SECTION 6.2: VOLUMES OF SOLIDS OF REVOLUTION: DISK / WASHER METHODS

LEARNING OBJECTIVES

- Find volumes of solids of revolution using Disk and Washer Methods.
- Prove formulas for volumes of cones, spheres, etc.

PART A: THE DISK METHOD ("*dx* SCAN")

A solid of revolution is obtained by revolving a plane (flat) region, called a generating region, about an axis of revolution.

The Disk and Washer Methods can be used to find the **volume** of such a solid.

Example 1 (Finding a Volume Using the Disk Method: "dx Scan")

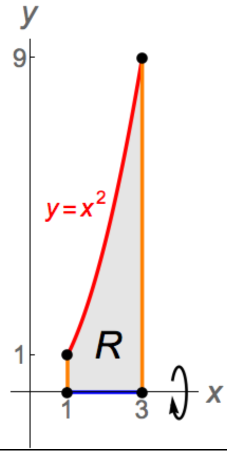
Sketch and shade in the **generating region** R bounded by the x-axis and the graphs of $y = x^2$, $x = 1$, and $x = 3$ in the usual xy-plane. Find the **volume** of the solid generated if R is revolved about the $\boldsymbol{x}$**-axis**. Lengths and distances are measured in **meters**.

§ Solution

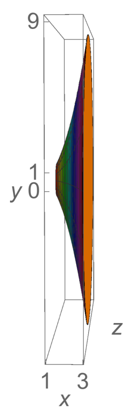
Steps may be reordered or done simultaneously.

Step 1: Sketch and shade in R.

- Indicate the **axis of revolution**. Here, it is the $\boldsymbol{x}$**-axis**.
- Find the **"corners"** of R, which are **intersection points**.
  - •• The point $(3, 9)$ is the intersection point between the graphs of $y = x^2$ and $x = 3$, because $(3, 9)$ is the only **solution of the system** $\begin{cases} y = x^2 \\ x = 3 \end{cases}$. When $x = 3$, $y = x^2 = (3)^2 = 9$.
  - •• Likewise, $(1, 1)$ is the intersection point between the graphs of $y = x^2$ and $x = 1$.

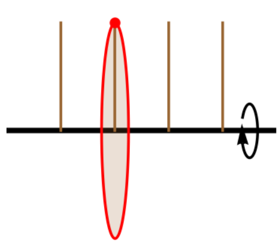
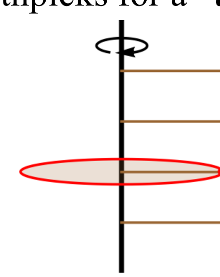
	It helps that the axis of revolution does **not** pass through the interior of *R*. See Example 2.

<u>Step 2</u>: Sketch the solid. (Optional.)

	The solid's <u>lateral surface</u> (which **excludes** the circular "lids" at the left and right ends) resembles the outer surface of a sideways volcano, bullhorn, or lampshade. Fill the inside with gray cement and attach orange lids at the left and right ends. The cement forms the desired solid.

<u>Step 3</u>: Select *dx* or *dy* "scan."

When using the **Disk or Washer Method**, we need to use <u>"toothpicks"</u> that are **perpendicular** to the axis of revolution.

<u>Horizontal Axis</u> (for example, the ***x*-axis**)	<u>Vertical Axis</u> (for example, the ***y*-axis**)
Use toothpicks for a **"*dx* scan."** 	Use toothpicks for a **"*dy* scan."** 

In this example, we have a **horizontal axis**, so use a **"*dx* scan."**

Step 4: Rewrite equations (if necessary).

Consider the **equations** of the boundaries of R that have **both** x and y in them.

- For a **"*dx* scan,"** solve them for y **in terms of *x***.
- For a **"*dy* scan,"** solve them for x **in terms of *y***.

In this example, we are doing a **"*dx* scan,"** so the equation $y = x^2$ is fine as-is. It is already solved for y **in terms of *x***.

Step 5: Find the area of [one face of] a cross section.

- Fix a **representative, generic** x-value in $(1, 3)$.

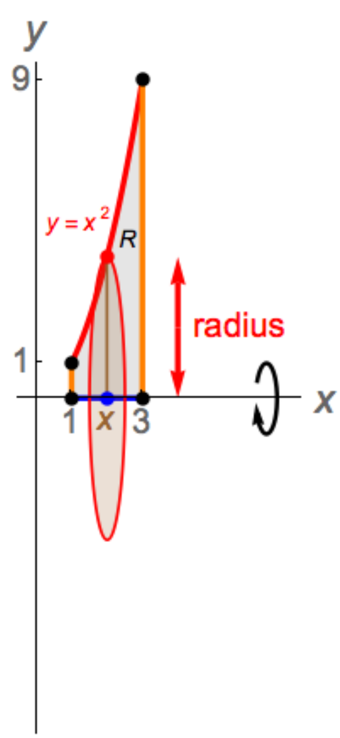
(We could have said $[1, 3]$ in this example.)

- Draw a **"toothpick"** across R at $x =$ (that x-value).

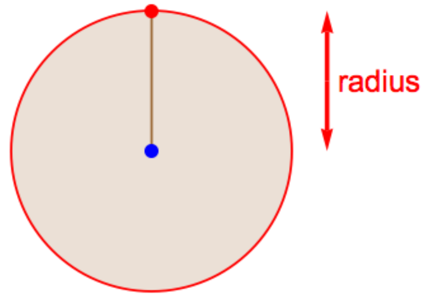
  - • The "toothpick" is actually a **thin rectangle**; we will discuss this when we discuss theory later.

- When we revolve the "toothpick" about the x-axis, we obtain a "thin disk." Think of a coin.

  - • Actually, we are revolving a **thin rectangle** and obtaining a disk with some thickness Δx.



Lie our "thin disk" (coin) down flat.



• Find the **radius** of [one face of] our "thin disk."

•• Look at the **red** and **blue** endpoints of our brown "toothpick" in the two previous figures. The radius is given by the ***y***-coordinate of the **top (red)** point **minus** the ***y***-coordinate of the **bottom (blue)** point.

•• **TIP 1: Radius as a length.** In this chapter, all lengths (including radii) will be determined by …

… a difference in ***y***-coordinates (Think: **"top – bottom"**), or
… a difference in ***x***-coordinates (Think: **"right – left"**).

A length cannot be negative.

$$\text{radius, } r(x) = y_{top} - y_{bottom} \quad \left[\text{or: } y_{top}(x) - y_{bottom}(x)\right]$$
$$= (x^2) - (0)$$
$$= x^2$$

•• **TIP 2: Grouping symbols can help.** Use grouping symbols if y_{bottom} has more than one term.

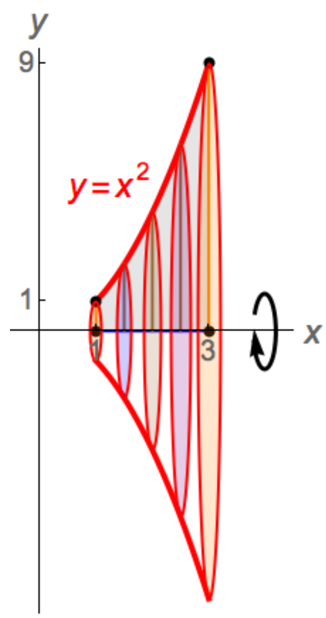
•• **WARNING 1: Don't just use *r*.** We will **later** use r to represent a **constant**. To avoid confusion, use "radius" or something like $r(x)$ to represent a radius that potentially **changes** for different values of x (related to our **"*dx* scan"**). There is less confusion with y_{top} and y_{bottom}.

• Find $A(x)$, the **area** of [one face of] our "thin disk."

$$A(x) = \pi(\text{radius})^2 = \pi(x^2)^2 = \pi x^4$$

Step 6: **Set up the integral(s)** for the volume of the solid.

• We perform a "dx scan" from $x = 1$ to $x = 3$, our **limits of integration**. Different "toothpicks" corresponding to different x-values in $[1, 3]$ generate different "thin disks" with different areas. Some sample "thin disks":



(Axes are scaled differently)

•• Look at the **"corner" points** $(1,1)$ and $(3,9)$. The fact that their $\boldsymbol{y}$-coordinates are 1 and 9 helps us with our **sketch**, but it **doesn't help us later** in the problem. **A "*dy* scan" would have had more use for those $\boldsymbol{y}$-coordinates.**

• **Integrate** the cross-sectional areas with respect to x **("*dx* scan")**.

$$\text{Volume, } V = \int_1^3 A(x)\,dx = \int_1^3 \pi x^4\,dx$$

Step 7: **Evaluate the integral(s)** to find the volume of the solid.

$$\text{Volume, } V = \pi\left[\frac{x^5}{5}\right]_1^3 = \pi\left(\left[\frac{(3)^5}{5}\right]-\left[\frac{(1)^5}{5}\right]\right) = \frac{242\pi}{5}\ \text{m}^3$$

• **WARNING 2: Units of volume.** Remember that, if distances and lengths are measured in meters, then your unit of volume will be **cubic meters**, or m^3. §

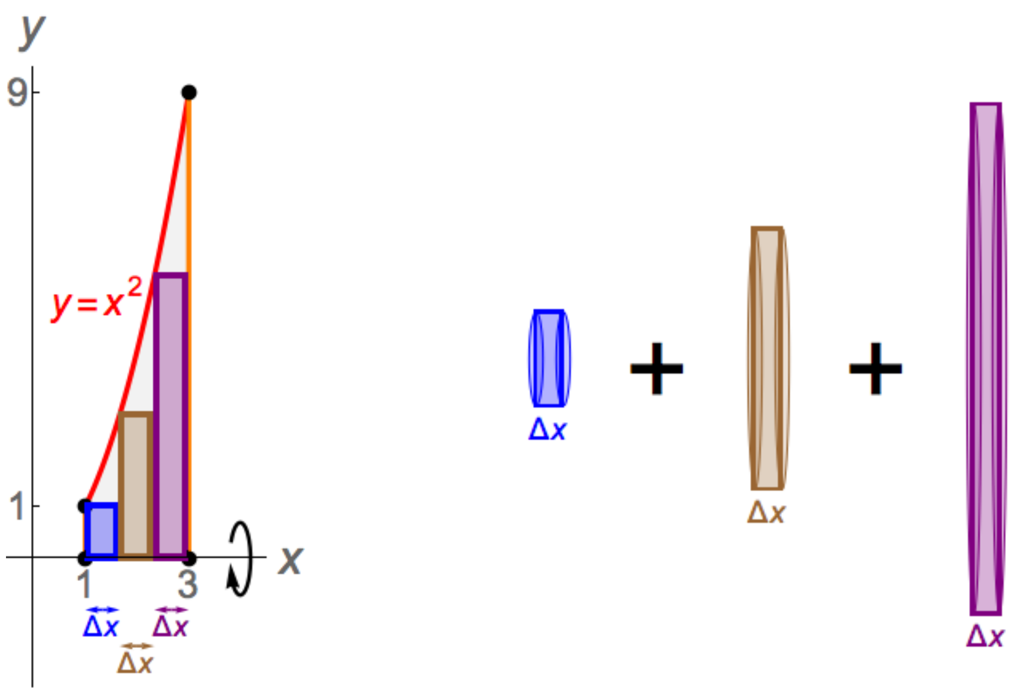
See Footnote 1 for a generalization of Example 1 … and what would happen if R were **below** the x-axis.

PART B: THE DISK METHOD (THEORY)

Let's approximate the generating region R from Example 1 by three **rectangles** of **width** Δx.

When a rectangle is revolved about the x-axis, we obtain a **disk** of **thickness** Δx. It resembles a hockey puck.

We approximate the volume of the solid from Example 1 by the **sum of the volumes** of the three disks.



$$\text{Volume, } V \approx \sum_{i=1}^{n} \left(\text{Volume of the } i^{\text{th}} \text{ disk}\right) \quad \left(n = 3 \text{ for the above}\right)$$

$$\approx \underbrace{\underbrace{\sum_{i=1}^{n} \left(\text{Area of a circular base of the } i^{\text{th}} \text{ disk}\right)}_{\text{in square meters } \left(\text{m}^2\right)} \underbrace{\left(\text{Width of the } i^{\text{th}} \text{ disk}\right)}_{\text{in meters } (\text{m})}}_{\text{in cubic meters } \left(\text{m}^3\right)}$$

$$\approx \sum_{i=1}^{n} A\left(x_{i-1}\right) \Delta x \quad \left(\sum_{i=1}^{n} \pi \left(x_{i-1}\right)^4 \Delta x \ \text{ for the above}\right)$$

This is a "Left-Hand" scheme, as in the figures above.

We would use $\sum_{i=1}^{n} A\left(x_i\right) \Delta x$ for a "Right-Hand" scheme.

To get the **exact** volume, let $\Delta x \to 0$.
For regular partitions, this implies that $\|P\| \to 0$.

$$\text{Volume, } V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} A(x_{i-1})\Delta x \qquad \left(\lim_{\|P\|\to 0} \sum_{i=1}^{n} \pi (x_{i-1})^4 \Delta x \ \text{ for Example 1} \right)$$

$$= \int_a^b A(x)\, dx \qquad \left(\int_1^3 \pi x^4\, dx \ \text{ for Example 1} \right)$$

assuming A is a continuous function on $[a, b]$.

Δx is replaced by dx in the integral.

PART C: THE DISK METHOD ("*dy* SCAN")

Example 2 (Finding a Volume Using the Disk Method: "dy Scan")

Sketch and shade in the **generating region** R bounded by the graphs of $y = x^2$, $y = 1$, and $y = 9$ in the usual *xy*-plane. Find the volume of the solid generated if R is revolved about the ***y*-axis**. Lengths and distances are measured in **meters**.

§ Solution

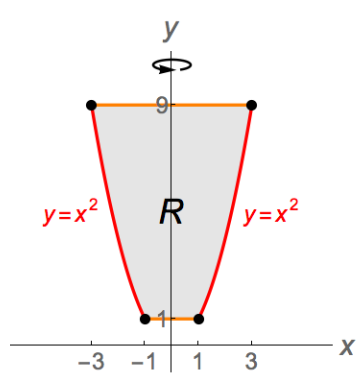
Steps may be reordered or done simultaneously.

Step 1: Sketch and shade in (and then modify) R.

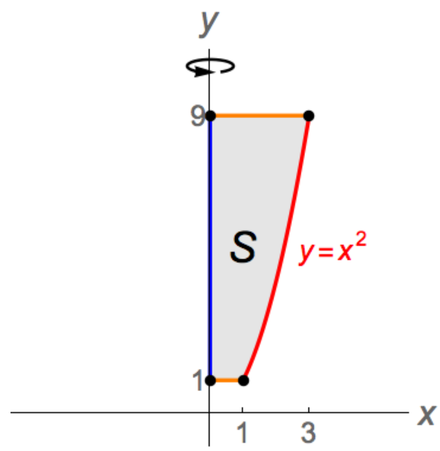
- Indicate the **axis of revolution**. Here, it is the ***y*-axis**.
- Find the **"corners"** of R, which are **intersection points**.

The **solutions of the system** $\begin{cases} y = x^2 \\ y = 1 \end{cases}$ are $(1, 1)$ and $(-1, 1)$.

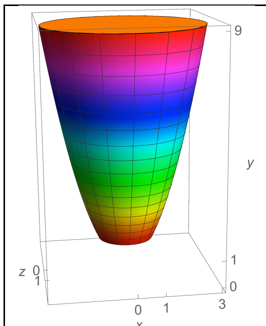
The **solutions of the system** $\begin{cases} y = x^2 \\ y = 9 \end{cases}$ are $(3, 9)$ and $(-3, 9)$.



• The **axis of revolution** passes through the interior of R, so we will need a **new generating region** we will call S. Fortunately, the symmetry of R about the axis allows us to simply take the **right half** of R as our S; we could have taken the **left half**, instead.
Think About It: What would you do in the absence of such symmetry?



Step 2: Sketch the solid. (Optional.)

	The lateral surface of the solid resembles that of a curved cup. Fill the cup with gray cement and attach orange lids at the top and the bottom. The cement forms the desired solid.

Step 3: Select dx or dy "scan."

When using the **Disk or Washer Method**, we need to use "toothpicks" that are **perpendicular** to the axis of revolution.

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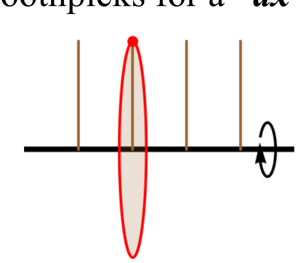
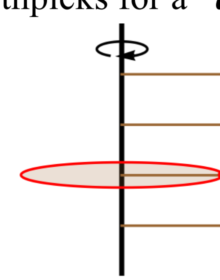
Use toothpicks for a **"*dx* scan."** 	Use toothpicks for a **"*dy* scan."** 

In this example, we have a **vertical axis**, so use a **"*dy* scan."**

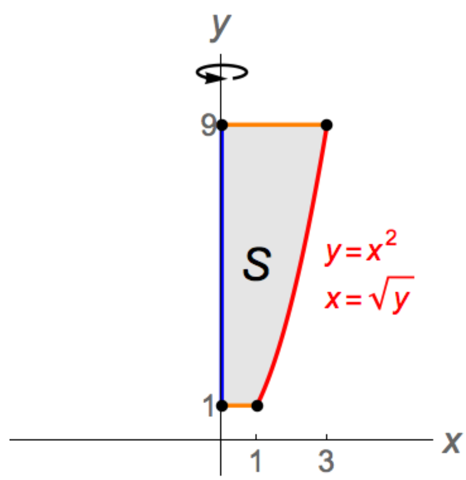
Step 4: Rewrite equations (if necessary).

We are doing a **"*dy* scan,"** so we must **solve** the equation $y = x^2$ for x **in terms of *y***.

$$y = x^2$$
$$x^2 = y$$
$$x = \pm\sqrt{y}$$

Along the red graph in the figure below, $x \geq 0$, so we will only use the nonnegative root:

$$x = \sqrt{y}$$



Step 5: Find the area of [one face of] a cross section.

• Fix a **representative, generic** y-value in $(1, 9)$.

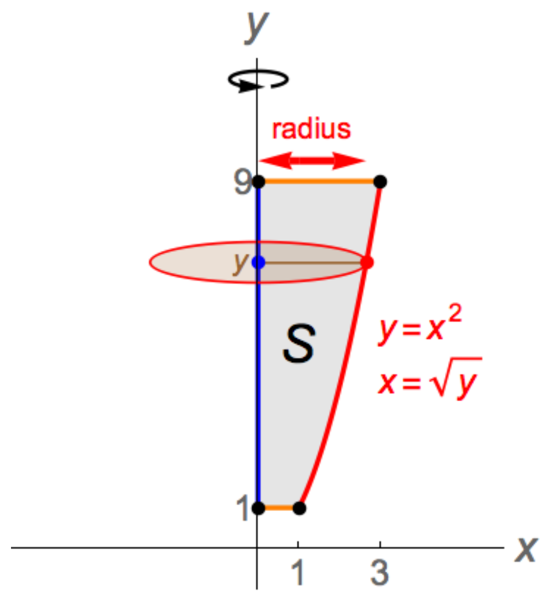
(We could have said $[1, 9]$ in this example.)

• Draw a **"toothpick"** across S at $y =$ (that y-value).

•• The "toothpick" is actually a **thin rectangle**.

• When we revolve the "toothpick" about the y-axis, we obtain a **"thin disk."**

•• Actually, we are revolving a **thin rectangle** and obtaining a **disk** with some thickness Δy.



• Find the **radius** of [one face of] our "thin disk."

•• Look at the **red** and **blue** endpoints of our brown "toothpick." The **radius** is given by the ***x***-coordinate of the **right (red)** point **minus** the ***x***-coordinate of the **left (blue)** point.

•• **See TIP 1:** Radii will be determined by a difference in ***x***-coordinates (Think: **"right – left"**).

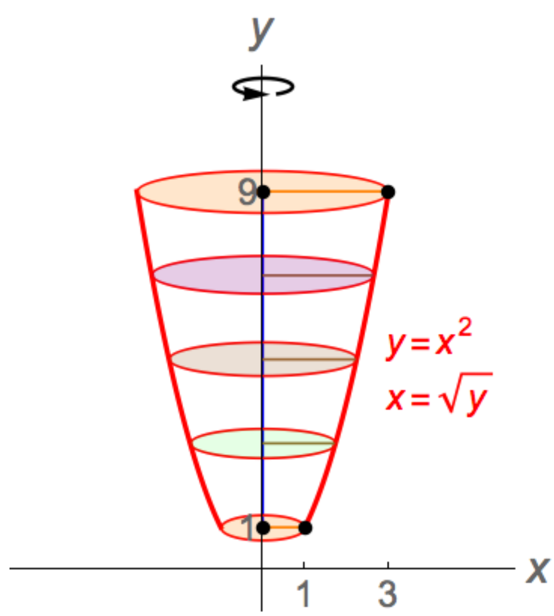
$$\text{radius, } r(y) = \boldsymbol{x}_{right} - \boldsymbol{x}_{left} \quad \left[\text{or } \boldsymbol{x}_{right}(\boldsymbol{y}) - \boldsymbol{x}_{left}(\boldsymbol{y})\right]$$
$$= \left(\sqrt{\boldsymbol{y}}\right) - (\mathbf{0})$$
$$= \sqrt{y}$$

• Find $A(x)$, the **area** of [one face of] our "thin disk."

$$A(x) = \pi(\text{radius})^2 = \pi\left(\sqrt{y}\right)^2 = \pi y$$

Step 6: **Set up the integral(s)** for the volume of the solid.

• We perform a "dy scan" from $y = 1$ to $y = 9$. Different "toothpicks" corresponding to different y-values in $(1, 9)$ generate different "thin disks" with different areas. Some sample "thin disks":



• **Integrate** the cross-sectional areas with respect to y **("*dy* scan")**.

$$\text{Volume, } V = \int_1^9 A(y)\,dy = \int_1^9 \pi y\,dy$$

Step 7: **Evaluate the integral(s)** to find the volume of the solid.

$$\text{Volume, } V = \pi\left[\frac{y^2}{2}\right]_1^9 = \pi\left(\left[\frac{(9)^2}{2}\right]-\left[\frac{(1)^2}{2}\right]\right) = 40\pi \text{ m}^3$$

§

See Footnote 2 for a generalization of Example 2.