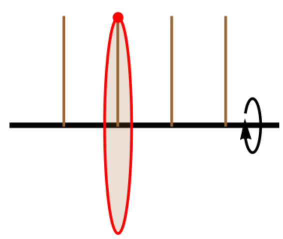
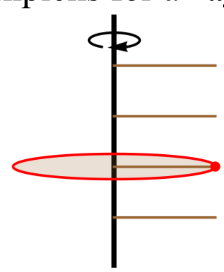
PART E: DISK METHOD vs. WASHER METHOD

When using the **Disk or Washer Method**, we need to use **"toothpicks"** that are **perpendicular** to the axis of revolution.

Which method do we use?

• In a way, we are always using the **Washer Method**, since the **Disk Method** is simply a special case of the **Washer Method** where $r_{in} = 0$.

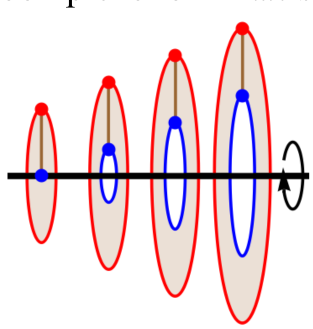
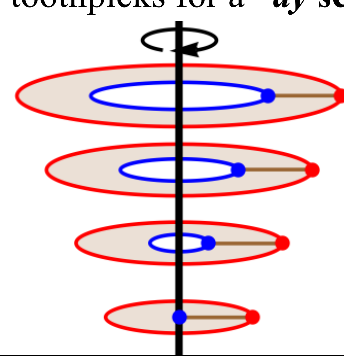
• We use the **DISK METHOD** when **all of our "toothpicks" extend all the way to the axis of revolution**. We at least need that **axis** to form a **boundary of the generating region**. See Examples 1 and 2, where the **generating regions** R and S are **flush against the axes of revolution, without gaps**.

Horizontal Axis (for example, the ***x*-axis**) Use toothpicks for a **"*dx* scan."**	Vertical Axis (for example, the ***y*-axis**) Use toothpicks for a **"*dy* scan."**
	

The **toothpicks may also lie on the other side of the axis**; see the last comment in Footnote 1.

• If that is not the case, then we use the **WASHER METHOD**. See Examples 3 and 4, where there are **gaps** between the **generating regions (*R*)** and the **axes of revolution**. These **gaps** lead to **holes** within our **washers**.

•• If $r_{in} = 0$ sometimes, then we obtain **some disks**; however, the Washer Method will still work.

Horizontal Axis (for example, the ***x*-axis**) Use toothpicks for a **"*dx* scan."**	Vertical Axis (for example, the ***y*-axis**) Use toothpicks for a **"*dy* scan."**
	

PART F: THE WASHER METHOD ("*dy* SCAN"); "WEIRD" AXES OF REVOLUTION

We will informally define a "weird" axis as a horizontal or vertical axis of revolution that is **neither** the x-axis nor the y-axis.

Example 5 (Finding a Volume Using the Washer Method: "dy Scan" and a "Weird" Axis of Revolution)

Sketch and shade in the **generating region** R bounded by the graphs of $y = x^2$, $y = 0$, and $x = 3$ in Quadrant I of the usual xy-plane. Find the **volume** of the solid generated if R is revolved about the line $x = 4$. Lengths and distances are measured in **meters**.

§ Solution

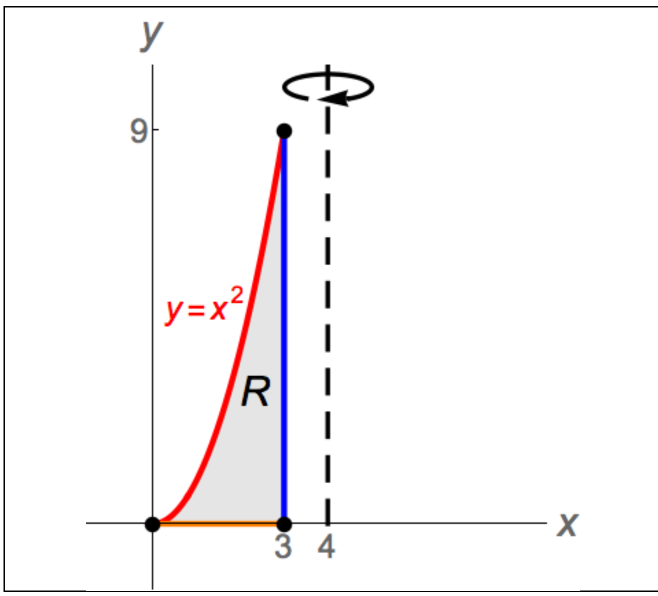
Steps may be reordered or done simultaneously.

Step 1: Sketch and shade in R.

• Indicate the **axis of revolution**. Here, it is the **vertical** line $x = 4$.

• Find the **"corners"** of R, which are **intersection points**.

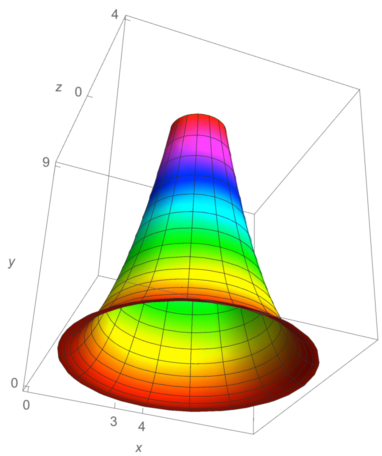
•• The **solution of the system** $\begin{cases} y = x^2 \\ x = 3 \end{cases}$ is $(3, 9)$. It turns out this intersection point's y-coordinate, 9, will help us later.



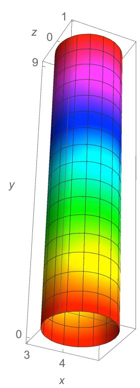
It helps that the axis of revolution does **not** pass through the interior of R.

Step 2: Sketch the solid. (Optional.)

See the previous figure. Imagine gray cement trapped between …

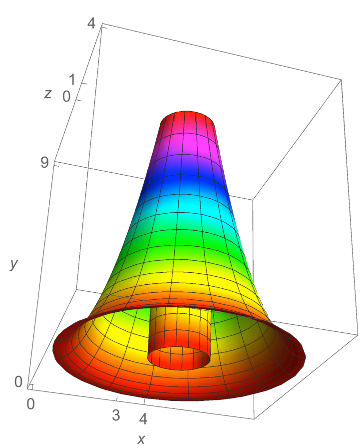


… the outer lateral surface of a volcano, bullhorn, or lampshade (from the **red** boundary of R, the **outer** boundary relative to the axis of revolution) and …

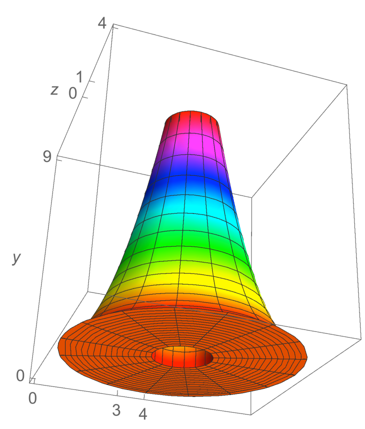


a right circular cylinder (from the **blue** boundary of R, the **inner** boundary relative to the axis of revolution).

The surfaces put together:



Fill the space between the lampshade and the cylinder with gray cement and attach an orange "lid" at the bottom. The cement forms the desired solid [with a gaping cylindrical hole].



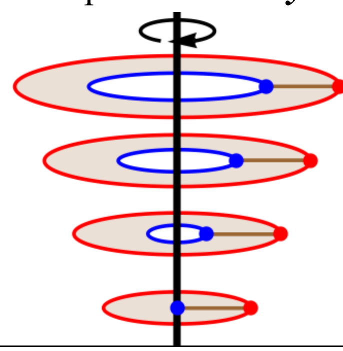
The variety of colors will help us visualize "*dy* scanning."
There is no top "lid"; the top is open.

Step 3: Select dx or dy "scan."

When using the **Disk or Washer Method**, we need to use **"toothpicks"** that are **perpendicular** to the axis of revolution.

Vertical Axis

Use toothpicks for a **"*dy* scan."**



Step 4: Rewrite equations (if necessary).

For a **"*dy* scan,"** we solve given equations for x **in terms of** $\boldsymbol{y}$.

$$y = x^2$$
$$x^2 = y$$
$$x = \pm\sqrt{y}$$

Take $x = \sqrt{y}$ because our only concern is Quadrant I.

• **TIP 3: Picking an equation to graph.** It is easier for students to [partially] graph $y = x^2$ instead of $x = \sqrt{y}$, even though $x = \sqrt{y}$ is the equation we will use afterwards.

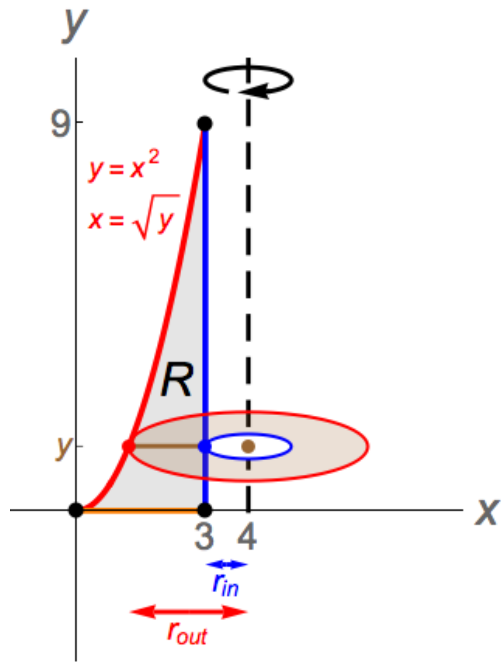
Step 5: Find the area of [one face of] a cross section.

• Fix a **representative, generic** y-value in $(0, 9)$.
Draw a **"toothpick"** across R at $y =$ (that y-value).

•• The "toothpick" is actually a **thin rectangle**.

• When we revolve the "toothpick" about the axis of revolution, we obtain a "**thin washer**."

•• Actually, we are revolving a **thin rectangle** and obtaining a **washer** with some thickness Δy.



• Find r_{out} and r_{in} for [one face of] our "thin washer."

•• Look at the **red** and **blue** endpoints of our brown "toothpick" and the **brown** point on the **axis of revolution**.

r_{out} and r_{in} are obtained by taking the **differences of their** ***x*-coordinates** … but which ones and how?

•• For both r_{out} and r_{in}, we think: **"right – left,"** although the **"left" point is different** for the r_{out} and r_{in} calculations.

•• r_{out} is given by the ***x***-coordinate of the **brown ("right")** point **minus** the ***x***-coordinate of the **red ("outer, left")** point.

$$\begin{aligned} r_{out} &= \boldsymbol{x}_{right} - \boldsymbol{x}_{left} \qquad \left[\text{or: } r_{out}(y) = \boldsymbol{x}_{right}(\boldsymbol{y}) - \boldsymbol{x}_{left}(\boldsymbol{y})\right] \\ &= (\mathbf{4}) - \left(\sqrt{\boldsymbol{y}}\right) \\ &= 4 - \sqrt{y} \end{aligned}$$

<u>Note</u>: **Squares are forgiving with respect to order here.** If we had mistakenly said that $r_{out} = \sqrt{y} - 4$, then y-values in $[0, 9]$ would have given us **negative radii**, which are technically **forbidden**. However, because **squares of opposites are equal**, $r_{out}^{\ 2} = \left(4 - \sqrt{y}\right)^2 = \left(\sqrt{y} - 4\right)^2$ and we can still get the correct volume. In this sense, using **"left – right"** instead of **"right – left"** will still lead to the correct volume, but an instructor may penalize us for "bad form."

•• r_{in} is given by the $\boldsymbol{x}$-coordinate of the **brown ("right")** point **minus** the $\boldsymbol{x}$-coordinate of the **blue ("inner, left")** point.

$$r_{in} = \boldsymbol{x}_{right} - \boldsymbol{x}_{left} \quad \left[\text{or: } r_{in}(y) = \boldsymbol{x}_{right}(\boldsymbol{y}) - \boldsymbol{x}_{left}(\boldsymbol{y})\right]$$
$$= (\mathbf{4}) - (\mathbf{3})$$
$$= 1$$

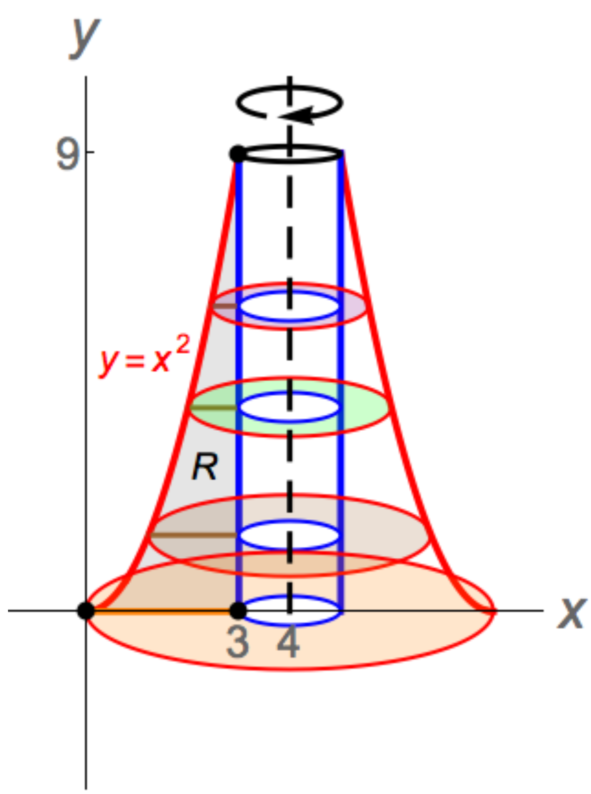
Note: For r_{out} and r_{in} here, the **"outer"** and **"inner"** points are **"left"** points, and the **brown** point on the **axis of revolution** is a **"right"** point. In other problems, their roles will **switch**.

• Find $A(y)$, the **area** of [one face of] our "thin washer."

$$A(y) = \pi r_{out}^{\ 2} - \pi r_{in}^{\ 2} \qquad \left[\text{or: } \pi r_{out}^{\ 2}(y) - \pi r_{in}^{\ 2}(y)\right]$$
$$= \pi\left(4 - \sqrt{y}\right)^2 - \pi(1)^2$$

Step 6: **Set up the integral(s)** for the volume of the solid.

• We perform a "*dy* scan" from $y = 0$ to $y = 9$.
Some sample "thin washers":



• **Integrate** the cross-sectional areas with respect to y **("*dy* scan")**.

•• Instead of being given the **upper limit of integration** (9), we obtained it from the ***y***-coordinate of an **intersection point** we found in Step 1. We use ***y***-coordinates because we are doing a **"*dy* scan."**

$$\begin{aligned}\text{Volume, } V &= \int_0^9 A(y)\,dy\\ &= \int_0^9 \left[\pi r_{out}^{\,2}(y) - \pi r_{in}^{\,2}(y)\right] dy\\ &= \int_0^9 \left[\pi\left(4-\sqrt{y}\right)^2 - \pi(1)^2\right] dy\end{aligned}$$

•• **Setup.** Ask your instructor if you need to simplify further.

Step 7: **Evaluate the integral(s)** to find the volume of the solid.

$$\begin{aligned}\text{Volume, } V &= \int_0^9 \left[\pi\left(4-\sqrt{y}\right)^2 - \pi(1)^2\right] dy\\ &= \pi\int_0^9 \left[\left(4-\sqrt{y}\right)^2 - (1)^2\right] dy\\ &= \pi\int_0^9 \left[\left(16-8\sqrt{y}+y\right) - (1)\right] dy\\ &= \pi\int_0^9 \left(15-8y^{1/2}+y\right) dy\\ &= \pi\left[15y - \frac{8y^{3/2}}{3/2} + \frac{y^2}{2}\right]_0^9\\ &= \pi\left[15y - \frac{16y^{3/2}}{3} + \frac{y^2}{2}\right]_0^9\\ &= \pi\left(\left[15(9) - \frac{16(9)^{3/2}}{3} + \frac{(9)^2}{2}\right] - \left[15(0) - \frac{16(0)^{3/2}}{3} + \frac{(0)^2}{2}\right]\right)\\ &= \pi\left(\left[135-144+\frac{81}{2}\right] - [0]\right)\\ &= \frac{63\pi}{2}\ \text{m}^3\end{aligned}$$

§

PART G: FAMOUS VOLUME FORMULAS

Example 6 (Finding the Volume of a Cone)

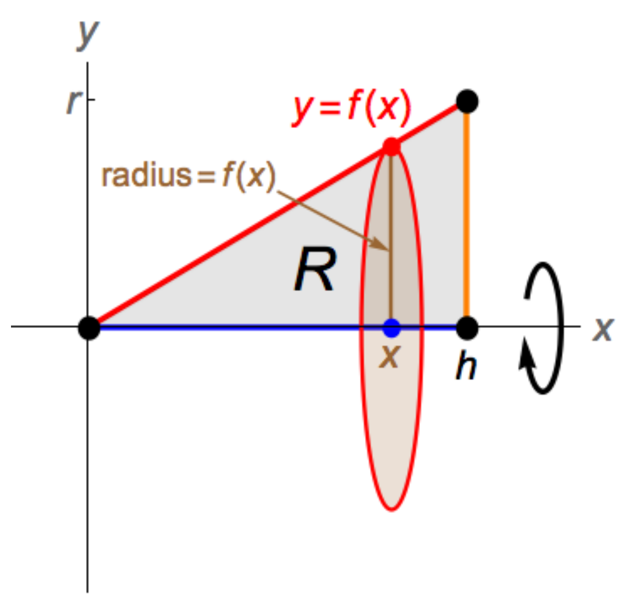
Use the Disk Method to find the volume of a right circular cone of altitude h and base radius r. (h and r are fixed but unknown **constants**.) Lengths and distances are measured in **meters**.

§ Partial Solution

• We obtain such a cone by revolving the triangular generating region R (see below) about the ***x*-axis**. Other choices for R and the axis of revolution also work.

• We are revolving R about a **horizontal** axis, so the Disk Method requires a **"*dx* scan."**

• Observe that the axis of revolution does **not** pass through the interior of R.



• Find the function f such that $y = f(x)$ models the slanted line segment.

Approach 1: Use Slope-Intercept Form of a Line [Segment].

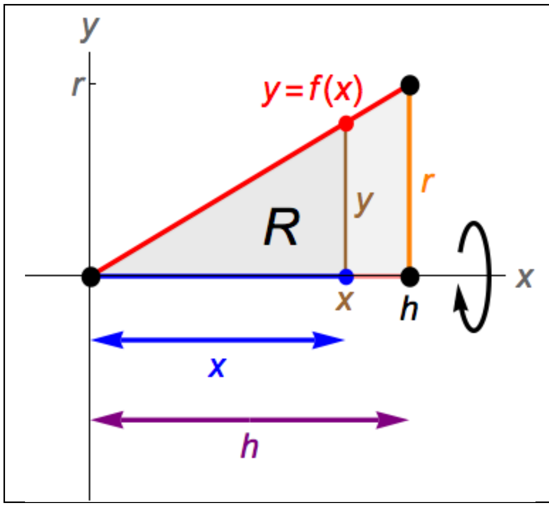
$$y = mx + b$$

$$y = \left(\frac{\text{rise}}{\text{run}}\right)x + 0$$

$$y = \frac{r}{h}x \quad (0 \le x \le h)$$

The right-hand side defines $f(x)$.

Approach 2: Use Side-Proportionality of Similar Triangles.



$$\frac{y}{x} = \frac{r}{h}$$

$$y = \frac{r}{h}x \quad \left(0 \le x \le h\right)$$

The right side defines $f\left(x\right)$.

• We apply the **Disk Method** to find the volume (V) of the cone.

$$V = \int_0^h \pi\left(\text{radius}\right)^2 dx = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx$$

The rest is left as an exercise for the reader.

WARNING 5: ***r* and *h* are constants here.** Treat them as you would 7 or π, say. §

Example 7 (Finding the Volume of a Sphere)

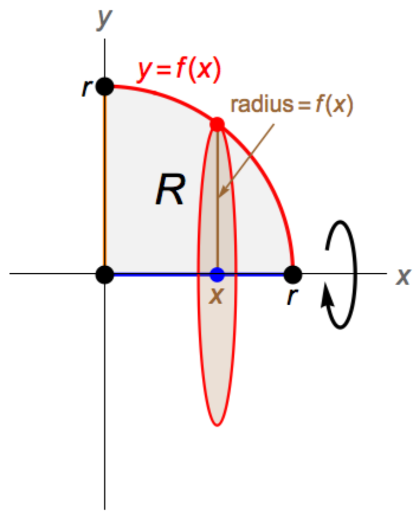
Use the Disk Method to find the volume of a sphere of radius r. (r is a fixed but unknown **constant**.) Lengths and distances are measured in **meters**.

§ Partial Solution

• The desired volume is **twice** the volume of the **hemisphere** obtained by revolving the quarter-circular generating region R (see below) about the ***x*-axis**. Other choices for R and the axis of revolution also work.

• We are revolving R about a **horizontal** axis, so the Disk Method requires a **"*dx* scan."**

• Observe that the axis of revolution does **not** pass through the interior of R.



• Find the function f such that $y = f(x)$ models the quarter-circle.

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

Take $y = \sqrt{r^2 - x^2},\ 0 \le x \le r$ (giving the indicated quarter-circle).

• We apply the **Disk Method** to find the volume (V) of the sphere.

$$V = 2\left(\text{Volume of hemisphere}\right) = 2\int_0^r \pi\left(\text{radius}\right)^2 dx$$
$$= 2\int_0^r \pi\left(\sqrt{r^2 - x^2}\right)^2 dx$$

The rest is left as an exercise for the reader.

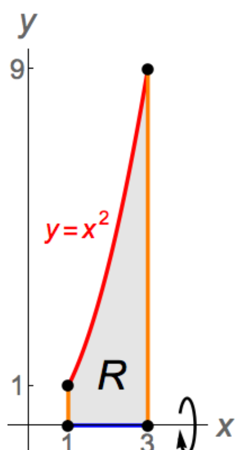
WARNING 5: Don't forget the "2" above.
WARNING 6: *r* is a constant here. §

FOOTNOTES

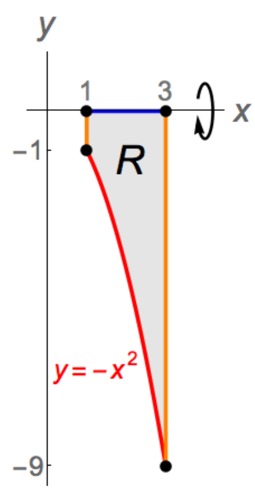
1. **Disk Method: Basic "*dx*" case.** Example 1 is basic, because:
   - The generating region R is in the usual xy-plane.
   - R is being revolved about the x-axis.
   - R is bounded by the x-axis, vertical lines $x = a$ and $x = b$ such that $a < b$, and the graph of $y = f(x)$ for a function f.
   - f is a continuous function on the x-interval $[a, b]$.
   - It helps that f is a simple [polynomial] function.
   - Since $a < b$, the volume of the resulting solid is given by: $V = \int_a^b \pi \left[f(x) \right]^2 dx$
   - f is nonnegative on $[a, b]$, but the formula works even if f is sometimes (or always) negative in value (due to the square). The radius of a "thin disk" would be $\left| f(x) \right|$.

   In Example 1, observe that we would have obtained the same solid with the same volume if we had replaced $y = x^2$ with $y = -x^2$. This is because:

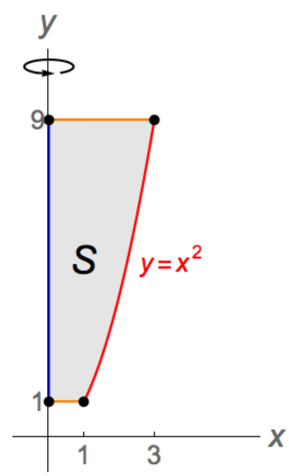
   radius, $r(x) = y_{top}(x) - y_{bottom}(x) = (0) - \left(-x^2\right) = x^2$ once again.



(See Footnote 1.)



(See Footnote 1.)



(See Footnote 2.)

2. **Disk Method: Basic "*dy*" case.** Example 2 is basic for similar reasons.
   - The generating region S is in the usual xy-plane.
   - S is being revolved about the y-axis.
   - S is bounded by the y-axis, horizontal lines $y = c$ and $y = d$, such that $c < d$, and the graph of $x = g(y)$ for a function g.
   - g is a continuous function on the y-interval $[c, d]$.
   - It helps that g is a simple function.
   - Since $c < d$, the volume of the resulting solid is given by: $V = \int_c^d \pi \left[g(y) \right]^2 dy$

3. Washer Method: Basic "*dx*" case. Example 3 is basic, because:

- The generating region R is in the usual xy-plane.
- R is being revolved about the x-axis.
- R is bounded by the graphs of $y = f(x)$ and $y = g(x)$ for functions f and g.

In other basic examples, R is also bounded by one or two vertical lines $x = a$ and/or $x = b$. In Example 3, instead of vertical lines, the intersection points of the graphs of $y = f(x) = -\sqrt[3]{x}$ and $y = g(x) = x^2$ provide values for a and b. Assume $a < b$.

- f and g are continuous functions on the x-interval $[a, b]$.
- $f(x) \geq g(x) \geq 0$ on $[a, b]$. Since the x-axis is the axis of revolution, this means that $y = f(x)$ is the "outer" graph and $y = g(x)$ is the "inner" graph. Also, $r_{out}(x) = f(x)$ and $r_{in}(x) = g(x)$.
- It helps that f and g are simple functions.
- Since $a < b$, the volume of the resulting solid is given by:

$$V = \int_a^b \left(\pi \left[f(x) \right]^2 - \pi \left[g(x) \right]^2 \right) dx = \pi \int_a^b \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right) dx$$ (See Warning 3.)