

## **SECTION 3.5: DIFFERENTIALS and LINEARIZATION OF FUNCTIONS**

### **LEARNING OBJECTIVES**

- Use differential notation to express the approximate change in a function on a small interval.
- Find linear approximations of function values.
- Analyze how errors can be propagated through functional relationships.

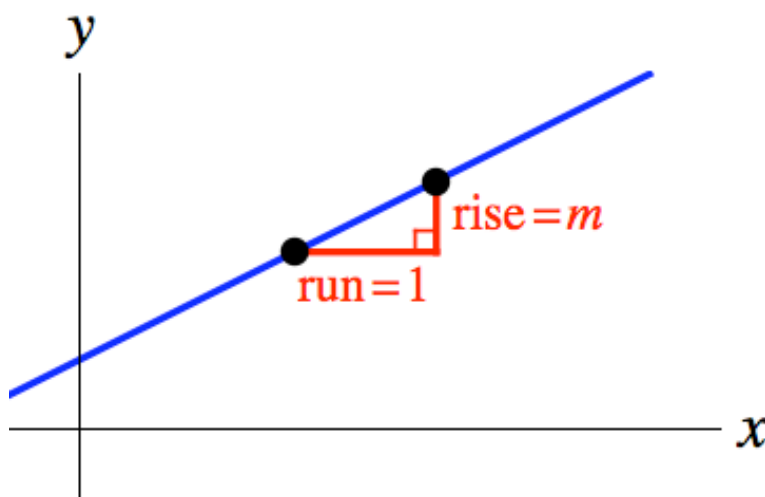
### **PART A: INTERPRETING SLOPE AS MARGINAL CHANGE**

Our story begins with **lines**. We know lines well, and we will use them to (locally) model graphs we don't know as well.

#### Interpretation of Slope $m$ as Marginal Change

For every unit increase in  $x$  along a line,  $y$  changes by  $m$ .

- If  $m < 0$ , then  $y$  **drops** in value.



If run = 1, then slope  $m = \frac{\text{rise}}{\text{run}} = \frac{\text{rise}}{1} = \text{rise}.$

**PART B: DIFFERENTIALS and CHANGES ALONG A LINE**

$dx$  and  $dy$  are the differentials of  $x$  and  $y$ , respectively.  
 They correspond to “small” changes in  $x$  and  $y$  along a **tangent line**.  
 We associate  $dy$  with “rise” and  $dx$  with “run.”

- If  $dy < 0$ , we move **down** along the line.
- If  $dx < 0$ , we move **left** along the line.

The following is key to this section:

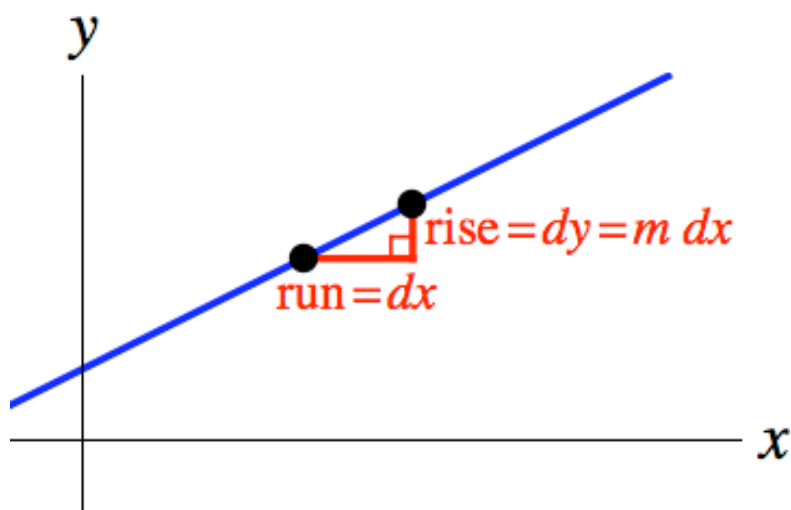
Rise Along a Line

slope =  $\frac{\text{rise}}{\text{run}}$ , so:

(rise) = (slope)(run)

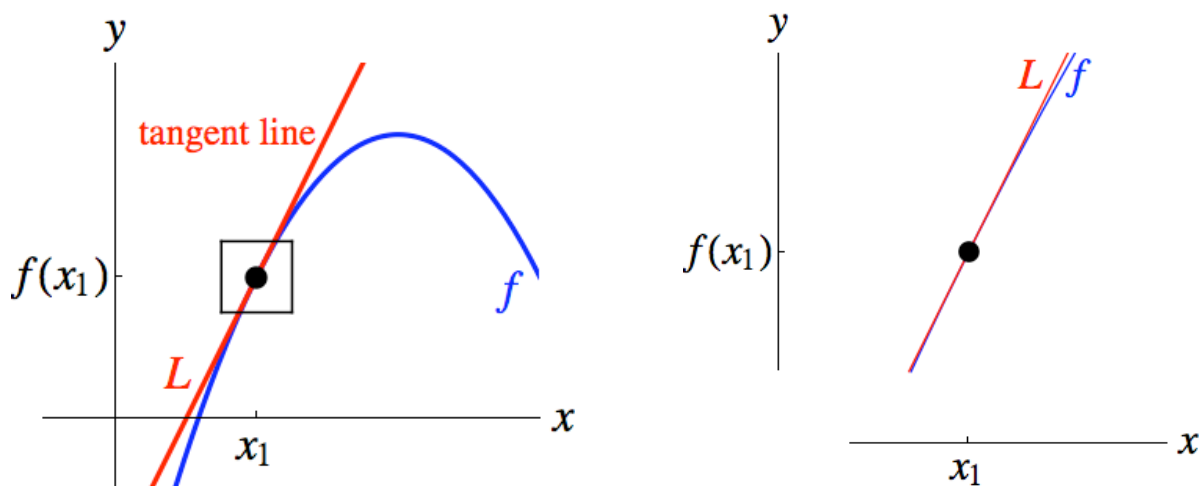
$m = \frac{dy}{dx}$  as a quotient of differentials, so:

$dy = m dx$



**PART C: LINEARIZATION OF FUNCTIONS**

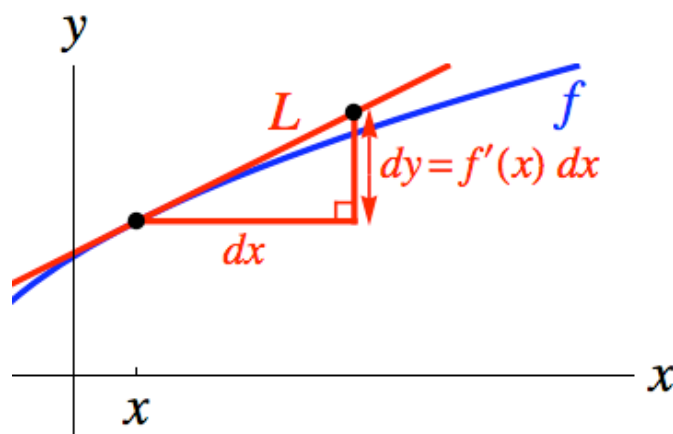
Remember the **Principle of Local Linearity** from Section 3.1. Assume that a function  $f$  is differentiable at  $x_1$ , which we will call the “**seed**.” Then, the **tangent line** to the graph of  $f$  at the point  $(x_1, f(x_1))$  represents the function  $L$ , the **best local linear approximation** to  $f$  close to  $x_1$ .  $L$  models (or “linearizes”)  $f$  locally on a small interval containing  $x_1$ .



The **slope** of the tangent line is given by  $f'(x_1)$  or, more generically, by  $f'(x)$ , so changes along the tangent line are related by the following formulas:

$dy = m \, dx$ $dy = f'(x) \, dx$
-----------------------------------

- This formula, written in differential form, is used to relate  $dx$  and  $dy$  as **variables**.
- In Leibniz notation, this can be written as  $dy = \frac{dy}{dx} \, dx$ , though those who see  $\frac{dy}{dx}$  as an inseparable entity may object to the appearance of “cancellation.”



$\Delta x$  and  $\Delta y$  are the increments of  $x$  and  $y$ , respectively.

They represent **actual changes** in  $x$  and  $y$  along the graph of  $f$ .

- If  $x$  changes  $\Delta x$  units from, say,  $x_1$  (“**seed, old  $x$** ”) to  $x_1 + \Delta x$  (“**new  $x$** ”), then  $y$  (or  $f$ ) changes  $\Delta y$  units from  $f(x_1)$  to  $f(x_1) + \Delta y$ , or  $f(x_1 + \Delta x)$ .

The “new” **function value**  $f(x_1 + \Delta x) = f(x_1) + \Delta y$ .

Informally,  $f(\text{new } x) = f(\text{old } x) + (\text{actual rise } \Delta y)$ .

$L(x_1 + \Delta x)$ , our **linear approximation** of  $f(x_1 + \Delta x)$ , is given by:

$$L(x_1 + \Delta x) = f(x_1) + dy$$

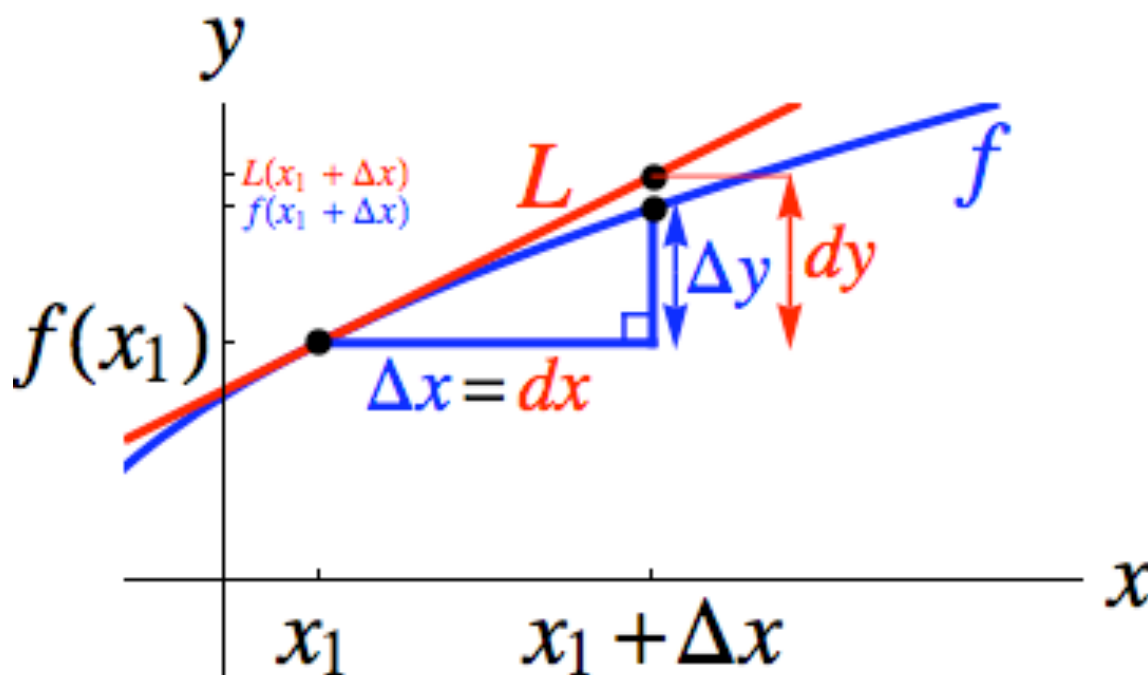
$$L(x_1 + \Delta x) = f(x_1) + [f'(x_1)][dx]$$

Informally,  $L(\text{new } x) = f(\text{old } x) + (\text{tangent rise } dy)$

$$L(\text{new } x) = f(\text{old } x) + [\text{slope}][\text{run } dx]$$

When finding  $L(x_1 + \Delta x)$ , we set  $dx = \Delta x$ , and then we hope that  $dy \approx \Delta y$ .

Then,  $L(x_1 + \Delta x) \approx f(x_1 + \Delta x)$ .



**PART D: EXAMPLES***Example 1 (Linear Approximation of a Function Value)*

Find a **linear approximation** of  $\sqrt{9.1}$  by using the value of  $\sqrt{9}$ .

Give the exact value of the linear approximation, and also give a decimal approximation rounded off to six significant digits.

**In other words:** Let  $f(x) = \sqrt{x}$ . Find a linear approximation  $L(9.1)$  for  $f(9.1)$  if  $x$  changes from 9 to 9.1.

*§ Solution Method 1 (Using Differentials)*

- $f$  is **differentiable** on  $(0, \infty)$ , which includes both 9 and 9.1, so this method is appropriate.
- We know that  $f(9) = \sqrt{9} = 3$  **exactly**, so 9 is a reasonable choice for the “seed”  $x_1$ .
- Find  $f'(9)$ , the **slope** of the tangent line at the “seed point”  $(9, 3)$ .

$$f(x) = \sqrt{x} = x^{1/2} \Rightarrow$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow$$

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$$

- Find the **run**  $dx$  (or  $\Delta x$ ).

$$\begin{aligned} \text{run } dx &= \text{"new } x" - \text{"old } x" \\ &= 9.1 - 9 \\ &= 0.1 \end{aligned}$$

- Find  $dy$ , the **rise** along the tangent line.

$$\begin{aligned}
 \text{rise } dy &= (\text{slope}) \cdot (\text{run}) \\
 &= [f'(9)] \cdot [dx] \\
 &= \left[\frac{1}{6}\right] [0.1] \\
 &= \frac{1}{60}
 \end{aligned}$$

- Find  $L(9.1)$ , our **linear approximation** of  $f(9.1)$ .

$$\begin{aligned}
 L(9.1) &= f(9) + dy \\
 &= 3 + \frac{1}{60} \\
 &= \frac{181}{60} \quad (\text{exact value}) \\
 &\approx 3.01667
 \end{aligned}$$

**WARNING 1:** Many students would forget to add  $f(9)$  and simply give  $dy$  as the approximation of  $\sqrt{9.1}$ . This mistake can be avoided by observing that  $\sqrt{9.1}$  should be very different from  $\frac{1}{60}$ .

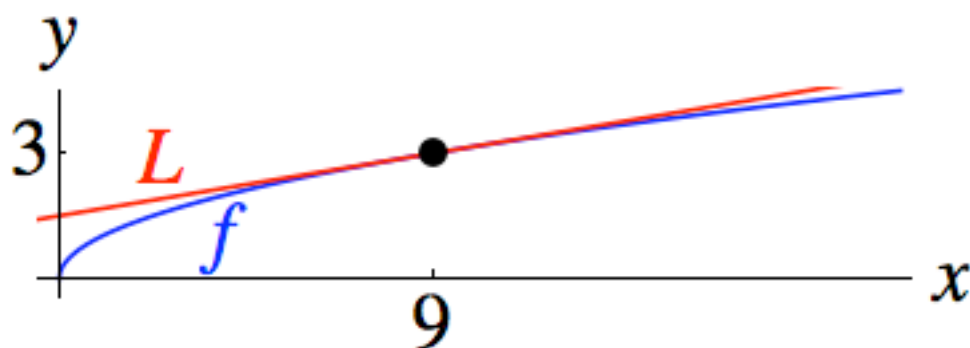
- In fact,  $f(9.1) = \sqrt{9.1} \approx 3.01662$ , so our approximation was accurate to five significant digits. Also, the **actual change in y** is given by:

$$\begin{aligned}
 \Delta y &= f(9.1) - f(9) \\
 &= \sqrt{9.1} - \sqrt{9} \\
 &\approx 3.01662 - 3 \\
 &\approx 0.01662
 \end{aligned}$$

This was approximated by  $dy = \frac{1}{60} \approx 0.01667$ .

The **error** is given by:  $dy - \Delta y \approx 0.01667 - 0.01662 \approx 0.00005$ .  
(See Example 5 for details on relative error, or percent error.)

- Our approximation  $L(9.1)$  was an **overestimate** of  $f(9.1)$ , because the graph of  $f$  curves downward (it is “concave down”; see Chapter 4).



- Differentials can be used to quickly find **other** linear approximations close to  $x = 9$ . (The approximations may become unreliable for values of  $x$  far from 9.)

Find a linear approximation of $f(9 + \Delta x)$	run $dx = \Delta x$	rise $dy = \left(\frac{1}{6}\right) dx$	Linear approximation, $L(9 + \Delta x)$
$f(8.8) = \sqrt{8.8}$	-0.2	$-\frac{1}{30} \approx -0.03333$	$L(8.8) = 3 - \frac{1}{30}$ $\approx 2.96667$
$f(8.9) = \sqrt{8.9}$	-0.1	$-\frac{1}{60} \approx -0.01667$	$L(8.9) = 3 - \frac{1}{60}$ $\approx 2.98333$
$f(9.1) = \sqrt{9.1}$ (We just did this.)	0.1	$\frac{1}{60} \approx 0.01667$	$L(9.1) = 3 + \frac{1}{60}$ $\approx 3.01667$
$f(9.2) = \sqrt{9.2}$	0.2	$\frac{1}{30} \approx 0.03333$	$L(9.2) = 3 + \frac{1}{30}$ $\approx 3.03333$

§ Solution Method 2 (Finding an Equation of the Tangent Line First:  $y = L(x)$ )

- Although this method may seem easier to many students, it does not stress the idea of **marginal change** the way that Method 1 does. Your instructor may demand Method 1.
- The **tangent line** at the “seed point”  $(9, 3)$  has **slope**  $f'(9) = \frac{1}{6}$ , as we saw in Method 1. Its equation is given by:

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{1}{6}(x - 9)$$

$L(x)$ , or  $y = 3 + \frac{1}{6}(x - 9)$ , which takes the form (with variable  $dx$ ):

$$L(x) = f(9) + \underbrace{[f'(9)][dx]}_{dy}$$

Also,  $L(x) = 3 + \frac{1}{6}(x - 9)$  simplifies as:  $L(x) = \frac{1}{6}x + \frac{3}{2}$ .

- In particular,  $L(9.1) = 3 + \frac{1}{6}(9.1 - 9)$  or  $\frac{1}{6}(9.1) + \frac{3}{2} \approx 3.01667$ , as before. §



Example 2 (Linear Approximation of a Trigonometric Function Value)

Find a **linear approximation** of  $\tan(42^\circ)$  by using the value of  $\tan(45^\circ)$ .  
Give the exact value of the linear approximation, and also give a decimal approximation rounded off to six significant digits.

§ Solution Method 1 (Using Differentials)

Let  $f(x) = \tan x$ .

**WARNING 2: Convert to radians.** In this example, we need to compute the **run**  $dx$  using radians. If  $f(x)$  or  $f'(x)$  were  $x \tan x$ , for example, then we would also need to convert to radians when evaluating **function values** and **derivatives**. (Also, a Footnote in Section 3.6 will discuss why the differentiation rules for trigonometric functions given in Section 3.4 do **not** apply if  $x$  is measured in degrees.)

• Converting to **radians**,  $45^\circ = \frac{\pi}{4}$  and  $42^\circ = \frac{7\pi}{30}$ . (As a practical matter, it turns out that we don't have to do these conversions, as long as we know what the **run**  $dx$  is in radians.) We want to find a linear approximation

$L\left(\frac{7\pi}{30}\right)$  for  $f\left(\frac{7\pi}{30}\right)$  if  $x$  changes from  $\frac{\pi}{4}$  to  $\frac{7\pi}{30}$ .

•  $f$  is **differentiable** on, among other intervals,  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , which includes both  $\frac{\pi}{4}$  and  $\frac{7\pi}{30}$ , so this method is appropriate.

• We know that  $f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$  **exactly**.

• Find  $f'\left(\frac{\pi}{4}\right)$ , the **slope** of the tangent line at the “seed point”  $\left(\frac{\pi}{4}, 1\right)$ .

$$f(x) = \tan x \Rightarrow$$

$$f'(x) = \sec^2 x \Rightarrow$$

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$$

- Find the **run**  $dx$  (or  $\Delta x$ ). It is usually easier to subtract the degree measures before converting to **radians**. Since it turns out  $dx < 0$  here, we run **left** (as opposed to right) along the tangent line.

$$\begin{aligned}
 \text{run } dx &= \text{"new } x" - \text{"old } x" \\
 &= 42^\circ - 45^\circ \\
 &= -3^\circ \\
 &= (-3^\circ) \left( \frac{\pi}{180^\circ} \right) \quad (\text{Converting to radians}) \\
 &= -\frac{\pi}{60}
 \end{aligned}$$

- Find  $dy$ , the **rise** along the tangent line. Actually, since it turns out  $dy < 0$  here, it really corresponds to a **drop**.

$$\begin{aligned}
 \text{rise } dy &= (\text{slope}) \cdot (\text{run}) \\
 &= \left[ f' \left( \frac{\pi}{4} \right) \right] \cdot [dx] \\
 &= [2] \left[ -\frac{\pi}{60} \right] \\
 &= -\frac{\pi}{30}
 \end{aligned}$$

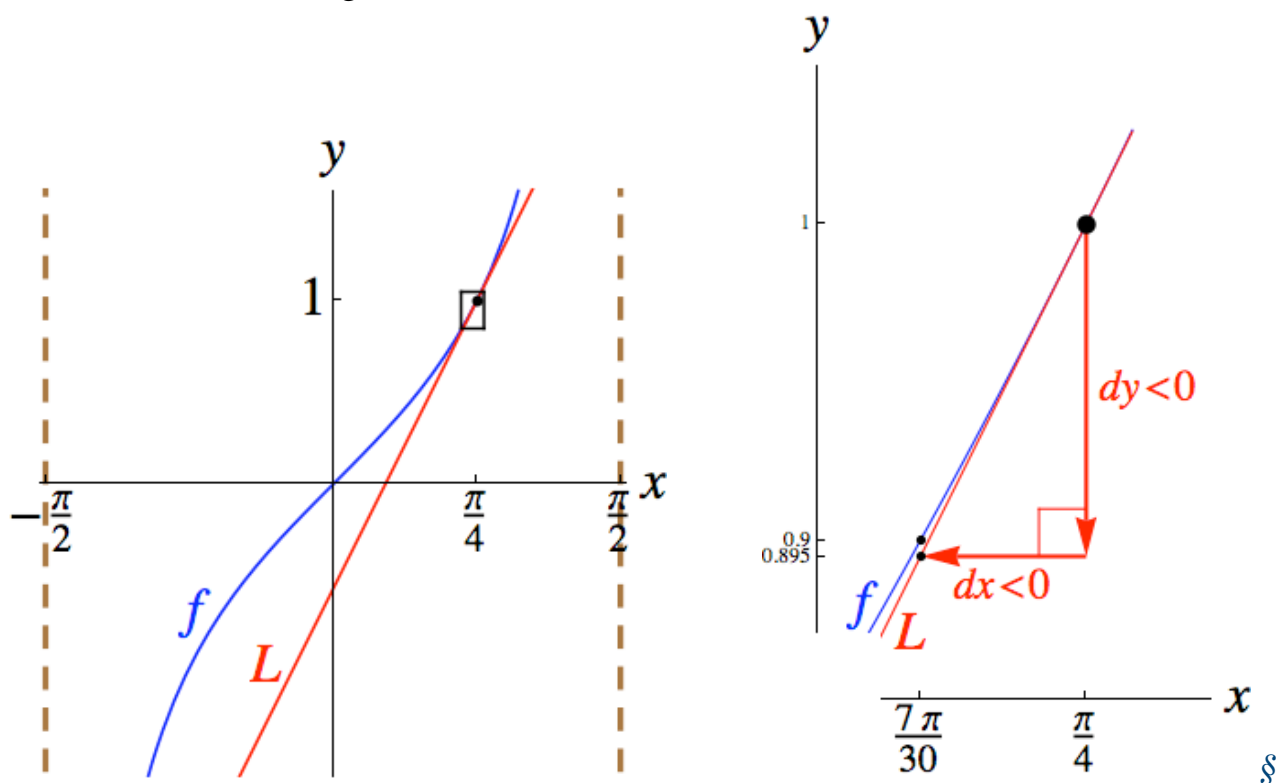
- Find  $L\left(\frac{7\pi}{30}\right)$ , our **linear approximation** of  $\tan(42^\circ)$ , or  $f\left(\frac{7\pi}{30}\right)$ .

$$\begin{aligned}
 L\left(\frac{7\pi}{30}\right) &= f\left(\frac{\pi}{4}\right) + dy \\
 &= 1 + \left(-\frac{\pi}{30}\right) \\
 &= \frac{30 - \pi}{30} \quad (\text{exact value}) \\
 &\approx 0.895280
 \end{aligned}$$

- In fact,  $f\left(\frac{7\pi}{30}\right) = \tan(42^\circ) \approx 0.900404$ .

- Our approximation  $L\left(\frac{7\pi}{30}\right)$  was an **underestimate** of  $f\left(\frac{7\pi}{30}\right)$ .

See the figures below.



§ Solution Method 2 (Finding an Equation of the Tangent Line First:  $y = L(x)$ )

- The **tangent line** at the “seed point”  $\left(\frac{\pi}{4}, 1\right)$  has **slope**  $f'\left(\frac{\pi}{4}\right) = 2$ , as we saw in Method 1. Its equation is given by:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$L(x)$ , or  $y = 1 + 2\left(x - \frac{\pi}{4}\right)$ , which takes the form (with variable  $dx$ ):

$$L(x) = f\left(\frac{\pi}{4}\right) + \underbrace{\left[f'\left(\frac{\pi}{4}\right)\right][dx]}_{dy}. \text{ Simplified, } L(x) = 2x + \frac{2 - \pi}{2}.$$

- In particular,  $L\left(\frac{7\pi}{30}\right) = 1 + 2\left(\frac{7\pi}{30} - \frac{\pi}{4}\right) = 1 + 2\left(-\frac{\pi}{60}\right) \approx 0.895280$ , or

$$L\left(\frac{7\pi}{30}\right) = 2\left(\frac{7\pi}{30}\right) + \frac{2 - \pi}{2} \approx 0.895280, \text{ as before. } §$$

**PART E: APPLICATIONS**

Since a typical calculator has a square root button and a tangent button, the previous examples might not seem useful.

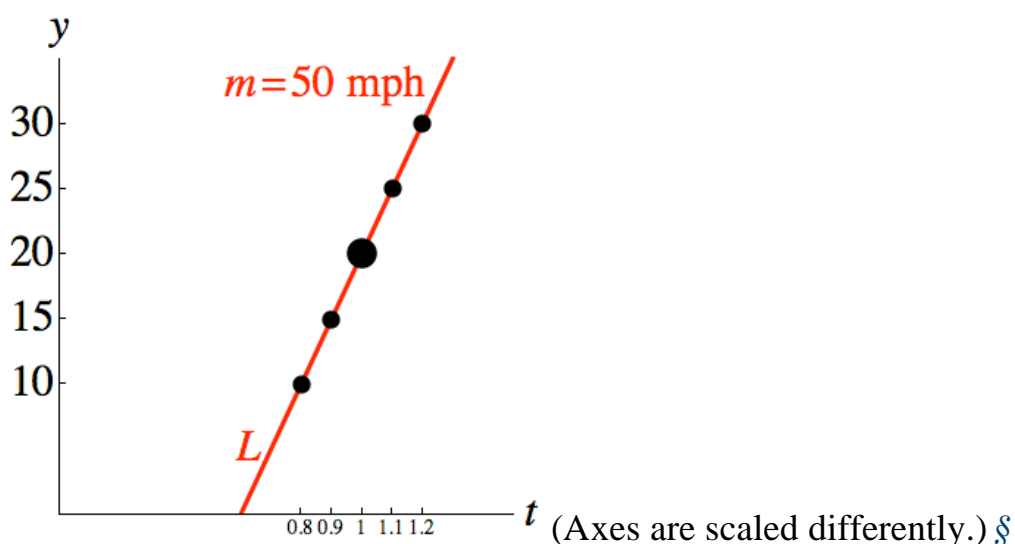
The table on Page 3.5.7 demonstrates how differentials can be used to quickly find **multiple approximations** of function values on a **small interval** around the “seed.”

Differentials can be used even if we **do not have a formula** for the function we are approximating, as we now demonstrate.

Example 3 (Approximating Position Values in the Absence of Formulas)

A car is moving on a coordinate line. Let  $y = s(t)$ , the **position** of the car (in miles)  $t$  hours after noon. We are given that  $s(1) = 20$  miles and  $v(1) = s'(1) = 50$  mph.

Find a linear approximation of $s(1 + \Delta t)$	<b>run</b> $dt = \Delta t$	<b>rise</b> $dy = 50 \, dt$	<b>Linear approximation,</b> $L(1 + \Delta t)$
$s(0.8)$	$-0.2$	$-10$	$L(0.8) = 20 - 10$ $= 10$ mi
$s(0.9)$	$-0.1$	$-5$	$L(0.9) = 20 - 5$ $= 15$ mi
$s(1.1)$	$0.1$	$5$	$L(1.1) = 20 + 5$ $= 25$ mi
$s(1.2)$	$0.2$	$10$	$L(1.2) = 20 + 10$ $= 30$ mi



**PART F: MEASUREMENT ERROR and PROPAGATED ERROR**

Let  $x$  be the **actual** (or **exact**) length, weight, etc. that we are trying to measure.

Let  $\Delta x$  (or  $dx$ ) be the **measurement error**. The error could be due to a poorly calibrated instrument or merely random chance.

Example 4 (Measurement Error)

The radius of a circle (unknown to us) is 10.7 inches (the **actual value**), but we measure it as 10.5 inches (the **measured value**). Sources differ on the definition of **measurement error**.

- 1) If we let measurement error = (measured value) – (actual value), then the “seed”  $x = 10.7$  inches, and  $\Delta x = 10.5 - 10.7 = -0.2$  inches.

This approach is more consistent with our usual notion of “error.”

- 2) If we let measurement error = (actual value) – (measured value), then the “seed”  $x = 10.5$  inches, and  $\Delta x = 10.7 - 10.5 = 0.2$  inches.

**We will adopt this approach**, because we know the measured value but not the actual value in an exercise such as Example 5. This suggests that the **measured value** is more appropriate than the actual value as the “seed.”

(See Footnote 1.) §

If  $y = f(x)$  for some function  $f$ , then an error in measuring  $x$  may lead to **propagated error** in  $y$ , denoted by  $\Delta y$ . As before, we approximate  $\Delta y$  by  $dy$  for convenience.

Example 5 (Propagated Error)

Let  $x$  be the radius of a circle, and let  $y$  be its area. Then,  $y$ , or  $f(x) = \pi x^2$ . We measure the **radius** using an instrument that may be “off” by as much as 0.5 inches; more precisely, it has a maximum possible absolute value of **measurement error** of 0.5 inches. We use the instrument to obtain a **measured value** of 10.5 inches. **Estimate** the maximum possible absolute value of the **propagated error** that we will obtain for the **area** of the circle. Use differentials and give an approximation written out to five significant digits.

§ Solution

$$\begin{aligned} dy &= f'(x) dx \\ &= 2\pi x dx \end{aligned}$$

We use the “seed”  $x_1 = 10.5$  inches, the **measured value**, for  $x$ .

Let the **run**  $dx$  be the maximum possible absolute value of the **measurement error**, which is 0.5 inches. (Really,  $-0.5 \text{ in} \leq dx \leq 0.5 \text{ in}$ .)

Then,  $|dy|$ , our approximation for the maximum possible absolute value of the **propagated error** in  $y$ , is given by:

$$\begin{aligned} |dy| &= |2\pi x \, dx| \\ &\leq |2\pi(10.5)(0.5)| \\ &\leq 32.987 \text{ in}^2 \end{aligned}$$

That is, we estimate that the **propagated error** in the **area** of the circle will be “off” by no more than  $32.987 \text{ in}^2$  in either direction (high or low).

• **Estimates of the area.** If we take the measured value of 10.5 inches for the radius, we will obtain the following **measured value** for the area:

$$\begin{aligned} y, \text{ or } f(10.5) &= \pi(10.5)^2 \\ &\approx 346.361 \text{ in}^2 \end{aligned}$$

Since we estimate that we are “off” by no more than  $32.987 \text{ in}^2$ , we estimate that the **actual value** of the area is between  $313.374 \text{ in}^2$  and  $379.348 \text{ in}^2$ .

Without differentials, we would say that the actual value of the area is between  $f(10.0)$  and  $f(11.0)$ , or between  $314.159 \text{ in}^2$  and  $380.133 \text{ in}^2$ . (The benefit of using differentials is more apparent when the function involved is more complicated.)

• **Relative error and percent error.** Is  $32.987 \text{ in}^2$  “bad” or “good”? The relative error and the percent error give us some context to decide.

$$\begin{aligned} \text{Relative error} &= \frac{dy}{y} \left( \text{really, } \frac{\text{maximum } |dy|}{\text{measured } y} \right) \\ &\approx \frac{32.987}{346.361} \\ &\approx 0.095239, \text{ or } 9.5239\% \text{ (percent error)} \end{aligned}$$

(If the measured area had been something like  $y = 1,000,000 \text{ in}^2$ , then  $32.987 \text{ in}^2$  probably wouldn’t be a big deal.)

- **Actual propagated error.** If the actual value of the **radius** is 10.7 inches, then the actual value of the **area** is  $f(10.7) = \pi(10.7)^2 \approx 359.681 \text{ in}^2$ , and the actual propagated error is given by:

$$\begin{aligned}\Delta y &= f(10.7) - f(10.5) \\ &= \pi(10.7)^2 - \pi(10.5)^2 \\ &\approx 13.320 \text{ in}^2\end{aligned}$$

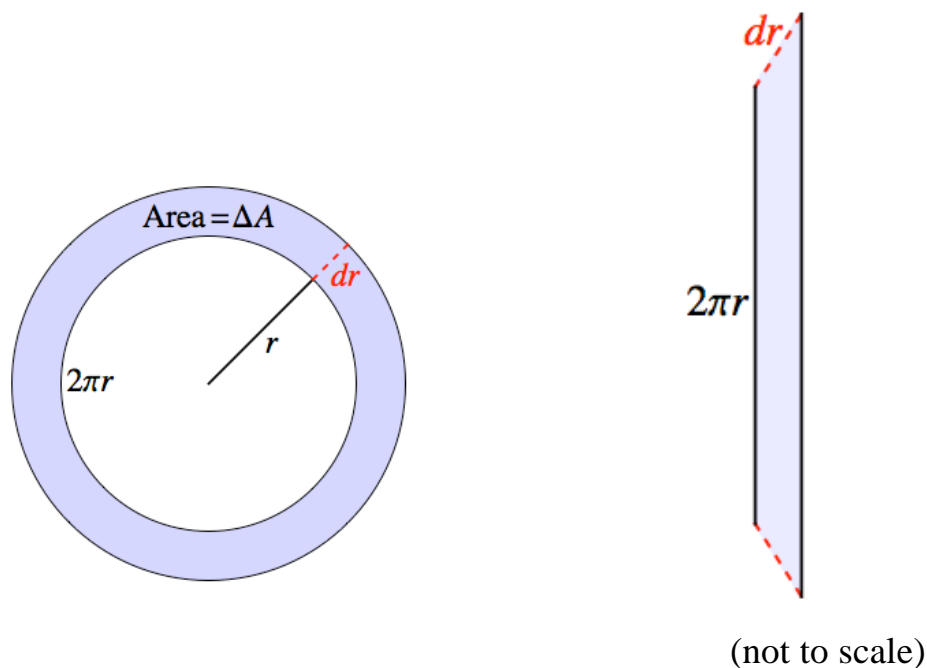
- **Notation.** If we let  $r$  = the radius and  $A$  = the area, then we obtain the more familiar formula  $A = \pi r^2$ . Also,  $dA = 2\pi r \, dr$ .

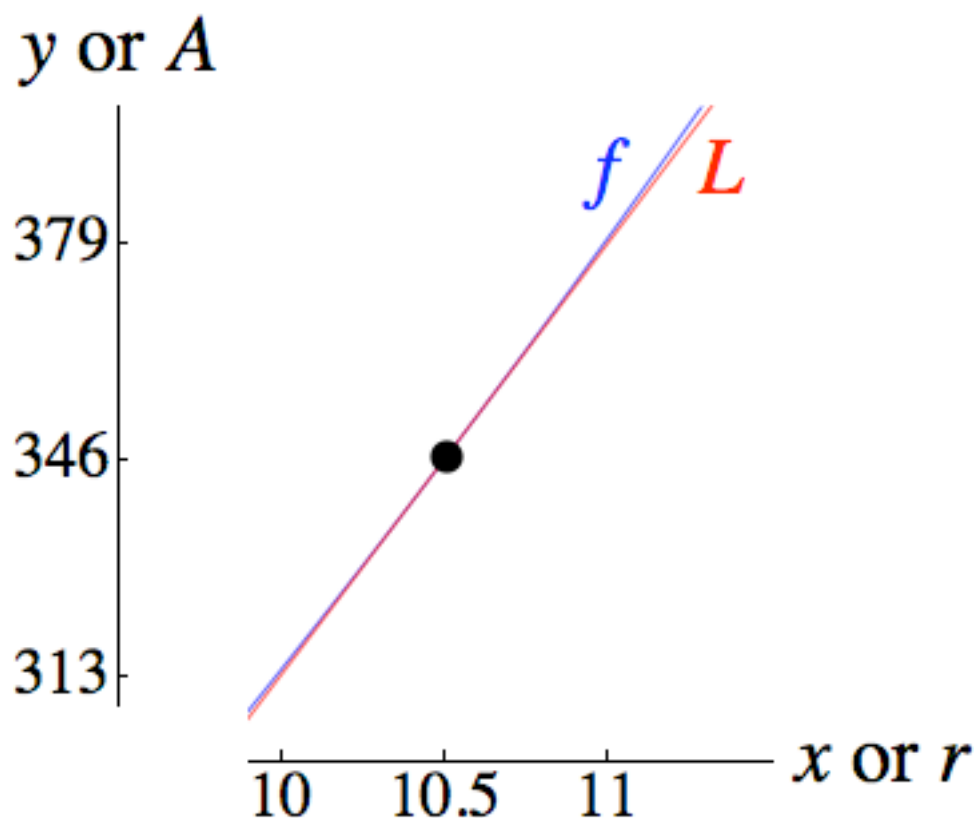
- **Geometric approach.** Observe that:

$$dA = (\text{circumference of circle of radius } r) (\text{thickness of ring})$$

This approximates  $\Delta A$ , the actual propagated error in the area. The figures below demonstrate why this approximation makes sense.

Imagine cutting the shaded ring below along the dashed slit and straightening it out. We obtain a trapezoid that is approximately a rectangle with dimensions  $2\pi r$  and  $dr$ . The area of both shaded regions is  $\Delta A$ , and we approximate it by  $dA$ , where  $dA = 2\pi r \, dr$ .





(Axes are scaled differently.) §

## FOOTNOTES

1. **Defining measurement (or absolute) error.** Definition 1) in Example 4 is used in *Wolfram MathWorld*, namely that measurement error = (measured value) – (actual value). Sometimes, absolute value is taken. See “Absolute Error,” *Wolfram Mathworld*, Web, 25 July 2011, <<http://mathworld.wolfram.com/AbsoluteError.html>>.
  - Larson in his calculus text (9<sup>th</sup> ed.) uses Definition 2):  
measurement error = (actual value) – (measured value). On top of the rationale given in Example 4, this is also more consistent with the notion of error when studying confidence intervals and regression analysis in statistics.



## SECTION 3.6: CHAIN RULE

### LEARNING OBJECTIVES

- Understand the Chain Rule and use it to differentiate composite functions.
- Know when and how to apply the Generalized Power Rule and the Generalized Trigonometric Rules, which are based on the Chain Rule.

### PART A: THE IDEA OF THE CHAIN RULE

Yul, Uma, and Xavier run in a race. Let  $y$ ,  $u$ , and  $x$  represent their positions (in miles), respectively.

- Assume that Yul always runs **twice** as fast as Uma. That is,  $\frac{dy}{du} = 2$ .  
(If Uma runs  $\Delta u$  miles, then Yul runs  $\Delta y$  miles, where  $\Delta y = 2 \Delta u$ .)
- Assume that Uma always runs **three** times as fast as Xavier. That is,  $\frac{du}{dx} = 3$ .
- Therefore, Yul always runs **six** times as fast as Xavier. That is,  $\frac{dy}{dx} = 6$ .

This is an example of the Chain Rule, which states that:  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ .

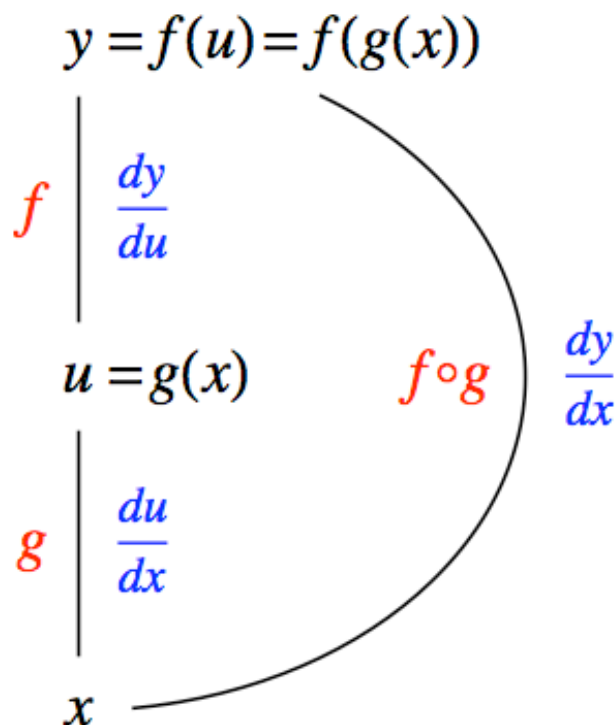
Here,  $6 = 2 \cdot 3$ .

**WARNING 1:** The Chain Rule is a calculus rule, not an algebraic rule, in that the “ $du$ ”s should **not** be thought of as “canceling.”

We can think of  $y$  as a function of  $u$ , which, in turn, is a function of  $x$ .

Call these functions  $f$  and  $g$ , respectively.

Then,  $y$  is a **composite function** of  $x$ ; this function is denoted by  $f \circ g$ .



• **In multivariable calculus**, you will see bushier trees and more complicated forms of the Chain Rule where you add products of derivatives along paths, extending what we have done above.

**TIP 1:** The Chain Rule is used to differentiate **composite functions** such as  $f \circ g$ . The derivative of a **product** of functions is **not** necessarily the product of the derivatives (see Section 3.3 on the Product Rule), but the derivative of a **composition** of functions **is** the product of the derivatives. (Composite functions were reviewed in Chapter 1.)

**PART B: FORMS OF THE CHAIN RULE**Chain Rule

Let  $y = f(u)$  and  $u = g(x)$ , where  $f$  and  $g$  are differentiable “where we care.” Then,

$$\text{Form 1) } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\text{Form 2) } (f \circ g)'(x) = [f'(u)][g'(x)]$$

$$\text{Form 3) } y' = [f'(u)][u']$$

- Essentially, the derivative of a **composite function** is obtained by taking the derivative of the “outer function” at  $u$  **times** the derivative of the “inner function” at  $x$ .
- Following *How to Ace Calculus* by Adams, Thompson, and Hass (Times, 1998), we will refer to the derivative of the **inner** function as the “**tail**.” In the forms above, the tail is denoted by  $\frac{du}{dx}$ ,  $g'(x)$ , and  $u'$ .

**WARNING 2:** Forgetting the “**tail**” is a very common error students make when applying the Chain Rule.

- See Footnote 1 for a partial proof.
- See Footnote 2 on a controversial form.

Many differentiation rules, such as the Generalized Power Rule and the Generalized Trigonometric Rules we will introduce in this section, are **based on the Chain Rule**.

**PART C: GENERALIZED POWER RULE**

The **Power Rule of Differentiation**, which we introduced in Part B of Section 3.2, can be used to find  $D_x(x^7)$ . However, it **cannot** be used to find  $D_x[(3x^2 + 4)^7]$  without expanding the indicated seventh power, something we would rather not do.

*Example 1 (Using the Chain Rule to Motivate the Generalized Power Rule)*

Use the **Chain Rule** to find  $D_x[(3x^2 + 4)^7]$ .

*§ Solution*

Let  $y = (3x^2 + 4)^7$ . We will treat  $y$  as a **composite function** of  $x$ .

$y = (f \circ g)(x) = f(g(x))$ , where:

$$u = g(x) = 3x^2 + 4 \quad (g \text{ is the "inner function"})$$

$$y = f(u) = u^7 \quad (f \text{ is the "outer function"})$$

Observe that  $\frac{dy}{du}$  and  $\frac{du}{dx}$  can be readily found using **basic rules**.

We can then find  $\frac{dy}{dx}$  using the **Chain Rule**.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= [D_u(u^7)][D_x(3x^2 + 4)] \\ &= [7u^6][6x] \end{aligned}$$

(Since  $u$  was our creation, we must express  $u$  in terms of  $x$ .)

$$\begin{aligned} &= [7(3x^2 + 4)^6][6x] \quad (\text{See Example 2 for a short cut.}) \\ &= 42x(3x^2 + 4)^6 \end{aligned}$$

Example 1 suggests the following short cut.

### Generalized Power Rule

Let  $u$  be a function of  $x$  that is differentiable “where we care.”

Let  $n$  be a real constant.

$$D_x(u^n) = (nu^{n-1}) \underbrace{(D_x u)}_{\text{"tail"}}$$

• **Rationale.** If  $y = u^n$ , then, by the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \\ D_x(u^n) &= [D_u(u^n)] \cdot [D_x u] \\ &= [nu^{n-1}] \cdot [D_x u] \end{aligned}$$

### Example 2 (Using the Generalized Power Rule; Revisiting Example 1)

Use the **Generalized Power Rule** to find  $D_x[(3x^2 + 4)^7]$ .

### § Solution

$$\begin{aligned} D_x[(3x^2 + 4)^7] &= [7(3x^2 + 4)^6][D_x(3x^2 + 4)] \quad (\text{See Warning 3.}) \\ &= [7(3x^2 + 4)^6][6x] \\ &= 42x(3x^2 + 4)^6 \end{aligned}$$

**WARNING 3: Copy the base.** The base  $u$ , which is  $(3x^2 + 4)$  here, is **copied** under the exponent. Do **not** differentiate it until you get to the “tail.”

**WARNING 4: Remember the exponent.** Many students forget to write the exponent, 6, because the base can take some time to write. You may want to write the exponent **before** writing out the base.

**WARNING 5: Identifying “tails.”** The  $D_x$  notation helps us keep track of “how far” to unravel tails. **A tail may have a tail of its own.** If we forget tails, we’re **not going far enough**. If we attach inappropriate tails (such as an additional “6” after the “6x” above), we’re going **too far**. §

Example 3 (Using the Generalized Power Rule in Conjunction with the Quotient Rule or the Product Rule)

Find  $D_x \left( \frac{x^3}{\sqrt{2x-1}} \right)$ .

§ Solution Method 1 (Using the Quotient Rule)

$$\begin{aligned}
 D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) &= \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{(\text{Lo})^2, \text{ the square of what's below}} \\
 &= \frac{(\sqrt{2x-1}) \cdot [D_x(x^3)] - (x^3) \cdot (D_x[(2x-1)^{1/2}])}{(\sqrt{2x-1})^2} \\
 &= \frac{(\sqrt{2x-1}) \cdot [3x^2] - (x^3) \cdot \left[ \frac{1}{2}(2x-1)^{-1/2} \right] \cdot [D_x(2x-1)]}{2x-1} \\
 &= \frac{(\sqrt{2x-1}) \cdot [3x^2] - (x^3) \cdot \left[ \frac{1}{2}(2x-1)^{-1/2} \right] \cdot [\cancel{2}]}{2x-1} \\
 &\quad \text{(We could factor the numerator at this point.)} \\
 &= \frac{(\sqrt{2x-1}) \cdot [3x^2] - \frac{x^3}{\sqrt{2x-1}}}{2x-1} \\
 &= \frac{\left[ (\sqrt{2x-1}) \cdot [3x^2] - \frac{x^3}{\sqrt{2x-1}} \right] \cdot \sqrt{2x-1}}{2x-1 \cdot \sqrt{2x-1}}
 \end{aligned}$$

**WARNING 6: Distribute before canceling.** Do not cancel in the numerators until we have distributed  $\sqrt{2x-1}$  through the first numerator.

$$\begin{aligned}
 &= \frac{(2x-1) \cdot [3x^2] - x^3}{(2x-1)^{3/2}} \\
 &= \frac{6x^3 - 3x^2 - x^3}{(2x-1)^{3/2}} \\
 &= \frac{5x^3 - 3x^2}{(2x-1)^{3/2}}, \text{ or } \frac{x^2(5x-3)}{(2x-1)^{3/2}}
 \end{aligned}$$

§ Solution Method 2 (Using the Product Rule)

If we had forgotten the Quotient Rule, we could have rewritten:

$$D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) = D_x \left[ x^3 (2x-1)^{-1/2} \right] \text{ and applied the **Product Rule**.}$$

We would then use the **Generalized Power Rule** to find  $D_x \left[ (2x-1)^{-1/2} \right]$ .

The key **drawback** here is that we obtain two terms, and students may find it difficult to combine them into a **single, simplified fraction**. Observe:

$$\begin{aligned} D_x \left( \frac{x^3}{\sqrt{2x-1}} \right) &= D_x \left[ x^3 (2x-1)^{-1/2} \right] \quad (\text{Rewriting}) \\ &= \left[ D_x (x^3) \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left( D_x \left[ (2x-1)^{-1/2} \right] \right) \\ &\quad (\text{by the Product Rule}) \\ &= \left[ 3x^2 \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left( \left[ -\frac{1}{2} (2x-1)^{-3/2} \right] \cdot \left[ D_x (2x-1) \right] \right) \\ &\quad (\text{by the Generalized Power Rule}) \\ &= \left[ 3x^2 \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left( \left[ -\frac{1}{2} (2x-1)^{-3/2} \right] \cdot \left[ \cancel{2} \right] \right) \\ &= \left[ 3x^2 \right] \cdot \left[ (2x-1)^{-1/2} \right] + (x^3) \cdot \left[ - (2x-1)^{-3/2} \right] \\ &= \frac{3x^2}{(2x-1)^{1/2}} - \frac{x^3}{(2x-1)^{3/2}} \\ &= \frac{3x^2}{(2x-1)^{1/2}} \cdot \frac{(2x-1)}{(2x-1)} - \frac{x^3}{(2x-1)^{3/2}} \\ &\quad \text{Build up the first fraction to obtain the LCD, } (2x-1)^{3/2}. \\ &= \frac{3x^2(2x-1) - x^3}{(2x-1)^{3/2}} \\ &= \frac{6x^3 - 3x^2 - x^3}{(2x-1)^{3/2}} \\ &= \frac{5x^3 - 3x^2}{(2x-1)^{3/2}}, \text{ or } \frac{x^2(5x-3)}{(2x-1)^{3/2}} \quad (\text{as in Method 1}) \end{aligned}$$

Example 4 (Using the Generalized Power Rule to Differentiate a Power of a Trigonometric Function)

Let  $f(\theta) = \sec^5 \theta$ . Find  $f'(\theta)$ .

§ Solution

First, rewrite  $f(\theta)$ :

$$\begin{aligned} f(\theta) &= \sec^5 \theta \\ &= (\sec \theta)^5 \end{aligned}$$

**WARNING 7: Rewriting before differentiating.** When differentiating a **power of a trigonometric function**, rewrite the power in this way. Students get very confused otherwise. Also, do **not** write  $\sec \theta^5$  here; that is equivalent to  $\sec(\theta^5)$ , not  $(\sec \theta)^5$ .

$$\begin{aligned} f'(\theta) &= \left[ 5(\sec \theta)^4 \right] \cdot \left[ D_\theta (\sec \theta) \right] \\ &\quad \text{(by the Generalized Power Rule)} \\ &= \left[ 5(\sec \theta)^4 \right] \cdot [\sec \theta \tan \theta] \\ &= 5(\sec \theta)^5 \tan \theta \\ &= 5 \sec^5 \theta \tan \theta \end{aligned}$$

§

Example 5 (Using the Generalized Power Rule to Prove the Reciprocal Rule)

Prove the **Reciprocal Rule** from Section 3.3:  $D_x \left[ \frac{1}{g(x)} \right] = -\frac{g'(x)}{[g(x)]^2}$ .

§ Solution

$$D_x \left[ \frac{1}{g(x)} \right] = D_x \left( [g(x)]^{-1} \right)$$

**WARNING 8:** “ $-1$ ” here denotes a reciprocal, **not** a function inverse.

$$\begin{aligned} &= \left( -[g(x)]^{-2} \right) \cdot [g'(x)] \quad \text{(by the Generalized Power Rule)} \\ &= -\frac{g'(x)}{[g(x)]^2} \end{aligned}$$

§



**PART D: GENERALIZED TRIGONOMETRIC RULES**

The **Basic Trigonometric Rules of Differentiation**, which we introduced in Section 3.4, can be used to find  $D_x(\sin x)$ . However, they **cannot** be used to find  $D_x[\sin(x^2)]$ .

Example 6 (Using the Chain Rule to Motivate the Generalized Trigonometric Rules)

Use the **Chain Rule** to find  $D_x[\sin(x^2)]$ .

§ Solution

Let  $y = \sin(x^2)$ . We will treat  $y$  as a **composite function** of  $x$ .

$y = (f \circ g)(x) = f(g(x))$ , where:

$$u = g(x) = x^2 \quad (g \text{ is the "inner function"})$$

$$y = f(u) = \sin u \quad (f \text{ is the "outer function"})$$

Observe that  $\frac{dy}{du}$  and  $\frac{du}{dx}$  can be readily found using **basic rules**.

We can then find  $\frac{dy}{dx}$  using the **Chain Rule**.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= [D_u(\sin u)][D_x(x^2)] \\ &= [\cos u][2x] \end{aligned}$$

(Since  $u$  was our creation, we must express  $u$  in terms of  $x$ .)

$$\begin{aligned} &= [\cos(x^2)][2x] \quad (\text{See Example 7 for a short cut.}) \\ &= 2x \cos(x^2) \end{aligned}$$

§

Example 6 suggests the following short cuts.

### Generalized Trigonometric Rules

Let  $u$  be a function of  $x$  that is differentiable “where we care” (see Footnote 4).

$$\begin{array}{ll} D_x(\sin u) = (\cos u) \underbrace{(D_x u)}_{\text{"tail"}} & D_x(\cos u) = (-\sin u) \underbrace{(D_x u)}_{\text{"tail"}} \\ D_x(\tan u) = (\sec^2 u) \underbrace{(D_x u)}_{\text{"tail"}} & D_x(\cot u) = (-\csc^2 u) \underbrace{(D_x u)}_{\text{"tail"}} \\ D_x(\sec u) = (\sec u \tan u) \underbrace{(D_x u)}_{\text{"tail"}} & D_x(\csc u) = (-\csc u \cot u) \underbrace{(D_x u)}_{\text{"tail"}} \end{array}$$

**WARNING 9:** In the **bottom two rules**, the “**tail**” is still written only **once**. The “tail” is the derivative of the common argument  $u$ .

- **Radians.** See Footnote 3 on how these rules encourage us to use radians (as opposed to degrees) when differentiating trigonometric functions.

### Example 7 (Using the Generalized Trigonometric Rules; Revisiting Example 6)

Use the **Generalized Trigonometric Rules** to find  $D_x[\sin(x^2)]$ .

#### § Solution

$$\begin{aligned} D_x[\sin(x^2)] &= [\cos(x^2)][D_x(x^2)] \quad (\text{See Warning 10.}) \\ &= [\cos(x^2)][2x] \\ &= 2x \cos(x^2) \end{aligned}$$

**WARNING 10: Copy the argument.** The sine function’s argument  $u$ , which is  $(x^2)$  here, is **copied** as the cosine function’s argument. **Do not** differentiate it until you get to the “**tail**.” §

**TIP 2: Consistency with the Basic Trigonometric Rules.** Observe:

$$\begin{aligned} D_x(\sin x) &= [\cos x][D_x(x)] \\ &= [\cos x][1] \\ &= \cos x \end{aligned}$$

The “tail” is simply 1 when the argument ( $x$  here) is just the variable of differentiation, so we can ignore the tail in the Basic Trigonometric Rules.

Example 8 (Using the Generalized Trigonometric Rules)

Let  $g(\theta) = \sec(9\theta^2 - \theta)$ . Find  $g'(\theta)$ .

§ Solution

$$\begin{aligned}
 g'(\theta) &= D_\theta [\sec(9\theta^2 - \theta)] \\
 &= [\sec(9\theta^2 - \theta) \tan(9\theta^2 - \theta)] \cdot [D_\theta(9\theta^2 - \theta)] \\
 &\quad \text{(See Warning 9.)} \\
 &= [\sec(9\theta^2 - \theta) \tan(9\theta^2 - \theta)] \cdot [18\theta - 1] \\
 &= (18\theta - 1) \sec(9\theta^2 - \theta) \tan(9\theta^2 - \theta)
 \end{aligned}$$

§

Example 9 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)

Let  $f(x) = \cos^5(7x)$ . Find  $f'(x)$ .

§ Solution

First, rewrite  $f(x)$ :

$$\begin{aligned}
 f(x) &= \cos^5(7x) \\
 &= [\cos(7x)]^5 \quad \text{(See Warning 7.)}
 \end{aligned}$$

Overall, we are differentiating a **power**, so we will first apply the **Generalized Power Rule**.

$$\begin{aligned}
 f'(x) &= \left(5[\cos(7x)]^4\right) \cdot (D_x[\cos(7x)]) \quad \text{(See Warning 5 and Tip 3.)} \\
 &\quad \text{(by the Generalized Power Rule)} \\
 &= \left(5[\cos(7x)]^4\right) \cdot [-\sin(7x)] \cdot [D_x(7x)] \\
 &\quad \text{(by the Generalized Trigonometric Rules)} \\
 &= \left(5[\cos(7x)]^4\right) \cdot [-\sin(7x)] \cdot [7] \\
 &= -35 \cos^4(7x) \sin(7x)
 \end{aligned}$$

§

**TIP 3: Linear arguments.** If  $a$  is a real constant, then  $D_x[\sin(ax)] = a \cos(ax)$ ,  $D_x[\cos(ax)] = -a \sin(ax)$ ,  $D_x[\sec(ax)] = a \sec(ax) \tan(ax)$ , etc. In Example 9, we saw that:  $D_x[\cos(7x)] = -7 \sin(7x)$ . These can be very useful short cuts.

• More generally,  $D_x[\sin(ax + b)] = a \cos(ax + b)$ , and so forth; the “tail” is still the **coefficient of  $x$**  in the linear argument.

Example 10 (Using the Generalized Power Rule, Followed by the Generalized Trigonometric Rules)

Show that  $D_x[\tan^4(\pi x)] = 4\pi \tan^3(\pi x) \sec^2(\pi x)$ .

(Left to the reader.) The solution is similar to that in Example 9.

Hint: First rewrite:  $D_x[\tan^4(\pi x)] = D_x([\tan(\pi x)]^4)$ . §

Example 11 (Using the Generalized Trigonometric Rules, Followed by the Generalized Power Rule)

Find  $D_x\left(\csc\left[(x^2 + 1)^3\right]\right)$ .

§ Solution

Overall, we are differentiating a **trigonometric function**, so we will first apply the **Generalized Trigonometric Rules**.

$$\begin{aligned}
 D_x\left(\csc\left[(x^2 + 1)^3\right]\right) &= \left(-\csc\left[(x^2 + 1)^3\right] \cot\left[(x^2 + 1)^3\right]\right) \cdot \left(D_x\left[(x^2 + 1)^3\right]\right) \\
 &\quad \text{(by the Generalized Trigonometric Rules)} \\
 &= \left(-\csc\left[(x^2 + 1)^3\right] \cot\left[(x^2 + 1)^3\right]\right) \cdot \left[3(x^2 + 1)^2\right] \cdot \left[D_x(x^2 + 1)\right] \\
 &\quad \text{(by the Generalized Power Rule)} \\
 &= \left(-\csc\left[(x^2 + 1)^3\right] \cot\left[(x^2 + 1)^3\right]\right) \cdot \left[3(x^2 + 1)^2\right] \cdot [2x] \\
 &= -6x(x^2 + 1)^2 \csc\left[(x^2 + 1)^3\right] \cot\left[(x^2 + 1)^3\right]
 \end{aligned}$$

§

**PART E: EXAMPLES WITH TANGENT LINES***Example 12 (Finding Horizontal Tangent Lines to a Polynomial Graph)*

Let  $f(x) = (x^2 - 9)^7$ . Find the  $x$ -coordinates of all points on the graph of  $y = f(x)$  where the **tangent line** is **horizontal**.

*§ Solution*

- We must find where the **slope** of the tangent line to the graph is 0. We must solve the equation:

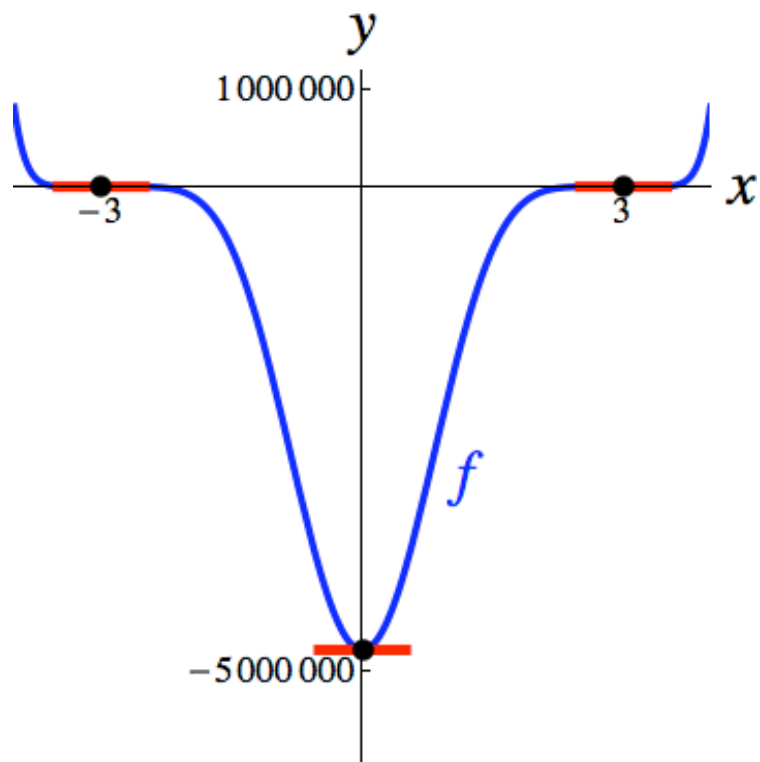
$$\begin{aligned}
 f'(x) &= 0 \\
 D_x \left[ (x^2 - 9)^7 \right] &= 0 \\
 \left[ 7(x^2 - 9)^6 \right] \cdot \left[ D_x (x^2 - 9) \right] &= 0 \quad (\text{by the Generalized Power Rule}) \\
 \left[ 7(x^2 - 9)^6 \right] \cdot [2x] &= 0 \\
 14x(x^2 - 9)^6 &= 0
 \end{aligned}$$

- The **Generalized Power Rule** is a great help here. The alternative? We could have expanded  $(x^2 - 9)^7$  by the Binomial Theorem, differentiated the result term-by-term, and then factored the result as  $14x(x^2 - 9)^6$  or as  $14x(x + 3)^6(x - 3)^6$  ... after quite a bit of work!
- Instead of factoring further, we will apply the **Zero Factor Property** directly:

$$\begin{aligned}
 x = 0 \quad \text{or} \quad x^2 - 9 &= 0 \\
 x^2 &= 9 \\
 x &= \pm 3
 \end{aligned}$$

The desired  $x$ -coordinates are:  $-3$ ,  $0$ , and  $3$ .

- Why does the graph of  $y = (x^2 - 9)^7$  below make sense?
  - Observe that  $f$  is an **even** function.
  - $f(x) = (x^2 - 9)^7 = (x + 3)^7 (x - 3)^7$ , which means that  $-3$  and  $3$  are zeros of  $f$  of multiplicity 7 (see Chapter 2 of the Precalculus notes). As a result, the graph has  **$x$ -intercepts** at  $(-3, 0)$  and  $(3, 0)$ , and the higher multiplicity indicates greater **flatness** around those points. Because the multiplicities are odd, the graph “**cuts through**” the  **$x$ -axis** at the  $x$ -intercepts, instead of “bouncing off” of the  $x$ -axis there.
  - The  **$y$ -intercept** is extremely low, because  $f(0) = (-9)^7 = -4,782,969$ .
- The **red tangent lines** below are truncated.



(Axes are scaled differently.)

Example 13 (Finding Horizontal Tangent Lines to a Trigonometric Graph)

Let  $f(x) = x + \cos(2x)$ . Find the  $x$ -coordinates of all points on the graph of  $y = f(x)$  where the **tangent line** is **horizontal**.

§ Solution

- We must find where the **slope** of the tangent line to the graph is 0.  
We must solve the equation:

$$\begin{aligned}
 f'(x) &= 0 \\
 D_x [x + \cos(2x)] &= 0 \\
 1 + [-\sin(2x)] \cdot [D_x(2x)] &= 0 \quad (\text{by Generalized Trigonometric Rules}) \\
 1 + [-\sin(2x)] \cdot [2] &= 0 \\
 1 - 2\sin(2x) &= 0 \quad (\text{See Tip 3 for a short cut.}) \\
 -2\sin(2x) &= -1 \\
 \sin(\underbrace{2x}_{\theta}) &= \frac{1}{2}
 \end{aligned}$$

Use the **substitution**  $\theta = 2x$ .

$$\sin \theta = \frac{1}{2}$$

Our solutions for  $\theta$  are:

$$\theta = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad \theta = \frac{5\pi}{6} + 2\pi n \quad (n \in \mathbb{Z})$$

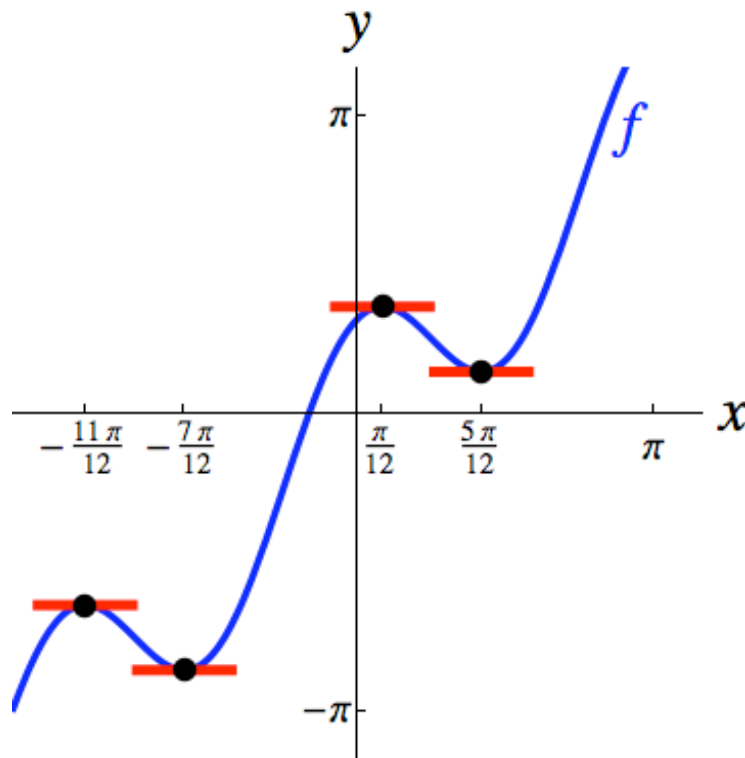
To find our solutions for  $x$ , replace  $\theta$  with  $2x$ , and solve for  $x$ .

$$\begin{aligned}
 2x &= \frac{\pi}{6} + 2\pi n & \text{or} & & 2x &= \frac{5\pi}{6} + 2\pi n & (n \in \mathbb{Z}) \\
 x &= \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \pi n & \text{or} & & x &= \left(\frac{1}{2}\right)\left(\frac{5\pi}{6}\right) + \pi n & (n \in \mathbb{Z}) \\
 x &= \frac{\pi}{12} + \pi n & \text{or} & & x &= \frac{5\pi}{12} + \pi n & (n \in \mathbb{Z})
 \end{aligned}$$

The desired  $x$ -coordinates are given by:

$$\left\{ x \in \mathbb{R} \left| x = \frac{\pi}{12} + \pi n, \text{ or } x = \frac{5\pi}{12} + \pi n, (n \in \mathbb{Z}) \right. \right\}.$$

- Observe that there are **infinitely many** points on the graph where the tangent line is horizontal.
- Why does the graph of  $y = x + \cos(2x)$  below make sense? The “ $x$ ” term leads to upward drift; the graph oscillates about the line  $y = x$ .
- The **red tangent lines** below are truncated.





**FOOTNOTES**

1. **Partial proof of the Chain Rule.** Assume that  $g$  is differentiable at  $a$ , and  $f$  is differentiable at  $g(a)$ . Let  $b = g(a)$ . More generally, let  $u = g(x)$ . As an optional step, we can let  $p = f \circ g$ . Then,  $p(x) = (f \circ g)(x) = f(g(x))$ . We will show that  $p$ , or  $f \circ g$ , is differentiable at  $a$ , with  $p'(a) = (f \circ g)'(a) = [f'(b)][g'(a)]$ .

$$\begin{aligned}
 p'(a) &= \lim_{x \rightarrow a} \frac{p(x) - p(a)}{x - a} \\
 &= \lim_{x \rightarrow a} \left[ \frac{p(x) - p(a)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right], \quad [g(x) \neq g(a)] \quad (\text{See Note 1 below.}) \\
 &= \lim_{x \rightarrow a} \left[ \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right] \\
 &= \left[ \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \right] \cdot \left[ \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \right] \\
 &= \left[ \lim_{u \rightarrow b} \frac{f(u) - f(b)}{u - b} \right] \cdot \left[ \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \right] \quad (\text{See Note 2 below.}) \\
 &= [f'(b)] \cdot [g'(a)]
 \end{aligned}$$

**Note 1:** We have a problem if  $g(x) = g(a)$  “near”  $x = a$ ; that is, the partial proof fails if  $g(x) = g(a)$  somewhere on every “punctured” open interval about  $x = a$ . The function in Footnote 4 exhibits this problem, where  $a = 0$ . Larson gives a more general proof in Appendix A of his calculus text (9<sup>th</sup> ed., p.A8). It is not for the faint of heart!

**Note 2:** We assume that  $g$  is differentiable (and thus **continuous**) at  $a$ . Therefore, as  $x \rightarrow a$ , then  $u \rightarrow b$ , since  $\lim_{x \rightarrow a} u = \lim_{x \rightarrow a} g(x) = g(a) = b$ .

2. **A controversial form of the Chain Rule.** Some sources give the Chain Rule as:

$$(f \circ g)'(x) = [f'(g(x))][g'(x)].$$

However, some object to the use of the notation  $f'(g(x))$ .

3. **Radians.** The proofs in Section 3.4 showing that  $D_x(\sin x) = \cos x$  and  $D_x(\cos x) = -\sin x$

utilized the limit statement  $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$ , which was proven in Footnote 1 of Section 3.4

under the assumption that  $h$  was measured in radians (or as “pure” real numbers).

• Define the “sind” and “cosd” functions as follows:

$\text{sind}(x)$  = the sine of  $x$  degrees, and  $\text{cosd}(x)$  = the cosine of  $x$  degrees.

$$\text{Now, } x \text{ degrees} = \left( x \cancel{\text{ degrees}} \right) \left( \frac{\pi \text{ [radians]}}{180 \cancel{\text{ degrees}}} \right) = \frac{\pi}{180} x \text{ [radians]}.$$

$$\text{Therefore, } \text{sind}(x) = \sin\left(\frac{\pi}{180} x\right), \text{ and } \text{cosd}(x) = \cos\left(\frac{\pi}{180} x\right).$$

- Unfortunately,  $D_x[\text{ Sind}(x)]$  is **not** simply  $\text{cosd}(x)$ , as demonstrated below:

$$\begin{aligned}
 D_x[\text{ Sind}(x)] &= D_x\left[\sin\left(\frac{\pi}{180}x\right)\right] \\
 &= \left[\cos\left(\frac{\pi}{180}x\right)\right] \cdot \left[D_x\left(\frac{\pi}{180}x\right)\right] \quad (\text{by Generalized Trigonometric Rules}) \\
 &= \left[\cos\left(\frac{\pi}{180}x\right)\right] \cdot \left[\frac{\pi}{180}\right] \\
 &= \frac{\pi}{180} \cos\left(\frac{\pi}{180}x\right) \\
 &= \frac{\pi}{180} \text{cosd}(x)
 \end{aligned}$$

Therefore, we prefer the use of our original sine and cosine functions, together with radian measure.

- See *The Math Forum @ Drexel* on the web: <http://mathforum.org/>, particularly <http://mathforum.org/library/drmath/view/53779.html> with Dr. Peterson.

4. **Applicability of the Chain Rule and short cuts.** In Section 3.2, Footnote 7, we defined a

piecewise-defined function  $f$  as follows:  $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ . It turns out that

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

The Product, Power, and Generalized Trigonometric

Rules give us the top rule for  $f'(x)$  when  $x \neq 0$ . However, these rules do **not** apply when

$x = 0$ , since it is not true that  $f(x) = x^2 \sin\left(\frac{1}{x}\right)$  when  $x = 0$ ; in fact, we would have had a

problem using these methods at  $x = 0$  if there were no open interval containing  $x = 0$  throughout which the rule applied. Nevertheless,  $f'(0)$  does exist! In Section 3.2,

Footnote 7, we showed that  $f'(0) = 0$  using the Limit Definition of the Derivative.