PART D: THE WASHER METHOD ("*dx* SCAN")

Example 3 (Finding a Volume Using the Washer Method: "dx Scan")

Sketch and shade in the **generating region** R bounded by the graphs of $x + y^3 = 0$ and $x^2 - y = 0$ in the usual xy-plane. Find the **volume** of the solid generated if R is revolved about the ***x*-axis**. Lengths and distances are measured in **meters**.

§ Solution

Steps may be reordered or done simultaneously. It would help to **solve** the given equations **for *x* or *y***, but for which? Let's do Steps 3 and 4 first.

Step 3: Select dx or dy "scan."

When using the **Disk or Washer Method**, we need to use **"toothpicks"** that are **perpendicular** to the axis of revolution.

(See Part E for more on this.)

In this example, we have a **horizontal axis**, so use a **"*dx* scan."**

Step 4: Rewrite equations (if necessary).

For a **"*dx* scan,"** we solve the given equations for y **in terms of *x***.

$x + y^3 = 0$ $y^3 = -x$ $y = \sqrt[3]{-x}$ $y = -\sqrt[3]{x}$ The last step is justified, because the cube root function is odd.	$x^2 - y = 0$ $y = x^2$

<u>(Back to) Step 1</u>: Sketch and shade in R.

• Indicate the **axis of revolution**. Here, it is the ***x*-axis**.

• Find the **"corners"** of R, which are **intersection points**.
Solve the system:

$$\begin{cases} x + y^3 = 0 \\ x^2 - y = 0 \end{cases} \Leftrightarrow \begin{cases} y = -\sqrt[3]{x} \\ y = x^2 \end{cases} \quad \text{(Step 4 helps here.)} \quad \Rightarrow$$

$-\sqrt[3]{x} = x^2$ $\left(-\sqrt[3]{x}\right)^3 = \left(x^2\right)^3$ $-x = x^6$ $0 = x^6 + x$ $x^6 + x = 0$ $x\left(x^5 + 1\right) = 0$	$x = 0$ **or** $x^5 + 1 = 0$ $x^5 = -1$ $x = \sqrt[5]{-1}$ $x = -1$

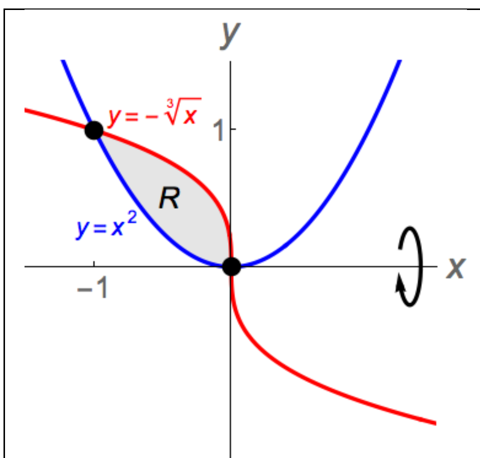
<u>Note</u>: -1 has five complex fifth roots, but only one of them is **real**, namely the **principal** fifth root, -1.

•• Find the corresponding ***y***-coordinates. We will use $y = x^2$, though we could have used $y = -\sqrt[3]{x}$.

$x = 0 \quad \Rightarrow \quad y = x^2 = (0)^2 = 0 \quad \Rightarrow \quad (0, 0)$ is an **intersection point**.

$x = -1 \quad \Rightarrow \quad y = x^2 = (-1)^2 = 1 \quad \Rightarrow \quad (-1, 1)$ is an **intersection point**.

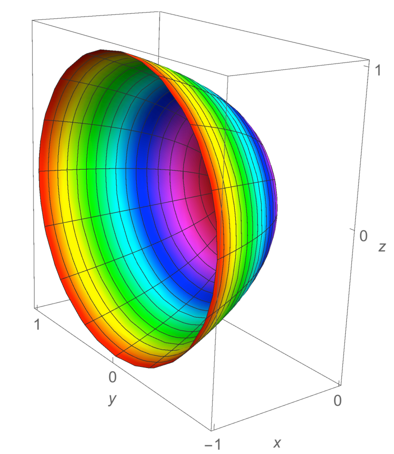
•• Technically, these solutions should be **checked** in the original system, but our **sketch** will help verify these.



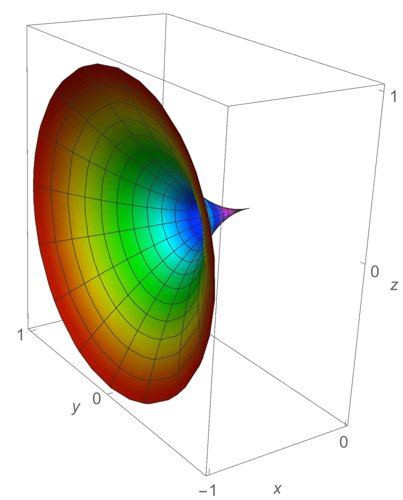
It helps that the axis of revolution does **not** pass through the interior of R.

Step 2: Sketch the solid. (Optional.)

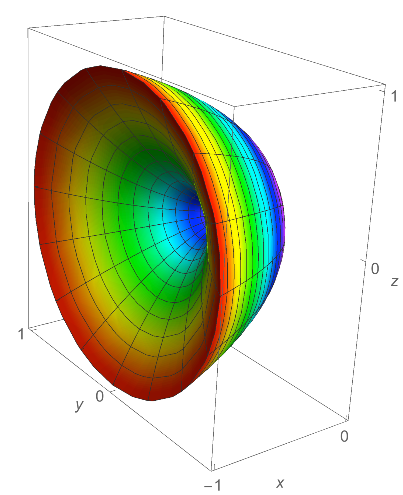
See the previous figure. Imagine gray cement trapped between …



… a sideways bowl (from the **red** boundary of R, the **outer** boundary relative to the axis of revolution) and …



a sideways funnel (from the **blue** boundary of R, the **inner** boundary relative to the axis of revolution).



← Put together. The gray cement forms our desired solid.

The variety of colors will help us visualize "dx scanning."

<u>(Ahead to) Step 5</u>: Find the area of [one face of] a cross section.

- Fix a **representative, generic** x-value in $\left(-1, 0\right)$.

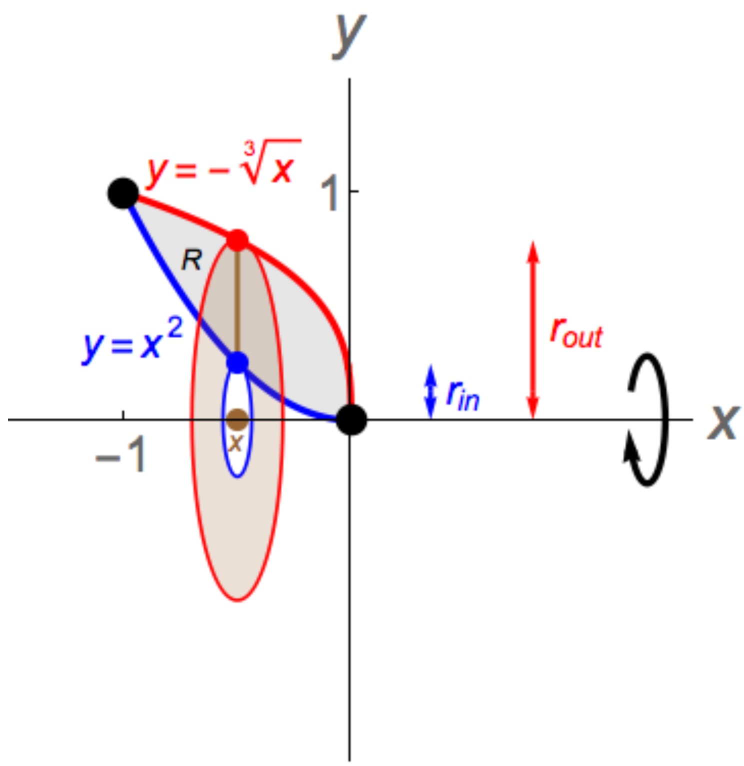
Avoid $x=-1$ and $x=0$; they are **not** representative.

- Draw a **"toothpick"** across R at $x =$ (that x-value).

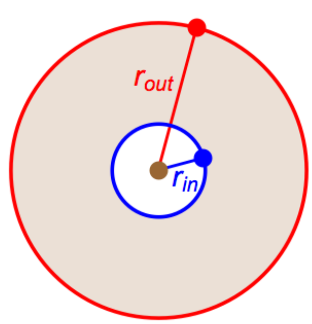
    •• The "toothpick" is actually a **thin rectangle**.

- When we revolve the "toothpick" about the x-axis, we obtain a "<u>thin washer</u>." Think of a DVD.

    •• Actually, we are revolving a **thin rectangle** and obtaining a <u>washer</u> with some thickness Δx .



Lie our "thin washer" (DVD) down flat.
For clarity, let's rotate the **red** and **blue** points.

	r_{out} is the <u>outer radius</u>, the radius of the larger, **outer, red** circle. r_{in} is the <u>inner radius</u>, the radius of the smaller, **inner, blue** circle.

- Find r_{out} and r_{in} for [one face of] our "thin washer."

  •• Look at the **red** and **blue** endpoints of our brown "toothpick" in the middle of the previous page. Also look at the **brown** point on the axis of revolution (the *x*-axis). **These three points lie on a straight line**; the **brown** point is like a star and the other two points are like planets moving in circular (not elliptical – sorry, Kepler) orbits about the star. r_{out} and r_{in} are obtained by taking the **differences of their *y*-coordinates** … but which ones and how?

  •• For both r_{out} and r_{in}, we think: **"top – bottom,"** although the **"top" point is different** for the r_{out} and r_{in} calculations.

  •• r_{out} is given by the ***y***-coordinate of the **red ("top")** point **minus** the ***y***-coordinate of the **brown ("bottom")** point.

$$\begin{aligned} r_{out} &= \boldsymbol{y}_{top} - \boldsymbol{y}_{bottom} \qquad \left[\text{or: } r_{out}(x) = \boldsymbol{y}_{top}(\boldsymbol{x}) - \boldsymbol{y}_{bottom}(\boldsymbol{x})\right] \\ &= \left(-\sqrt[3]{\boldsymbol{x}}\right) - (\boldsymbol{0}) \\ &= -\sqrt[3]{x} \end{aligned}$$

  ••• **Don't be surprised by the opposite sign.** Since we only consider *x*-values in $[-1, 0]$, the values of $-\sqrt[3]{x}$ are actually **nonnegative** as we perform our "*dx* scan."

Note 1: We use the **red** point for r_{out} because the **red** point is on the **outer** boundary of *R* **relative to the axis of revolution** (the **red** graph is **further away** from the *x*-axis than the blue graph is).

Note 2: The **brown** point lies on the **axis of revolution** and is used for both r_{out} and r_{in}.

Note 3: Here, the **red ("outer")** point is a **"top"** point, and the **brown** point on the **axis of revolution** is a **"bottom"** point. In Example 4, their roles will **switch**.

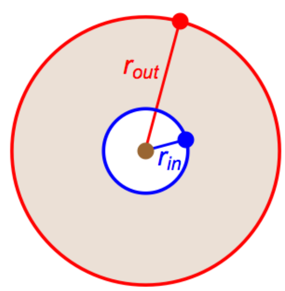
•• r_{in} is given by the **y**-coordinate of the **blue ("top")** point **minus** the **y**-coordinate of the **brown ("bottom")** point.

$$r_{in} = y_{top} - y_{bottom} \quad \left[\text{or: } r_{in}(x) = y_{top}(x) - y_{bottom}(x)\right]$$
$$= \left(x^2\right) - (0)$$
$$= x^2$$

Note 1: We use the **blue** point for r_{in} because the **blue** point is on the **inner** boundary of R **relative to the axis of revolution** (the **blue** graph is **closer to** the x-axis than the red graph is).

Note 2: Here, the **blue ("inner")** point is a **"top"** point, and the **brown** point on the **axis of revolution** is a **"bottom"** point. In Example 4, their roles will **switch**.

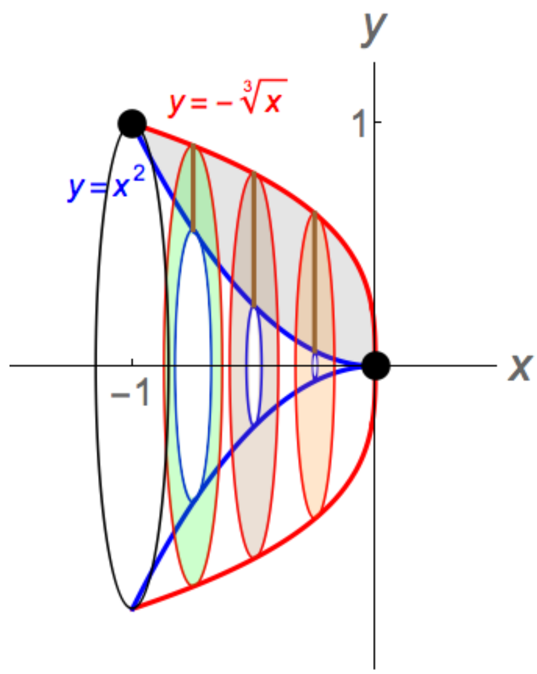
• Find $A(x)$, the **area** of [one face of] our "thin washer."



$$A(x) = (\text{Area within large red circle}) - (\text{Area within small blue circle})$$
$$\text{or } (\text{Big area}) - (\text{Small area})$$
$$\text{or } (\text{Area of "whole"}) - (\text{Area of "hole"})$$
$$= \pi r_{out}^{\ 2} - \pi r_{in}^{\ 2} \qquad \left[\text{or: } \pi r_{out}^{\ 2}(x) - \pi r_{in}^{\ 2}(x)\right]$$
$$= \pi\left(-\sqrt[3]{x}\right)^2 - \pi\left(x^2\right)^2$$

Step 6: **Set up the integral(s)** for the volume of the solid.

• We perform a "*dx* scan" from $x = -1$ to $x = 0$. Different "toothpicks" corresponding to different *x*-values in $(-1, 0)$ generate different "thin washers" with different areas. Some sample "thin washers":



• **Integrate** the cross-sectional areas with respect to *x* **("*dx* scan")**.

•• Instead of being given the **limits of integration**, we obtained them from the ***x***-coordinates of the **intersection points** we found in Step 1. We use the ***x***-coordinates because we are doing a **"*dx* scan."**

$$\text{Volume, } V = \int_{-1}^{0} A(x)\,dx$$
$$= \int_{-1}^{0} \left[\pi r_{out}^{\,2}(x) - \pi r_{in}^{\,2}(x)\right] dx$$

•• **WARNING 3: Squares.** Although π may be factored out of the integrand and "popped out" of the integral, be careful not to misplace the squares. Books present the generic formula:

$$V = \pi \int_{a}^{b} \left[r_{out}^{\,2}(x) - r_{in}^{\,2}(x)\right] dx$$

Many students **miswrite** this as:

$$V = \pi \int_{a}^{b} \left[r_{out}(x) - r_{in}(x)\right]^2 dx \quad \textbf{(NO!)}$$

$$\text{Volume, } V = \int_{-1}^{0} \left[\pi\left(-\sqrt[3]{x}\right)^2 - \pi\left(x^2\right)^2 \right] dx$$

•• **Setup.** If you are asked to simply **"set up" the integral(s)** for the desired volume, then the above may be sufficient. Ask your instructor; you may need to simplify further.

Step 7: **Evaluate the integral(s)** to find the volume of the solid.

$$\begin{aligned}
\text{Volume, } V &= \int_{-1}^{0} \pi \left[\left(-\sqrt[3]{x}\right)^2 - \left(x^2\right)^2 \right] dx \\
&= \pi \int_{-1}^{0} \left(x^{2/3} - x^4 \right) dx \\
&= \pi \left[\frac{x^{5/3}}{5/3} - \frac{x^5}{5} \right]_{-1}^{0} \\
&= \pi \left[\frac{3x^{5/3}}{5} - \frac{x^5}{5} \right]_{-1}^{0} \\
&= \pi \left(\left[\frac{3(0)^{5/3}}{5} - \frac{(0)^5}{5} \right] - \left[\frac{3(-1)^{5/3}}{5} - \frac{(-1)^5}{5} \right] \right) \\
&= \pi \left([0] - \left[-\frac{3}{5} + \frac{1}{5} \right] \right) \\
&= \frac{2\pi}{5} \text{ m}^3
\end{aligned}$$

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See Footnote 3 for a generalization of Example 3.

WARNING 4: Don't switch r_{out} and r_{in}. If we had **miswritten** the volume integral as $V = \pi \int_a^b \left[r_{in}^{\,2}(x) - r_{out}^{\,2}(x) \right] dx$ (**NO!**), we would have obtained $-\frac{2\pi}{5}$ m^3, the opposite of the correct answer. **We cannot have negative volumes**, so if you get one, check to see if you made this error.

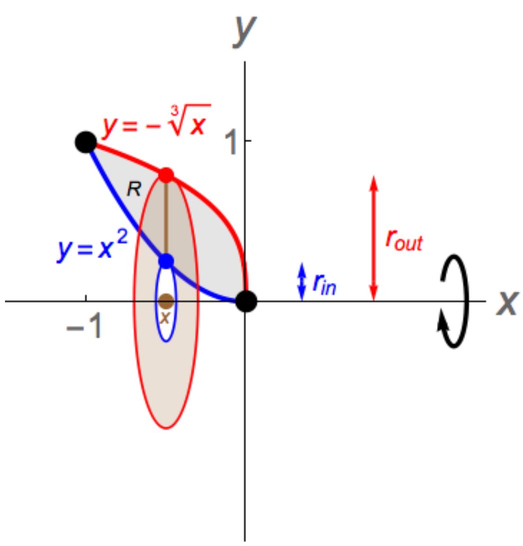
Example 4 (Finding a Volume Using the Washer Method: "dx Scan"; Revisiting Example 3)

Sketch and shade in the **generating region** R bounded by the graphs of $y = \sqrt[3]{x}$ and $y = -x^2$ in the usual xy-plane. Find the **volume** of the solid generated if R is revolved about the $\boldsymbol{x}$**-axis**. Lengths and distances are measured in **meters**.

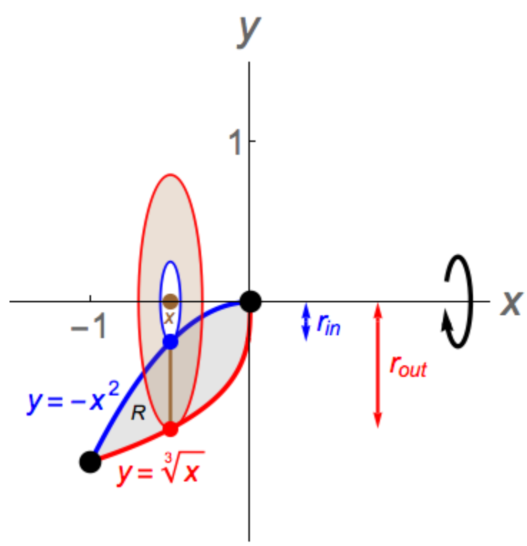
§ Partial Solution

It turns out that R is the **reflection** of the region from Example 3 about the $\boldsymbol{x}$**-axis**. The solid we obtain is the **same** as the one for Example 3.

Example 3 (before)



Example 4 (now)



As in Example 3,

- The **red** graph (this time, of $y = \sqrt[3]{x}$) is the **"outer"** graph relative to the axis of revolution, and
- The **blue** graph (this time, of $y = -x^2$) is the **"inner"** graph relative to the axis of revolution.

Also,

- The **"outer," red** graph generates the same **"sideways bowl"** we saw in Step 2 of Example 3.
- The **"inner," blue** graph generates the same **"sideways funnel"** we saw in Step 2 of Example 3.

Look at the aligned **brown**, **blue**, and **red** points for Example 4 (now). Here are differences from Example 3:

- The **brown** point on the **axis of revolution** will be our **"top"** point for r_{out} and r_{in}, not our "bottom" point (as was the case in Example 3).

- r_{out} is given by the $\boldsymbol{y}$-coordinate of the **brown ("top")** point on the axis **minus** the $\boldsymbol{y}$-coordinate of the **red ("bottom")** point.

$$\begin{aligned} r_{out} &= \boldsymbol{y}_{top} - \boldsymbol{y}_{bottom} \qquad \left[\text{or: } r_{out}(x) = \boldsymbol{y}_{top}(\boldsymbol{x}) - \boldsymbol{y}_{bottom}(\boldsymbol{x})\right] \\ &= (\mathbf{0}) - \left(\sqrt[3]{\boldsymbol{x}}\right) \\ &= -\sqrt[3]{x} \end{aligned}$$

- r_{in} is given by the $\boldsymbol{y}$-coordinate of the **brown ("top")** point on the axis **minus** the $\boldsymbol{y}$-coordinate of the **blue ("bottom")** point.

$$\begin{aligned} r_{in} &= \boldsymbol{y}_{top} - \boldsymbol{y}_{bottom} \qquad \left[\text{or: } r_{in}(x) = \boldsymbol{y}_{top}(\boldsymbol{x}) - \boldsymbol{y}_{bottom}(\boldsymbol{x})\right] \\ &= (\mathbf{0}) - \left(-\boldsymbol{x}^2\right) \\ &= x^2 \end{aligned}$$

In fact, we end up with the **same area formula** as for Example 3. $A(x)$, the **area** of [one face of] our "thin washer," is again given by:

$$\begin{aligned} A(x) &= \pi r_{out}{}^2 - \pi r_{in}{}^2 \qquad \left[\text{or: } \pi r_{out}{}^2(x) - \pi r_{in}{}^2(x)\right] \\ &= \pi\left(-\sqrt[3]{x}\right)^2 - \pi\left(x^2\right)^2 \end{aligned}$$

Squares are forgiving with respect to signs. If we had **mistakenly** said that $r_{out} = \sqrt[3]{x}$ and $r_{in} = -x^2$, then x-values in $[-1, 0)$ would have given us **negative radii**, which are technically **forbidden**. However, because **squares of opposites are equal**,

$$\pi\left(\sqrt[3]{x}\right)^2 - \pi\left(-x^2\right)^2 = \pi\left(-\sqrt[3]{x}\right)^2 - \pi\left(x^2\right)^2 = A(x)$$

and we can still get the correct volume. In this sense, using **"bottom – top"** instead of **"top – bottom"** will still lead to the correct volume, but an instructor may penalize us for "bad form."

See Example 3 for the rest of the solution. §