SECTION 3.5: DIFFERENTIALS and LINEARIZATION OF FUNCTIONS

LEARNING OBJECTIVES

• Use differential notation to express the approximate change in a function on a small interval.
• Find linear approximations of function values.
• Analyze how errors can be propagated through functional relationships.

PART A: INTERPRETING SLOPE AS MARGINAL CHANGE

Our story begins with lines. We know lines well, and we will use them to (locally) model graphs we don’t know as well.

Interpretation of Slope $m$ as Marginal Change

For every unit increase in $x$ along a line, $y$ changes by $m$.

• If $m < 0$, then $y$ drops in value.

If run = 1, then slope $m = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{1} = \text{rise}$.
**PART B: DIFFERENTIALS and CHANGES ALONG A LINE**

\( dx \) and \( dy \) are the differentials of \( x \) and \( y \), respectively. They correspond to “small” changes in \( x \) and \( y \) along a tangent line. We associate \( dy \) with “rise” and \( dx \) with “run.”

- If \( dy < 0 \), we move **down** along the line.
- If \( dx < 0 \), we move **left** along the line.

The following is key to this section:

**Rise Along a Line**

\[
\text{slope} = \frac{\text{rise}}{\text{run}}, \quad m = \frac{dy}{dx} \quad \text{as a quotient of differentials, so:}
\]

\[
\text{rise} = dy = m \, dx
\]

\[
\text{run} = dx
\]
PART C: LINEARIZATION OF FUNCTIONS

Remember the **Principle of Local Linearity** from Section 3.1. Assume that a function $f$ is differentiable at $x_1$, which we will call the “seed.” Then, the tangent line to the graph of $f$ at the point $(x_1, f(x_1))$ represents the function $L$, the **best local linear approximation** to $f$ close to $x_1$. $L$ models (or “linearizes”) $f$ locally on a small interval containing $x_1$.

The **slope** of the tangent line is given by $f'(x_1)$ or, more generically, by $f'(x)$, so changes along the tangent line are related by the following formulas:

\[
dy = m \, dx = f'(x) \, dx
\]

- This formula, written in **differential form**, is used to relate $dx$ and $dy$ as **variables**.
- In Leibniz notation, this can be written as $dy = \frac{dy}{dx} \, dx$, though those who see $\frac{dy}{dx}$ as an inseparable entity may object to the appearance of “cancellation.”
Δx and Δy are the increments of x and y, respectively. They represent actual changes in x and y along the graph of f.

- If x changes Δx units from, say, x₁ ("seed, old x") to x₁ + Δx ("new x"), then y (or f) changes Δy units from f(x₁) to f(x₁) + Δy, or f(x₁ + Δx).

The "new" function value \( f(x_1 + \Delta x) = f(x_1) + \Delta y \).

Informally, \( f(\text{new } x) = f(\text{old } x) + (\text{actual rise } \Delta y) \).

L(\(x_1 + \Delta x\)), our linear approximation of \( f(x_1 + \Delta x) \), is given by:

\[
L(x_1 + \Delta x) = f(x_1) + dy \\
L(x_1 + \Delta x) = f(x_1) + \left[ f'(x_1) \right][dx]
\]

Informally, \( L(\text{new } x) = f(\text{old } x) + (\text{tangent rise } \Delta y) \)

\[
L(\text{new } x) = f(\text{old } x) + [\text{slope}][\text{run } dx]
\]

When finding \( L(x_1 + \Delta x) \), we set \( dx = \Delta x \), and then we hope that \( dy \approx \Delta y \).

Then, \( L(x_1 + \Delta x) \approx f(x_1 + \Delta x) \).
Example 1 (Linear Approximation of a Function Value)

Find a linear approximation of \( \sqrt{9.1} \) by using the value of \( \sqrt{9} \). Give the exact value of the linear approximation, and also give a decimal approximation rounded off to six significant digits.

In other words: Let \( f(x) = \sqrt{x} \). Find a linear approximation \( L(9.1) \) for \( f(9.1) \) if \( x \) changes from 9 to 9.1.

§ Solution Method 1 (Using Differentials)

- \( f \) is differentiable on \((0, \infty)\), which includes both 9 and 9.1, so this method is appropriate.
- We know that \( f(9) = \sqrt{9} = 3 \) exactly, so 9 is a reasonable choice for the “seed” \( x_1 \).
- Find \( f'(9) \), the slope of the tangent line at the “seed point” \((9, 3)\).

\[
f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \\
f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \\
f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}
\]
- Find the run \( dx \) (or \( \Delta x \)).

\[
run \ dx = "new \ x" - "old \ x" \\
= 9.1 - 9 \\
= 0.1
\]
• Find $dy$, the rise along the tangent line.

$$ \text{rise } dy = \left( \text{slope} \right) \cdot (\text{run})$$

$$= \left[ f'(9) \right] \cdot [dx]$$

$$= \left[ \frac{1}{6} \right] [0.1]$$

$$= \frac{1}{60}$$

• Find $L(9.1)$, our linear approximation of $f(9.1)$.

$$L(9.1) = f(9) + dy$$

$$= 3 + \frac{1}{60}$$

$$= \frac{181}{60} \quad \text{(exact value)}$$

$$\approx 3.01667$$

**WARNING 1:** Many students would forget to add $f(9)$ and simply give $dy$ as the approximation of $\sqrt{9.1}$. This mistake can be avoided by observing that $\sqrt{9.1}$ should be very different from $\frac{1}{60}$.

• In fact, $f(9.1) = \sqrt{9.1} \approx 3.01662$, so our approximation was accurate to five significant digits. Also, the actual change in $y$ is given by:

$$\Delta y = f(9.1) - f(9)$$

$$= \sqrt{9.1} - \sqrt{9}$$

$$\approx 3.01662 - 3$$

$$\approx 0.01662$$

This was approximated by $dy = \frac{1}{60} \approx 0.01667$.

The error is given by: $dy - \Delta y \approx 0.01667 - 0.01662 \approx 0.00005$.

(See Example 5 for details on relative error, or percent error.)
Our approximation \( L(9.1) \) was an overestimate of \( f(9.1) \), because the graph of \( f \) curves downward (it is “concave down”; see Chapter 4).  

Differentials can be used to quickly find other linear approximations close to \( x = 9 \). (The approximations may become unreliable for values of \( x \) far from 9.)

<table>
<thead>
<tr>
<th>Find a linear approximation of ( f(9 + \Delta x) )</th>
<th>( \text{run} ) ( dx = \Delta x )</th>
<th>( \text{rise} ) ( dy = \left( \frac{1}{6} \right) dx )</th>
<th>Linear approximation, ( L(9 + \Delta x) )</th>
</tr>
</thead>
</table>
| \( f(8.8) = \sqrt{8.8} \)                      | -0.2            | - \( \frac{1}{30} \) \( \approx -0.03333 \) | \( L(8.8) = 3 - \frac{1}{30} \)
  \( \approx 2.96667 \) |
| \( f(8.9) = \sqrt{8.9} \)                      | -0.1            | - \( \frac{1}{60} \) \( \approx -0.01667 \) | \( L(8.9) = 3 - \frac{1}{60} \)
  \( \approx 2.98333 \) |
| \( f(9.1) = \sqrt{9.1} \)                      | 0.1             | \( \frac{1}{60} \) \( \approx 0.01667 \)    | \( L(9.1) = 3 + \frac{1}{60} \)
  \( \approx 3.01667 \) |
| \( f(9.2) = \sqrt{9.2} \)                      | 0.2             | \( \frac{1}{30} \) \( \approx 0.03333 \)    | \( L(9.2) = 3 + \frac{1}{30} \)
  \( \approx 3.03333 \) |
\section*{Solution Method 2 (Finding an Equation of the Tangent Line First: \(y = L(x)\))}

- Although this method may seem easier to many students, it does not stress the idea of\textbf{ marginal change} the way that Method 1 does. Your instructor may demand Method 1.

- The \textbf{tangent line} at the “seed point” \((9, 3)\) has \textbf{slope} \(f'(9) = \frac{1}{6}\), as we saw in Method 1. Its equation is given by:

\begin{equation}
y - y_1 = m(x - x_1) \\
y - 3 = \frac{1}{6}(x - 9)
\end{equation}

\(L(x)\), or \(y = 3 + \frac{1}{6}(x - 9)\), which takes the form (with variable \(dx\)):

\begin{equation}
L(x) = f(9) + \left[ f'(9) \right][dx] \\
\end{equation}

Also, \(L(x) = 3 + \frac{1}{6}(x - 9)\) simplifies as: \(L(x) = \frac{1}{6}x + \frac{3}{2}\).

- In particular, \(L(9.1) = 3 + \frac{1}{6}(9.1 - 9)\) or \(\frac{1}{6}(9.1) + \frac{3}{2} \approx 3.01667\), as before.
Example 2 (Linear Approximation of a Trigonometric Function Value)

Find a linear approximation of \( \tan(42^\circ) \) by using the value of \( \tan(45^\circ) \).
Give the exact value of the linear approximation, and also give a decimal approximation rounded off to six significant digits.

§ Solution Method 1 (Using Differentials)

Let \( f(x) = \tan x \).

**WARNING 2: Convert to radians.** In this example, we need to compute the run \( dx \) using radians. If \( f(x) \) or \( f'(x) \) were \( x \tan x \), for example, then we would also need to convert to radians when evaluating function values and derivatives. (Also, a Footnote in Section 3.6 will discuss why the differentiation rules for trigonometric functions given in Section 3.4 do not apply if \( x \) is measured in degrees.)

- Converting to radians, \( 45^\circ = \frac{\pi}{4} \) and \( 42^\circ = \frac{7\pi}{30} \). (As a practical matter, it turns out that we don’t have to do these conversions, as long as we know what the run \( dx \) is in radians.) We want to find a linear approximation \( L\left(\frac{7\pi}{30}\right) \) for \( f\left(\frac{7\pi}{30}\right) \) if \( x \) changes from \( \frac{\pi}{4} \) to \( \frac{7\pi}{30} \).

- \( f \) is differentiable on, among other intervals, \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \), which includes both \( \frac{\pi}{4} \) and \( \frac{7\pi}{30} \), so this method is appropriate.

- We know that \( f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1 \) exactly.

- Find \( f'\left(\frac{\pi}{4}\right) \), the slope of the tangent line at the “seed point” \( \left(\frac{\pi}{4}, 1\right) \).

\[
\begin{align*}
f(x) &= \tan x \quad \Rightarrow \\
f'(x) &= \sec^2 x \quad \Rightarrow \\
f'\left(\frac{\pi}{4}\right) &= \sec^2 \left(\frac{\pi}{4}\right) = 2
\end{align*}
\]
• Find the **run** \( dx \) (or \( \Delta x \)). It is usually easier to subtract the degree measures before converting to **radians**. Since it turns out \( dx < 0 \) here, we run **left** (as opposed to right) along the tangent line.

\[
\text{run } dx = "\text{new } x" - "\text{old } x"
\]
\[
= 42^\circ - 45^\circ \\
= -3^\circ
\]
\[
= (-3^\circ) \left( \frac{\pi}{180^\circ} \right) \quad \text{(Converting to radians)}
\]
\[
= - \frac{\pi}{60}
\]

• Find \( dy \), the **rise** along the tangent line. Actually, since it turns out \( dy < 0 \) here, it really corresponds to a **drop**.

\[
\text{rise } dy = (\text{slope}) \cdot (\text{run})
\]
\[
= \left[ f'(\frac{\pi}{4}) \right] \cdot [dx]
\]
\[
= [2] \left[ -\frac{\pi}{60} \right]
\]
\[
= -\frac{\pi}{30}
\]

• Find \( L\left(\frac{7\pi}{30}\right) \), our **linear approximation** of \( \tan(42^\circ) \), or \( f\left(\frac{7\pi}{30}\right) \).

\[
L\left(\frac{7\pi}{30}\right) = f\left(\frac{\pi}{4}\right) + dy
\]
\[
= 1 + \left( -\frac{\pi}{30} \right)
\]
\[
= \frac{30 - \pi}{30} \quad \text{(exact value)}
\]
\[
\approx 0.895280
\]

• In fact, \( f\left(\frac{7\pi}{30}\right) = \tan(42^\circ) \approx 0.900404 \).
• Our approximation \( L\left(\frac{7\pi}{30}\right) \) was an **underestimate** of \( f\left(\frac{7\pi}{30}\right) \).

See the figures below.

\[ \begin{align*}
  \text{\# Solution Method 2 (Finding an Equation of the Tangent Line First: } y &= L(x) \\
  \text{\hspace{0.5cm}} &\text{\#} \\
\end{align*} \]

\[ \begin{align*}
  \text{\#} & \text{The tangent line at the “seed point” } \left(\frac{\pi}{4}, 1\right) \text{ has slope } f'\left(\frac{\pi}{4}\right) = 2, \text{ as we saw in Method 1. Its equation is given by:} \\
  &y - y_1 = m(x - x_1) \\
  &y - 1 = 2\left(x - \frac{\pi}{4}\right) \\
  &L(x), \text{ or } y = 1 + 2\left(x - \frac{\pi}{4}\right), \text{ which takes the form (with variable } dx): \\
  &L(x) = f\left(\frac{\pi}{4}\right) + \left[f'\left(\frac{\pi}{4}\right)\right] dx. \text{ Simplified, } L(x) = 2x + \frac{2 - \pi}{2}. \\
  \text{\#} & \text{In particular, } L\left(\frac{7\pi}{30}\right) = 1 + 2\left(\frac{7\pi}{30} - \frac{\pi}{4}\right) = 1 + 2\left(-\frac{\pi}{60}\right) \approx 0.895280, \text{ or} \\
  &L\left(\frac{7\pi}{30}\right) = 2\left(\frac{7\pi}{30}\right) + \frac{2 - \pi}{2} \approx 0.895280, \text{ as before. } \# \]
\]
PART E: APPLICATIONS

Since a typical calculator has a square root button and a tangent button, the previous examples might not seem useful.

The table on Page 3.5.7 demonstrates how differentials can be used to quickly find multiple approximations of function values on a small interval around the “seed.”

Differentials can be used even if we do not have a formula for the function we are approximating, as we now demonstrate.

Example 3 (Approximating Position Values in the Absence of Formulas)

A car is moving on a coordinate line. Let \( y = s(t) \), the position of the car (in miles) \( t \) hours after noon. We are given that \( s(1) = 20 \) miles and \( v(1) = s'(1) = 50 \) mph.

<table>
<thead>
<tr>
<th>Find a linear approximation of ( s(1 + \Delta t) )</th>
<th>( \text{run} ) ( \Delta t = dt )</th>
<th>( \text{rise} ) ( 50 ) ( dy = 50 ) ( dt )</th>
<th>Linear approximation, ( L(1 + \Delta t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(0.8) )</td>
<td>(-0.2)</td>
<td>(-10)</td>
<td>( L(0.8) = 20 - 10 = 10 ) mi</td>
</tr>
<tr>
<td>( s(0.9) )</td>
<td>(-0.1)</td>
<td>(-5)</td>
<td>( L(0.9) = 20 - 5 = 15 ) mi</td>
</tr>
<tr>
<td>( s(1.1) )</td>
<td>(0.1)</td>
<td>(5)</td>
<td>( L(1.1) = 20 + 5 = 25 ) mi</td>
</tr>
<tr>
<td>( s(1.2) )</td>
<td>(0.2)</td>
<td>(10)</td>
<td>( L(1.2) = 20 + 10 = 30 ) mi</td>
</tr>
</tbody>
</table>

(Axes are scaled differently.)
PART F: MEASUREMENT ERROR and PROPAGATED ERROR

Let $x$ be the actual (or exact) length, weight, etc. that we are trying to measure.

Let $\Delta x$ (or $dx$) be the measurement error. The error could be due to a poorly calibrated instrument or merely random chance.

**Example 4 (Measurement Error)**

The radius of a circle (unbeknownst to us) is 10.7 inches (the actual value), but we measure it as 10.5 inches (the measured value). Sources differ on the definition of measurement error.

1) If we let measurement error $= (\text{measured value}) - (\text{actual value})$, then the “seed” $x = 10.7$ inches, and $\Delta x = 10.5 - 10.7 = -0.2$ inches.

This approach is more consistent with our usual notion of “error.”

2) If we let measurement error $= (\text{actual value}) - (\text{measured value})$, then the “seed” $x = 10.5$ inches, and $\Delta x = 10.7 - 10.5 = 0.2$ inches.

We will adopt this approach, because we know the measured value but not the actual value in an exercise such as Example 5. This suggests that the measured value is more appropriate than the actual value as the “seed.”

(See Footnote 1.) §

If $y = f(x)$ for some function $f$, then an error in measuring $x$ may lead to propagated error in $y$, denoted by $\Delta y$. As before, we approximate $\Delta y$ by $dy$ for convenience.

**Example 5 (Propagated Error)**

Let $x$ be the radius of a circle, and let $y$ be its area. Then, $y$, or $f(x) = \pi x^2$.

We measure the radius using an instrument that may be “off” by as much as 0.5 inches; more precisely, it has a maximum possible absolute value of measurement error of 0.5 inches. We use the instrument to obtain a measured value of 10.5 inches. Estimate the maximum possible absolute value of the propagated error that we will obtain for the area of the circle. Use differentials and give an approximation written out to five significant digits.

*§ Solution*

$$dy = f'(x) \, dx = 2\pi x \, dx$$
We use the “seed” \( x_1 = 10.5 \) inches, the **measured value**, for \( x \).

Let the **run** \( dx \) be the maximum possible absolute value of the **measurement error**, which is 0.5 inches. (Really, \(-0.5 \text{ in} \leq dx \leq 0.5 \text{ in.}\))

Then, \(|dy|\), our approximation for the maximum possible absolute value of the **propagated error** in \( y \), is given by:

\[
|dy| = |2\pi x \, dx| \\
\leq |2\pi (10.5)(0.5)| \\
\leq 32.987 \text{ in}^2
\]

That is, we estimate that the **propagated error** in the **area** of the circle will be “off” by no more than 32.987 \( \text{in}^2 \) in either direction (high or low).

**• Estimates of the area.** If we take the measured value of 10.5 inches for the radius, we will obtain the following **measured value** for the area:

\[
y, \text{ or } f(10.5) = \pi (10.5)^2 \\
= 346.361 \text{ in}^2
\]

Since we estimate that we are “off” by no more than 32.987 \( \text{in}^2 \), we estimate that the **actual value** of the area is between 313.374 \( \text{in}^2 \) and 379.348 \( \text{in}^2 \).

Without differentials, we would say that the actual value of the area is between \( f(10.0) \) and \( f(11.0) \), or between 314.159 \( \text{in}^2 \) and 380.133 \( \text{in}^2 \). (The benefit of using differentials is more apparent when the function involved is more complicated.)

**• Relative error and percent error.** Is 32.987 \( \text{in}^2 \) “bad” or “good”? The relative error and the percent error give us some context to decide.

\[
\text{Relative error} = \frac{dy}{y} \left( \text{really, maximum } \frac{|dy|}{\text{measured } y} \right) \\
\approx \frac{32.987}{346.361} \\
\approx 0.095239, \text{ or } 9.5239\% \text{ (percent error)}
\]

(If the measured area had been something like \( y = 1,000,000 \text{ in}^2 \), then 32.987 \( \text{in}^2 \) probably wouldn’t be a big deal.)
• **Actual propagated error.** If the actual value of the *radius* is 10.7 inches, then the actual value of the *area* is \( f(10.7) = \pi (10.7)^2 \approx 359.681 \text{ in}^2 \), and the actual propagated error is given by:

\[
\Delta y = f(10.7) - f(10.5) \\
= \pi (10.7)^2 - \pi (10.5)^2 \\
= 13.320 \text{ in}^2
\]

• **Notation.** If we let \( r = \) the radius and \( A = \) the area, then we obtain the more familiar formula \( A = \pi r^2 \). Also, \( dA = 2\pi r \, dr \).

• **Geometric approach.** Observe that:

\[
dA = (\text{circumference of circle of radius } r) \cdot (\text{thickness of ring})
\]

This approximates \( \Delta A \), the actual propagated error in the area. The figures below demonstrate why this approximation makes sense.

Imagine cutting the shaded ring below along the dashed slit and straightening it out. We obtain a trapezoid that is approximately a rectangle with dimensions \( 2\pi r \) and \( dr \). The area of both shaded regions is \( \Delta A \), and we approximate it by \( dA \), where \( dA = 2\pi r \, dr \).
FOOTNOTES


   • Larson in his calculus text (9th ed.) uses Definition 2):
   
   measurement error = (actual value) \( - \) (measured value). On top of the rationale given in Example 4, this is also more consistent with the notion of error when studying confidence intervals and regression analysis in statistics.
SECTION 3.6: CHAIN RULE

LEARNING OBJECTIVES

• Understand the Chain Rule and use it to differentiate composite functions.
• Know when and how to apply the Generalized Power Rule and the Generalized Trigonometric Rules, which are based on the Chain Rule.

PART A: THE IDEA OF THE CHAIN RULE

Yul, Uma, and Xavier run in a race. Let $y$, $u$, and $x$ represent their positions (in miles), respectively.

• Assume that Yul always runs twice as fast as Uma. That is, $\frac{dy}{du} = 2$.
  (If Uma runs $\Delta u$ miles, then Yul runs $\Delta y$ miles, where $\Delta y = 2 \Delta u$.)

• Assume that Uma always runs three times as fast as Xavier. That is, $\frac{du}{dx} = 3$.

• Therefore, Yul always runs six times as fast as Xavier. That is, $\frac{dy}{dx} = 6$.

This is an example of the Chain Rule, which states that: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Here, $6 = 2 \cdot 3$.

**WARNING 1:** The Chain Rule is a calculus rule, not an algebraic rule, in that the “$du$’s should not be thought of as “canceling.”
We can think of $y$ as a function of $u$, which, in turn, is a function of $x$. Call these functions $f$ and $g$, respectively. Then, $y$ is a composite function of $x$; this function is denoted by $f \circ g$.

- **In multivariable calculus**, you will see bushier trees and more complicated forms of the Chain Rule where you add products of derivatives along paths, extending what we have done above.

**TIP 1:** The Chain Rule is used to differentiate composite functions such as $f \circ g$. The derivative of a product of functions is not necessarily the product of the derivatives (see Section 3.3 on the Product Rule), but the derivative of a composition of functions is the product of the derivatives. (Composite functions were reviewed in Chapter 1.)
PART B: FORMS OF THE CHAIN RULE

Chain Rule

Let \( y = f(u) \) and \( u = g(x) \), where \( f \) and \( g \) are differentiable “where we care.” Then,

Form 1) \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)

Form 2) \( (f \circ g)'(x) = [f'(u)] \cdot [g'(x)] \)

Form 3) \( y' = [f'(u)] \cdot [u'] \)

• Essentially, the derivative of a composite function is obtained by taking the derivative of the “outer function” at \( u \) times the derivative of the “inner function” at \( x \).

• Following How to Ace Calculus by Adams, Thompson, and Hass (Times, 1998), we will refer to the derivative of the inner function as the “tail.” In the forms above, the tail is denoted by \( \frac{du}{dx} \), \( g'(x) \), and \( u' \).

**WARNING 2:** Forgetting the “tail” is a very common error students make when applying the Chain Rule.

• See Footnote 1 for a partial proof.

• See Footnote 2 on a controversial form.

Many differentiation rules, such as the Generalized Power Rule and the Generalized Trigonometric Rules we will introduce in this section, are based on the Chain Rule.
PART C: GENERALIZED POWER RULE

The **Power Rule of Differentiation**, which we introduced in Part B of Section 3.2, can be used to find $D_x \left( x^7 \right)$. However, it **cannot** be used to find $D_x \left[ \left( 3x^2 + 4 \right)^7 \right]$ without expanding the indicated seventh power, something we would rather not do.

**Example 1 (Using the Chain Rule to Motivate the Generalized Power Rule)**

Use the **Chain Rule** to find $D_x \left[ \left( 3x^2 + 4 \right)^7 \right]$.

§ **Solution**

Let $y = \left( 3x^2 + 4 \right)^7$. We will treat $y$ as a **composite function** of $x$.

$y = (f \circ g)(x) = f \left( g(x) \right)$, where:

$u = g(x) = 3x^2 + 4$ \hspace{1em} (g is the “inner function”)

$y = f(u) = u^7$ \hspace{1em} (f is the “outer function”)

Observe that \( \frac{dy}{du} \) and \( \frac{du}{dx} \) can be readily found using **basic rules**.

We can then find \( \frac{dy}{dx} \) using the **Chain Rule**.

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[ D_u \left( u^7 \right) \right] \left[ D_x \left( 3x^2 + 4 \right) \right] = 7u^6 \cdot 6x = 42x \left( 3x^2 + 4 \right)^6
\]

(\text{Since } u \text{ was our creation, we must express } u \text{ in terms of } x.\text{)}

(See Example 2 for a short cut.)
Example 1 suggests the following short cut.

**Generalized Power Rule**

Let \( u \) be a function of \( x \) that is differentiable “where we care.” Let \( n \) be a real constant.

\[
D_x \left( u^n \right) = \left( nu^{n-1} \right) \left( D_x u \right)
\]

• **Rationale.** If \( y = u^n \), then, by the Chain Rule,

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \Rightarrow \\
D_x \left( u^n \right) = \left[ D_u \left( u^n \right) \right] \cdot \left[ D_x u \right]
\]

\[
= \left[ nu^{n-1} \right] \cdot \left[ D_x u \right]
\]

**Example 2 (Using the Generalized Power Rule; Revisiting Example 1)**

Use the **Generalized Power Rule** to find \( D_x \left[ \left( 3x^2 + 4 \right)^7 \right] \).

**Solution**

\[
D_x \left[ \left( 3x^2 + 4 \right)^7 \right] = \left[ 7 \left( 3x^2 + 4 \right)^6 \right] \left[ D_x \left( 3x^2 + 4 \right) \right] \quad \text{(See Warning 3.)}
\]

\[
= \left[ 7 \left( 3x^2 + 4 \right)^6 \right] \left[ 6x \right]
\]

\[
= 42x \left( 3x^2 + 4 \right)^6
\]

**WARNING 3:** Copy the base. The base \( u \), which is \( 3x^2 + 4 \) here, is copied under the exponent. Do **not** differentiate it until you get to the “tail.”

**WARNING 4:** Remember the exponent. Many students forget to write the exponent, 6, because the base can take some time to write. You may want to write the exponent **before** writing out the base.

**WARNING 5:** Identifying “tails.” The \( D_x \) notation helps us keep track of “how far” to unravel tails. A tail may have a tail of its own. If we forget tails, we’re **not going far enough**. If we attach inappropriate tails (such as an additional “6” after the “6x” above), we’re going **too far**.
Example 3 (Using the Generalized Power Rule in Conjunction with the Quotient Rule or the Product Rule)

Find \( D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) \).

\[ D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) = \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{(\text{Lo})^2}, \text{the square of what's below} \]

\[ = \frac{(\sqrt{2x-1}) \cdot [D_x (x^3)] - (x^3) \cdot \left( D_x \left( \frac{1}{2} (2x-1)^{-1/2} \right) \right]}{\sqrt{2x-1}} \]

\[ = \frac{2x-1}{2x-1} \cdot \frac{3x^2 - x^3 \cdot \left[ \frac{1}{2} (2x-1)^{-1/2} \right] \cdot [D_x (2x-1)]}{2x-1} \]

(We could factor the numerator at this point.)

\[ = \frac{(\sqrt{2x-1}) \cdot [3x^2] - x^3}{2x-1} \]

\[ = \frac{\left[ (\sqrt{2x-1}) \cdot [3x^2] - x^3 \right]}{2x-1} \cdot \frac{\sqrt{2x-1}}{\sqrt{2x-1}} \]

\[ = \frac{(2x-1) \cdot [3x^2] - x^3}{(2x-1)^{3/2}} \]

\[ = \frac{6x^3 - 3x^2 - x^3}{(2x-1)^{3/2}} \]

\[ = \frac{5x^3 - 3x^2}{(2x-1)^{3/2}} \text{, or } \frac{x^2 (5x-3)}{(2x-1)^{3/2}} \]

\[ \text{WARNING 6: Distribute before canceling. Do not cancel in the numerators until we have distributed } \sqrt{2x-1} \text{ through the first numerator.} \]
§ Solution Method 2 (Using the Product Rule)

If we had forgotten the Quotient Rule, we could have rewritten:

\[ D_x \left( \frac{x^3}{\sqrt{2x - 1}} \right) = D_x \left[ x^3 (2x - 1)^{-1/2} \right] \] and applied the Product Rule.

We would then use the Generalized Power Rule to find \( D_x \left[ (2x - 1)^{-1/2} \right] \).

The key drawback here is that we obtain two terms, and students may find it difficult to combine them into a single, simplified fraction. Observe:

\[
D_x \left( \frac{x^3}{\sqrt{2x - 1}} \right) = D_x \left[ x^3 (2x - 1)^{-1/2} \right] \quad \text{(Rewriting)}
\]

\[
= \left[ D_x (x^3) \right] \cdot \left[ (2x - 1)^{-1/2} \right] + \left( x^3 \right) \cdot \left( D_x \left[ (2x - 1)^{-1/2} \right] \right) \quad \text{(by the Product Rule)}
\]

\[
= \left[ 3x^2 \right] \cdot \left[ (2x - 1)^{-1/2} \right] + \left( x^3 \right) \cdot \left( \left[ -\frac{1}{2} (2x - 1)^{-3/2} \right] \cdot \left[ D_x (2x - 1) \right] \right) \quad \text{(by the Generalized Power Rule)}
\]

\[
= \left[ 3x^2 \right] \cdot \left[ (2x - 1)^{-1/2} \right] + \left( x^3 \right) \cdot \left( \left[ -\frac{1}{2} (2x - 1)^{-3/2} \right] \cdot \left[ 2 \right] \right)
\]

\[
= \left[ 3x^2 \right] \cdot \left[ (2x - 1)^{-1/2} \right] + \left( x^3 \right) \cdot \left( \left[ -\frac{3}{2} (2x - 1)^{-3/2} \right] \right)
\]

\[
= \frac{3x^2}{(2x - 1)^{1/2}} - \frac{x^3}{(2x - 1)^{3/2}}
\]

\[
= \frac{3x^2}{(2x - 1)^{1/2}} \cdot \frac{(2x - 1)}{(2x - 1)^{3/2}} - \frac{x^3}{(2x - 1)^{3/2}}
\]

Build up the first fraction to obtain the LCD, \( (2x - 1)^{3/2} \).

\[
= \frac{3x^2 (2x - 1) - x^3}{(2x - 1)^{3/2}}
\]

\[
= \frac{6x^3 - 3x^2 - x^3}{(2x - 1)^{3/2}}
\]

\[
= \frac{5x^3 - 3x^2}{(2x - 1)^{3/2}}, \text{ or } \frac{x^2 (5x - 3)}{(2x - 1)^{3/2}} \quad \text{(as in Method 1)}
\]
Example 4 (Using the Generalized Power Rule to Differentiate a Power of a Trigonometric Function)

Let \( f(\theta) = \sec^5 \theta \). Find \( f'(\theta) \).

§ Solution

First, rewrite \( f(\theta) \):

\[
f(\theta) = \sec^5 \theta = (\sec \theta)^5
\]

**WARNING 7**: Rewriting before differentiating. When differentiating a power of a trigonometric function, rewrite the power in this way. Students get very confused otherwise. Also, do **not** write \( \sec \theta^5 \) here; that is equivalent to \( \sec(\theta^5) \), not \( (\sec \theta)^5 \).

\[
f'(\theta) = 5(\sec \theta)^4 \cdot [D_\theta (\sec \theta)]
\]

(by the Generalized Power Rule)

\[
= 5(\sec \theta)^4 \cdot [\sec \theta \tan \theta]
\]

\[
= 5(\sec \theta)^5 \tan \theta
\]

\[
= 5 \sec^5 \theta \tan \theta
\]

§

Example 5 (Using the Generalized Power Rule to Prove the Reciprocal Rule)

Prove the **Reciprocal Rule** from Section 3.3: \( D_x \left[ \frac{1}{g(x)} \right] = - \frac{g'(x)}{[g(x)]^2} \).

§ Solution

\[
D_x \left[ \frac{1}{g(x)} \right] = D_x \left[ g(x)^{-1} \right]
\]

**WARNING 8**: “\(-1\)” here denotes a reciprocal, **not** a function inverse.

\[
= \left[ -g(x)^{-2} \right] \cdot [g'(x)] \quad (\text{by the Generalized Power Rule})
\]

\[
= -\frac{g'(x)}{[g(x)]^2}
\]

§
PART D: GENERALIZED TRIGONOMETRIC RULES

The Basic Trigonometric Rules of Differentiation, which we introduced in Section 3.4, can be used to find $D_x (\sin x)$. However, they cannot be used to find $D_x \left[ \sin \left( x^2 \right) \right]$.

Example 6 (Using the Chain Rule to Motivate the Generalized Trigonometric Rules)

Use the Chain Rule to find $D_x \left[ \sin \left( x^2 \right) \right]$.

Solution

Let $y = \sin \left( x^2 \right)$. We will treat $y$ as a composite function of $x$.

$y = (f \circ g) (x) = f(g(x))$, where:

$u = g(x) = x^2$  \hspace{1cm} (g is the “inner function”)

$y = f(u) = \sin u$  \hspace{1cm} (f is the “outer function”)

Observe that $\frac{dy}{du}$ and $\frac{du}{dx}$ can be readily found using basic rules.

We can then find $\frac{dy}{dx}$ using the Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left[ D_u (\sin u) \right] \left[ D_x \left( x^2 \right) \right]$$

$$= \left[ \cos u \right] \left[ 2x \right]$$

(Since $u$ was our creation, we must express $u$ in terms of $x$.)

$$= \left[ \cos \left( x^2 \right) \right] \left[ 2x \right]$$  \hspace{1cm} (See Example 7 for a short cut.)

$$= 2x \cos \left( x^2 \right)$$
Example 6 suggests the following short cuts.

**Generalized Trigonometric Rules**

Let $u$ be a function of $x$ that is differentiable “where we care” (see Footnote 4).

\[
\begin{align*}
D_x \sin u &= \cos u \frac{D_x u}{\text{"tail"}} \\
D_x \cos u &= -\sin u \frac{D_x u}{\text{"tail"}} \\
D_x \tan u &= \sec^2 u \frac{D_x u}{\text{"tail"}} \\
D_x \cot u &= -\csc^2 u \frac{D_x u}{\text{"tail"}} \\
D_x \sec u &= \sec u \tan u \frac{D_x u}{\text{"tail"}} \\
D_x \csc u &= -\csc u \cot u \frac{D_x u}{\text{"tail"}}
\end{align*}
\]

**WARNING 9:** In the **bottom two rules**, the “tail” is still written only once. The “tail” is the derivative of the common argument $u$.

- **Radians.** See Footnote 3 on how these rules encourage us to use radians (as opposed to degrees) when differentiating trigonometric functions.

**Example 7 (Using the Generalized Trigonometric Rules; Revisiting Example 6)**

Use the **Generalized Trigonometric Rules** to find $D_x \left[ \sin \left( x^2 \right) \right]$.

**§ Solution**

\[
D_x \left[ \sin \left( x^2 \right) \right] = \left[ \cos \left( x^2 \right) \right] \left[ D_x \left( x^2 \right) \right] \\
= \left[ \cos \left( x^2 \right) \right] \left[ 2x \right] \\
= 2x \cos \left( x^2 \right)
\]

**WARNING 10:** **Copy the argument.** The sine function’s argument $u$, which is $(x^2)$ here, is copied as the cosine function’s argument. Do not differentiate it until you get to the “tail.”

**TIP 2:** **Consistency with the Basic Trigonometric Rules.** Observe:

\[
D_x \left( \sin x \right) = \left[ \cos x \right] \left[ D_x \left( x \right) \right] \\
= \left[ \cos x \right] \left[ 1 \right] \\
= \cos x
\]

The “tail” is simply 1 when the argument ($x$ here) is just the variable of differentiation, so we can ignore the tail in the Basic Trigonometric Rules.
Example 8 (Using the Generalized Trigonometric Rules)

Let \( g(\theta) = \sec(9\theta^2 - \theta) \). Find \( g'(\theta) \).

\[ g'(\theta) = D_{\theta}\left[\sec(9\theta^2 - \theta)\right] \]
\[ = \left[\sec(9\theta^2 - \theta)\tan(9\theta^2 - \theta)\right]\cdot D_{\theta}\left(9\theta^2 - \theta\right) \]

(See Warning 9.)

\[ = \left[\sec(9\theta^2 - \theta)\tan(9\theta^2 - \theta)\right]\cdot [18\theta - 1] \]
\[ = (18\theta - 1)\sec(9\theta^2 - \theta)\tan(9\theta^2 - \theta) \]

Example 9 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)

Let \( f(x) = \cos^5(7x) \). Find \( f'(x) \).

\[ f'(x) = \left(5\cos^4(7x)\right)\cdot D_x\left[\cos(7x)\right] \]

(See Warning 5 and Tip 3.)

(by the Generalized Power Rule)

\[ = \left(5\cos^4(7x)\right)\cdot [-\sin(7x)]\cdot D_x\left(7x\right) \]

(by the Generalized Trigonometric Rules)

\[ = \left(5\cos^4(7x)\right)\cdot [-\sin(7x)]\cdot [7] \]
\[ = -35 \cos^4(7x)\sin(7x) \]
**TIP 3: Linear arguments.** If $a$ is a real constant, then $D_x[\sin(ax)] = a\cos(ax)$, $D_x[\cos(ax)] = -a\sin(ax)$, $D_x[\sec(ax)] = a\sec(ax)\tan(ax)$, etc. In Example 9, we saw that: $D_x[\cos(7x)] = -7\sin(7x)$. These can be very useful short cuts.

- More generally, $D_x[\sin(ax + b)] = a\cos(ax + b)$, and so forth; the “tail” is still the coefficient of $x$ in the linear argument.

**Example 10 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)**

Show that $D_x[\tan^4(\pi x)] = 4\pi \tan^3(\pi x)\sec^2(\pi x)$.

(Left to the reader.) The solution is similar to that in Example 9.

Hint: First rewrite: $D_x[\tan^4(\pi x)] = D_x\left([\tan(\pi x)]^4\right)$.

**Example 11 (Using the Generalized Trigonometric Rules, Followed by the Generalized Power Rule)**

Find $D_x\left(\csc\left[(x^2 + 1)^3\right]\right)$.

§ **Solution**

Overall, we are differentiating a trigonometric function, so we will first apply the Generalized Trigonometric Rules.

\[
D_x\left(\csc\left[(x^2 + 1)^3\right]\right) = \left(-\csc\left[(x^2 + 1)^3\right]\cot\left[(x^2 + 1)^3\right]\right) \cdot \left(D_x\left[(x^2 + 1)^3\right]\right)
\]

(by the Generalized Trigonometric Rules)

\[
= \left(-\csc\left[(x^2 + 1)^3\right]\cot\left[(x^2 + 1)^3\right]\right) \cdot \left[3(x^2 + 1)^2\right] \cdot \left[D_x\left(x^2 + 1\right)\right]
\]

(by the Generalized Power Rule)

\[
= \left(-\csc\left[(x^2 + 1)^3\right]\cot\left[(x^2 + 1)^3\right]\right) \cdot \left[3(x^2 + 1)^2\right] \cdot [2x]
\]

\[
= -6x(x^2 + 1)^2\csc\left[(x^2 + 1)^3\right]\cot\left[(x^2 + 1)^3\right]
\]

§
**PART E: EXAMPLES WITH TANGENT LINES**

*Example 12 (Finding Horizontal Tangent Lines to a Polynomial Graph)*

Let \( f(x) = (x^2 - 9)^7 \). Find the \( x \)-coordinates of all points on the graph of \( y = f(x) \) where the tangent line is horizontal.

§ *Solution*

- We must find where the slope of the tangent line to the graph is 0. We must solve the equation:
  \[
  f'(x) = 0
  \]
  \[
  D_x[(x^2 - 9)^7] = 0
  \]
  \[
  \left[7(x^2 - 9)^6\right] \cdot D_x(x^2 - 9) = 0 \quad \text{(by the Generalized Power Rule)}
  \]
  \[
  \left[7(x^2 - 9)^6\right] \cdot [2x] = 0
  \]
  \[
  14x(x^2 - 9)^6 = 0
  \]

- The Generalized Power Rule is a great help here. The alternative? We could have expanded \((x^2 - 9)^7\) by the Binomial Theorem, differentiated the result term-by-term, and then factored the result as \(14x(x^2 - 9)^6\) or as \(14x(x + 3)^6(x - 3)^6\) … after quite a bit of work!

- Instead of factoring further, we will apply the Zero Factor Property directly:
  \[
  x = 0 \quad \text{or} \quad x^2 - 9 = 0
  \]
  \[
  x^2 = 9
  \]
  \[
  x = \pm 3
  \]

The desired \( x \)-coordinates are: \(-3, 0,\) and \(3\).
• Why does the graph of \( y = (x^2 - 9)^7 \) below make sense?

  • Observe that \( f \) is an **even** function.

  • \( f(x) = (x^2 - 9)^7 = (x + 3)^7 (x - 3)^7 \), which means that \(-3\) and \(3\) are zeros of \( f \) of multiplicity 7 (see Chapter 2 of the Precalculus notes). As a result, the graph has **x-intercepts** at \((-3, 0)\) and \((3, 0)\), and the higher multiplicity indicates greater **flatness** around those points. Because the multiplicities are odd, the graph “**cuts through**” the **x-axis** at the x-intercepts, instead of “bouncing off” of the x-axis there.

  • The **y-intercept** is extremely low, because 
  \[ f(0) = (-9)^7 = -4,782,969. \]

• The **red tangent lines** below are truncated.

(Axes are scaled differently.)
**Example 13 (Finding Horizontal Tangent Lines to a Trigonometric Graph)**

Let \( f(x) = x + \cos(2x) \). Find the \( x \)-coordinates of all points on the graph of \( y = f(x) \) where the tangent line is horizontal.

**§ Solution**

- We must find where the slope of the tangent line to the graph is 0. We must solve the equation:

\[
\frac{df}{dx} = 0
\]

\[
D_x \left[ x + \cos(2x) \right] = 0
\]

\[
1 + \left[ -\sin(2x) \right] \cdot [D_x (2x)] = 0 \quad \text{(by Generalized Trigonometric Rules)}
\]

\[
1 + \left[ -\sin(2x) \right] \cdot \left[ 2 \right] = 0
\]

\[
1 - 2\sin(2x) = 0 \quad \text{(See Tip 3 for a short cut.)}
\]

\[
-2\sin(2x) = -1
\]

\[
\sin \left( \frac{2x}{\theta} \right) = \frac{1}{2}
\]

Use the substitution \( \theta = 2x \).

\[
\sin \theta = \frac{1}{2}
\]

Our solutions for \( \theta \) are:

\[
\theta = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad \theta = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z})
\]

To find our solutions for \( x \), replace \( \theta \) with \( 2x \), and solve for \( x \).

\[
2x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z})
\]

\[
x = \left( \frac{1}{2} \right) \left( \frac{\pi}{6} \right) + \pi n \quad \text{or} \quad x = \left( \frac{1}{2} \right) \left( \frac{5\pi}{6} \right) + \pi n \quad (n \in \mathbb{Z})
\]

\[
x = \frac{\pi}{12} + \pi n \quad \text{or} \quad x = \frac{5\pi}{12} + \pi n \quad (n \in \mathbb{Z})
\]
The desired $x$-coordinates are given by:

$$\left\{ x \in \mathbb{R} \bigg| x = \frac{\pi}{12} + \pi n, \text{ or } x = \frac{5\pi}{12} + \pi n, \ (n \in \mathbb{Z}) \right\}.$$

- Observe that there are infinitely many points on the graph where the tangent line is horizontal.

- Why does the graph of $y = x + \cos(2x)$ below make sense? The “$x$” term leads to upward drift; the graph oscillates about the line $y = x$.

- The red tangent lines below are truncated.
FOOTNOTES

1. **Partial proof of the Chain Rule.** Assume that \( g \) is differentiable at \( a \), and \( f \) is differentiable at \( g(a) \). Let \( b = g(a) \). More generally, let \( u = g(x) \). As an optional step, we can let \( p = f \circ g \). Then, \( p(x) = (f \circ g)(x) = f(g(x)) \). We will show that \( p \), or \( f \circ g \), is differentiable at \( a \), with \( p'(a) = (f \circ g)'(a) = \frac{[f'(b)] [g'(a)]}{g(x) - g(a)} \).

\[
p'(a) = \lim_{x \to a} \frac{p(x) - p(a)}{x - a}
= \lim_{x \to a} \left[ \frac{p(x) - p(a)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right], \quad \left[ g(x) \neq g(a) \right] \quad \text{(See Note 1 below.)}
= \lim_{x \to a} \left[ \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right]
= \left[ \lim_{x \to a} \frac{f(u) - f(b)}{u - b} \right] \cdot \left[ \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right] \quad \text{(See Note 2 below.)}
= \left[ f'(b) \right] \cdot \left[ g'(a) \right]
\]

**Note 1:** We have a problem if \( g(x) = g(a) \) “near” \( x = a \); that is, the partial proof fails if \( g(x) = g(a) \) somewhere on every “punctured” open interval about \( x = a \). The function in Footnote 4 exhibits this problem, where \( a = 0 \). Larson gives a more general proof in Appendix A of his calculus text (9th ed., p.A8). It is not for the faint of heart!

**Note 2:** We assume that \( g \) is differentiable (and thus continuous) at \( a \). Therefore, as \( x \to a \), then \( u \to b \), since \( \lim_{x \to a} u = \lim_{x \to a} g(x) = g(a) = b \).

2. **A controversial form of the Chain Rule.** Some sources give the Chain Rule as:
\[
(f \circ g)'(x) = \left[ f'(g(x)) \right] g'(x) \]. However, some object to the use of the notation \( f'(g(x)) \).

3. **Radians.** The proofs in Section 3.4 showing that \( D_x (\sin x) = \cos x \) and \( D_x (\cos x) = -\sin x \) utilized the limit statement \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \), which was proven in Footnote 1 of Section 3.4 under the assumption that \( h \) was measured in radians (or as “pure” real numbers).
- Define the “sind” and “cosd” functions as follows:
  \( \text{sind}(x) = \text{the sine of } x \text{ degrees}, \) and \( \text{cosd}(x) = \text{the cosine of } x \text{ degrees} \).

Now, \( x \text{ degrees} = \left( x \text{ degrees} \right) \frac{\pi [\text{radians}]}{180 \text{ degrees}} = \frac{\pi}{180} x \text{ [radians]} \).

Therefore, \( \text{sind}(x) = \sin \left( \frac{\pi}{180} x \right) \), and \( \text{cosd}(x) = \cos \left( \frac{\pi}{180} x \right) \).
• Unfortunately, \( D_x \left[ \text{sind}(x) \right] \) is not simply \( \cos(x) \), as demonstrated below:
\[
D_x \left[ \text{sind}(x) \right] = D_x \left[ \sin \left( \frac{\pi}{180} x \right) \right]
\]
\[
= \left[ \cos \left( \frac{\pi}{180} x \right) \right] \cdot \left[ D_x \left( \frac{\pi}{180} x \right) \right] \quad \text{(by Generalized Trigonometric Rules)}
\]
\[
= \left[ \cos \left( \frac{\pi}{180} x \right) \right] \cdot \left[ \frac{\pi}{180} \right]
\]
\[
= \frac{\pi}{180} \cos \left( \frac{\pi}{180} x \right)
\]
\[
= \frac{\pi}{180} \cos \left( x \right)
\]

Therefore, we prefer the use of our original sine and cosine functions, together with radian measure.


4. **Applicability of the Chain Rule and short cuts.** In Section 3.2, Footnote 7, we defined a piecewise-defined function \( f \) as follows:
\[
f(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right), & x \neq 0 \\
  0, & x = 0 
\end{cases}
\]
It turns out that
\[
f'(x) = \begin{cases} 
  2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right), & x \neq 0 \\
  0, & x = 0 
\end{cases}
\]
The Product, Power, and Generalized Trigonometric Rules give us the top rule for \( f'(x) \) when \( x \neq 0 \). However, these rules do **not** apply when \( x = 0 \), since it is not true that \( f(x) = x^2 \sin \left( \frac{1}{x} \right) \) when \( x = 0 \); in fact, we would have had a problem using these methods at \( x = 0 \) if there were no open interval containing \( x = 0 \) throughout which the rule applied. Nevertheless, \( f'(0) \) does exist! In Section 3.2, Footnote 7, we showed that \( f'(0) = 0 \) using the Limit Definition of the Derivative.