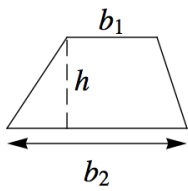
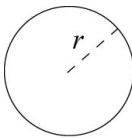
At the instant of interest, when the angle of elevation from the observer to the missile is 60° , the velocity of the missile is about 714.0 mph.

Step 9: Does the answer make sense?

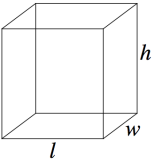
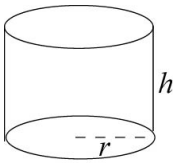
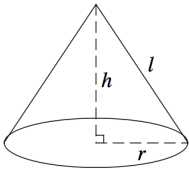
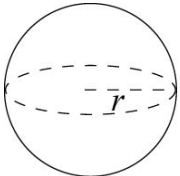
The value of $\frac{dh}{dt}$ better be **positive**, since the missile is rising, and it is.

(If it is negative, then this place may be in trouble!) §

PART H: FORMULAS FROM PLANE GEOMETRY

Description	Plane Figure	Formulas
Square with side length s		Area = s^2 Perimeter = $4s$ (the distance around)
Rectangle with base b and height h (covers Square)		Area = bh Perimeter = $2b + 2h$
Parallelogram with base b and height h (covers Rectangle, Square)		Area = bh
Triangle with base b and height h (think: half a Parallelogram)		Area = $\frac{1}{2}bh$
Trapezoid with bases b_1 and b_2 and height h		Area = $\left(\frac{b_1 + b_2}{2}\right)h$ (the average of the bases times the height)
Circle with radius r		Area = πr^2 Circumference = $2\pi r$ (the distance around)

PART I: FORMULAS FROM SOLID GEOMETRY

Description	Solid	Formulas
Rectangular Box with dimensions l , w , and h		Volume = lwh Surface Area = $2lw + 2wh + 2lh$ (See Note 1.)
Right Circular Cylinder with base radius r and height h		Volume = $\pi r^2 h$ Lateral Surface Area = $2\pi rh$ Total Surface Area = $2\pi rh + 2\pi r^2$ (See Note 2.)
Right Circular Cone with base radius r and height h		Volume = $\frac{1}{3}\pi r^2 h$ Lateral Surface Area = πrl , with slant height $l = \sqrt{r^2 + h^2}$ Total Surface Area = $\pi rl + \pi r^2$ (See Note 3.)
Sphere with radius r		Volume = $\frac{4}{3}\pi r^3$ Surface Area = $4\pi r^2$

- In Chapter 6, we may **prove** some of these formulas.
- We can use dimensional analysis to help check our formulas. If **lengths** are measured in **meters**, say, then **surface areas** are measured in **square meters**, and **volumes** are measured in **cubic meters**. For example, if the radius r of a sphere is measured in **meters**, then the volume formula $V = \frac{4}{3}\pi r^3$ does, in fact, yield a volume in **cubic meters**. This analysis prevents us from accidentally switching this formula with the formula for surface area.

Note 1 (Box)

The **volume** equals the rectangular **base area** times the **height**.

The **surface area** is the sum of the areas of the six sides.
Think of the walls, floor, and ceiling of a room.

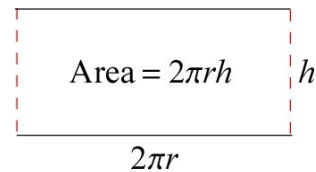
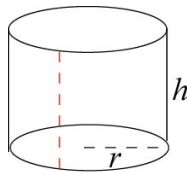
Note 2 (Cylinder)

The **volume** equals the circular **base area** times the **height**.

The **total surface area** equals the sum of the **lateral surface area** and the two circular **base areas**.

The **lateral surface area** equals the base **circumference** times the **height**.

- Consider the area of a soup can label. Imagine **slitting** the label along the red dashed line segment below and **spreading** it out as the rectangle on the right.

Note 3 (Cone)

The **volume** equals **one-third** of the volume of the right circular **cylinder** with the same base radius and height. (The cone “fits snugly” within this circumscribing cylinder.)

The **total surface area** equals the sum of the **lateral surface area** and the circular **base area**.