3.7: IMPLICIT DIFFERENTIATION (IMP. DIFF.)

(A) Implicit Functions

Note

Ex $y = 2x + 1$ describes y as an explicit function of x.

Form: $y = f(x)$

The equation $20x - 10y = -10$ describes y as an implicit function of x.

Why? Let $f(x) = 2x + 1$
If we plug in $f(x)$ for y...

$$20x - 10f(x) = -10$$
$$20x - 10(2x + 1) = -10$$
$$\cancel{20x - 20x} - 10 = -10$$
$$-10 = -10$$

We get an identity (true $\forall x \in Dom(f)$).
Then, f (or y) is an implicit function of x.
So, the graph of f is the same as part or all of the graph of $20x - 10y = -10$.

Here, all: $y = 2x + 1$ is equivalent to $20x - 10y = -10$.

Ex. $y = \frac{1}{x}$ describes y as an explicit func. of x
Form: $y = f(x)$

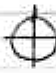
y buried in there. $xy = 1$ " " implicit func. " "

Ex (pp. 146-7)

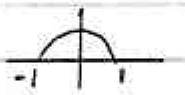
buried in there

$x^2+y^2=1$ determines many imp. funcs.

let's see if we can get some funcs out of here

 fails VLT, but...

$$y^2 = 1-x^2$$
$$y = \pm\sqrt{1-x^2}$$

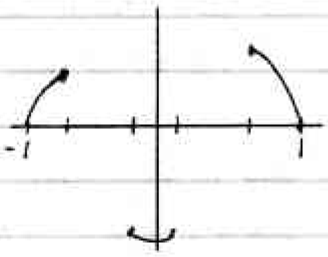
① $f_1(x) = \sqrt{1-x^2}$ 

② $f_2(x) = \sqrt{1-x^2}$, $x \geq 0$ 

③ $f_3(x) = -\sqrt{1-x^2}$ 

④
$$f_4(x) = \begin{cases} \sqrt{1-x^2}, & -1 \leq x \leq -0.7 \\ -\sqrt{1-x^2}, & -0.2 \leq x \leq 0.2 \\ \sqrt{1-x^2}, & 0.6 \leq x \leq 1 \end{cases}$$

Smiley!



For ①-④, [the coords. of] every point (x,y) on the graph satisfies $x^2+y^2=1$. (Their graphs are pieces of the original unit circle.)

(etc.)

How many? There is literally no point.

Ex $x^2+y^2=-1$ No imp. funcs!

(B) Implicit Diff'n (Imp. Diff.)

No $x^2+y^2=-1$

Assume that equations determine a diff'e' (where needed) implicit func. f.

$$y = f(x)$$

$$D_x y = y' \quad \text{(Notation)}$$

Warmups!

$$① \; D_x(7y) = 7y'$$

$$② \; D_x(\underbrace{y}_{\substack{f(x)\\ \text{imp.!}}}{}^2) = 2y\underbrace{y'}_{\text{tail}}$$

$$③ \; D_x(y^3) = 3y^2y'$$

Overall we're D_x... what rule?

$$④ \; D_x(x^3y^2) = (3x^2)(y^2) + (x^3)(2yy') \quad \text{(Product Rule)}$$

^ copy + copy ^

$$= 3x^2y^2 + 2x^3yy'$$

$$⑤ \; D_x(x+y)^4 = 4(x+y)^3 D_x(x+y) \quad \text{(Gen. Power Rule)}$$

$$= 4(x+y)^3(1+y')$$

$$⑥ \; D_x \cos^3(xy) = D_x[\cos(xy)]^3$$

$$= 3[\cos(xy)]^2 \underbrace{D_x[\cos(xy)]}_{[-\sin(xy)][\underbrace{D_x(xy)}_{[(1)y+xy']}]} \quad \text{(Gen. Power Rule)}$$

(Trig)

(Product Rule)

Let's glue e-thing together:

$$= -3(y+xy')\cos^2(xy)\sin(xy)$$

Put trig stuff at end - unambig w/ ()

Ex If $x^2 - 2x + y^2 + 6y = 15$,

(a) Find y'

Later, we'll see (CTS, etc.) how

Isolate y? YUK!
Use Imp. Diff.

(1) D_x [each term on] both sides.

$$D_x(x^2 - 2x + y^2 + 6y) = D_x(15)$$ careful!

Many write "15"

$$2x - 2 + 2yy' + 6y' = 0$$

Solve for y'...

(2) Isolate terms with y' on one side.

$$2yy' + 6y' = 2 - 2x$$

(3) Factor out y'

$$y'(2y + 6) = 2 - 2x$$

(4) Isolate y' by $\div$

$$y' = \frac{2 - 2x}{2y + 6}$$

what's different about this expr for y'?

$$y' = \frac{1 - x}{y + 3}$$

OK to have y, but not y'

Up to 17

(b) Find the slope of the tangent line to the graph of the eq. at $(4,1)$.

Don't have to do on HW. I'd have to be a jerk to give you another pt.

Check: $(4,1)$ is on the graph.

Can skip
$$\left(\begin{array}{l} x^2-2x+y^2+6y=15 \\ (4)^2-2(4)+(1)^2+6(1)\stackrel{?}{=}15 \checkmark \end{array}\right)$$

$$\text{Slope} = y' = \left[\frac{1-x}{y+3}\right]_{\substack{\text{eval at} \\ (4,1)}}$$

$$= \frac{1-4}{1+3}$$

$$= -\frac{3}{4}$$

Another Method

$$x^2-2x+y^2+6y=15 \quad D_x$$
$$2x-2+2yy'+6y'=0$$

Plug in $x=4, y=1$ NOW! (If you just need one deriv. value)

We don't need an expr. for y'. Formula helpful if need to find >1 deriv. value.

$$2(4)-2+2(1)y'+6y'=0$$
$$6+8y'=0$$
$$y'=-\frac{3}{4}$$

ⓒ Same for $(4,-7)$ ← ✓ on graph

$$y' = \left[\frac{1-x}{y+3}\right]_{(4,-7)}$$

$$= \left[\frac{1-4}{-7+3}\right]$$

$$= \frac{3}{4}$$

ⓓ Same for $(1,2)$ ← ✓ on graph

$$y' = \left[\frac{1-x}{y+3}\right]_{(1,2)}$$

$$= 0$$

ⓔ What's going on?

Few remembered! (I claim this is the eq. of a... circle. What technique can I use to show that?

$$x^2 - 2x + y^2 + 6y = 15$$

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = 15 + 1 + 9$$

$$\overset{\uparrow}{\text{CTS}} \qquad \overset{\uparrow}{\text{CTS}}$$

$$(x-1)^2 + (y+3)^2 = 25$$

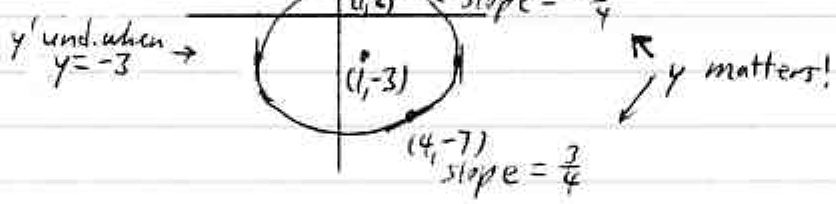
und → + → 0 → − → und

Circle, $\underbrace{\text{Center } (1,-3)}_{\text{Makes left side } = 0}$, $r = \sqrt{25} = 5$

$(x-h)^2 + (y-k)^2 = r^2$
Center: (h,k)
radius $= r$

We still "read" slopes left-to-right

y-coord like "tiebreaker"



You can get the center from $y' = \frac{1-x}{y+3}$

⋮

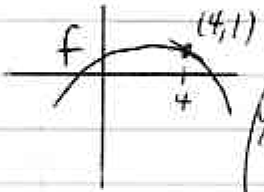
What if you want to use f notation?

Passes VLT

we're not supposed to ⟲

slopes left-to-right

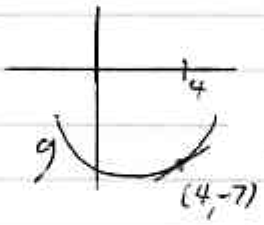
unambig.



$\left(y = f(x), \text{ where } f \text{ is an implicit func. determined by the eq.} \right)$

$$f'(4) = -\frac{3}{4}$$

Deriv. depends on y-coord. It matters what part of the original graph you're on.



$$g'(4) = \frac{3}{4}$$

SKIP

(f) Find y''

$$y' = \frac{1-x}{y+3}$$

$$y'' = D_x\left(\frac{1-x}{y+3}\right) \quad \text{Quotient Rule}$$

$$= \frac{(y+3)D_x(1-x) - (1-x)D_x(y+3)}{(y+3)^2}$$

$$= \frac{(y+3)(-1) - (1-x)(y')}{(y+3)^2}$$

$$= \frac{-(y+3) - (1-x)y'}{(y+3)^2}$$

Plug into y'

$$= \frac{-(y+3) - (1-x)\left(\frac{1-x}{y+3}\right)}{(y+3)^2} \cdot \frac{(y+3)}{(y+3)}$$

·each term in N,D by $(y+3)$

Can we still cancel? NO!

$$= \frac{-(y+3)^2 - (1-x)^2}{(y+3)^3}$$

3.8: RELATED RATES

Ⓐ Implicit Diff'n (Imp. Diff.)

In 3.7, $y = f(x)$. ← implicit

Review Ex Find $D_x(4x^3 + xy - 3y^2)$

$$= 12x^2 + (1)(y) + (x)(y') - 6y\underbrace{y'}_{\text{tail}}$$

Your book tends to combine
Factor out y'
May need to be more descriptive in this section

$$= 12x^2 + y + (x - 6y)y'$$

or $\frac{dy}{dx}$

Review Ex If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ when $x = 3$ and $y = 4$.

What's going to be the geom. interp. of our answer?

$$x^2 + y^2 = 25$$

If you plug in, you're just verifying that the pt. lies on the graph

Don't plug in YET!
D_x both sides implicitly

$$D_x(x^2) + D_x(y^2) = D_x(25)$$

$$2x + 2y\left(\frac{dy}{dx}\right) = 0$$

$$x + y\left(\frac{dy}{dx}\right) = 0$$

Method 1: $\frac{dy}{dx} = -\frac{x}{y}$ } Plug in!

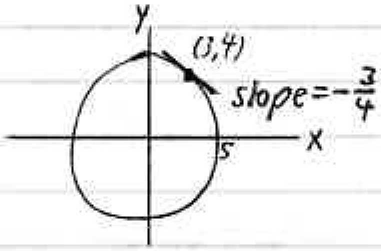
$$= -\frac{3}{4}$$

Method 2: Plug in NOW! (After D_x)

$$(3) + (4)\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -\frac{3}{4}$$

What's the graph of $x^2+y^2=25$?



In 252, trees!
z
x y
r s r s
func from 1st Rev. Ex.

$12x^2 \ldots$
(or is there more?)

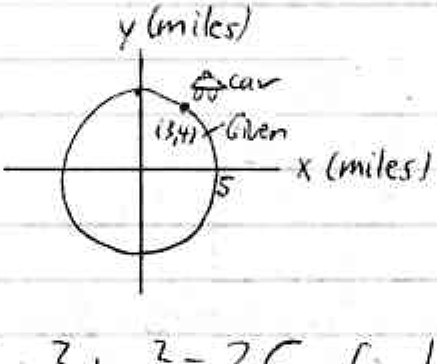
Now, x $x=f(t)$ t y $y=g(t)$

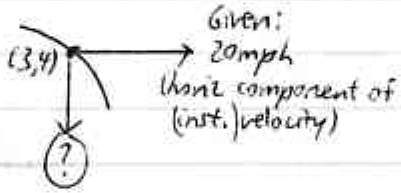
Ex find $D_t(4x^3+xy-3y^2)$

$$= 12x^2\underbrace{\left(\frac{dx}{dt}\right)}_{\text{tail}} + \left(\frac{dx}{dt}\right)(y) + (x)\left(\frac{dy}{dt}\right) - 6y\left(\frac{dy}{dt}\right)$$

NASCAR track
Tank!
Why I'm not an engineer... Just care W-E
In terms of hor. motion
What's a natural?
(Given: horiz. comp.)
Maybe a couple
Translate!
Interpret using notation!

Ex





If $x^2+y^2=25$, find $\frac{dy}{dt}$ if $\frac{dx}{dt}=20$ mph when $x=3$ and $y=4$.

$$x^2+y^2=25$$

Diff not wrt x but

D_t both sides implicitly

Helps! Esp. w/ 25 → 0.

$$D_t(x^2)+D_t(y^2)=D_t(25)$$

$$2x\left(\frac{dx}{dt}\right)+2y\left(\frac{dy}{dt}\right)=0$$

$$x\left(\frac{dx}{dt}\right)+y\left(\frac{dy}{dt}\right)=0$$

They're related by this Eq. Ugly to find formula for dy/dt, o.w. from Chain Rule

$\frac{dx}{dt}$, $\frac{dy}{dt}$ are <u>related rates</u>

Plug in.

$$(3)(20)+(4)\left(\frac{dy}{dt}\right)=0$$

$$60+4\left(\frac{dy}{dt}\right)=0$$

Why −?

$$\boxed{\frac{dy}{dt}=-15\text{ mph}}$$

↓ (S at a speed of 15 mph) (+ ↑ ↓ −)

vertical comp.

↘ Do you know what this vector is called in physics?

vel. along circle is what? 3-4-5 in proportion

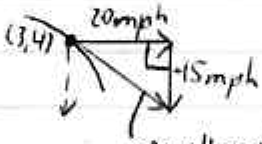
<u>Note</u>

At (3,4),

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}\leftarrow -15}{\frac{dx}{dt}\leftarrow 20}$$

$$=-\frac{3}{4}\quad \text{AGAIN!}$$

From: $\frac{dy}{dx}\frac{dx}{dt}=\frac{dy}{dt}$



resultant (slope $=-\frac{3}{4}$)

velocity along circle is 25 mph

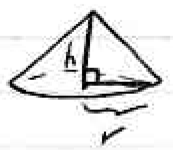
(B) <u>Word Problems</u>

slime, goo, toxic waste shape may change

<u>Ex</u> An object expands as a right circular cone at a rate of 20 ft.³/day. If the radius grows at a rate of 36 in./day, how fast does the height change at the instant that the radius is 10 ft. and the height is $\frac{6}{\pi}$ ft.?

① Read!

② Diagram



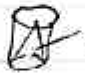
You have a rt. circ. cone. What could I ask you about the cone?

General- nothing specific to our "instant of interest." (Relevant to entire model/process.)

What's area(cyl)?
Says can know!
Look inside front cover.

③ Key Formula:

$$V = \frac{1}{3}\pi r^2 h \quad \text{(Know!)}$$

 $\frac{1}{3}$

Need to know what I'm diff'ing w.r.t. Other info helps.
What's another issue that may come up? Then,
What is 20 ft.³/day?

Unit issues compatibility, RAD

What do we want to find!

④ Given info at the instant of interest (and in general)
Find what?

$$\frac{dV}{dt} = 20 \left(\frac{ft.^3}{day}\right)$$

$$\frac{dr}{dt} = 36 \left(\frac{in.}{day}\right) \cdot \frac{1 \text{ ft.}}{12 \text{ in.}} = 3 \left(\frac{ft.}{day}\right)$$

Find $\frac{dh}{dt}$ when

$$r = 10 \text{ (ft.)}$$
$$h = \frac{6}{\pi} \text{ (ft.)}$$

⑤ D_t both sides of ③

$$D_t(V) = D_t\left(\frac{1}{3}\pi \underbrace{r^2 h}_{\text{Use Product Rule}}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[\left(2r\frac{dr}{dt}\right)(h) + (r^2)\left(\frac{dh}{dt}\right)\right]$$

You don't want to solve for $\frac{dh}{dt}$ as is! Don't need general formula.

⑥ Plug in values from ④, and
⑦ Solve for $\frac{dh}{dt}$ (units!)

$$20 = \frac{1}{3}\pi\left[2(10)(3)\left(\frac{6}{\pi}\right) + (10)^2\left(\frac{dh}{dt}\right)\right]$$

$$20 = 120 + \frac{100\pi}{3}\frac{dh}{dt}$$

$\frac{3}{4}x = \ldots$
Solve for x: $\cdot\frac{4}{3}$

$$-100 = \frac{100\pi}{3}\frac{dh}{dt}$$

$$-100\left(\frac{3}{100\pi}\right) = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{3}{\pi} \approx -0.9549 \ \frac{ft.}{day}$$

What does this mean? Why does this make sense? Even though $V\nearrow$, r is also $\nearrow$

slippery goo height shrinking as expands along floor

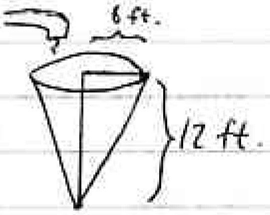
⑧ Conclusion (in English)

At the instant of interest, the height shrinks at a rate of $\frac{3}{\pi}$ $\frac{ft.}{day}$.

(Recap!) Interpret "−"

Ex 4 (p.156)

①,② Water tank



Water is pumped in at 10 $\frac{\text{gal.}}{\text{min.}}$ ($\approx 1.337 \frac{\text{ft.}^3}{\text{min.}}$).
Approximate the rate at which the water level rises when the water is 3 ft. deep.

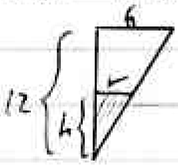
This is the general diagram.

similar cones/triangles

You must incorporate this relationship into your formula! You can't let r, h roam freely, independently. How is r related to h?

③ $V = \frac{1}{3}\pi r^2 h$ ← want in terms of h, alone
(We can do this, because regardless of how much water is in here, r and h have the same relationship)

Similar triangles



$\frac{r}{h} = \frac{6}{12}$

$r = \frac{1}{2}h$ (We must incorporate this dependency relationship into our formula.)

$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V = \frac{1}{12}\pi h^3$

If water is pumped out, $\frac{dV}{dt}$ −

④ Given: $\frac{dV}{dt} \approx 1.337$
Find $\frac{dh}{dt}$ when $h = 3$.

⑤ $V = \frac{1}{12}\pi h^3$

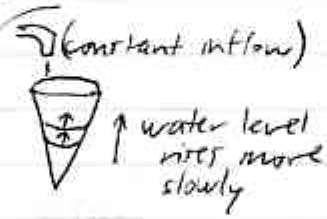
$\frac{dV}{dt} = \frac{1}{12}\pi\left(3h^2 \frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

⑥ $1.337 \approx \frac{1}{4}\pi(3)^2\left(\frac{dh}{dt}\right)$

⑦ $\frac{dh}{dt} \approx 0.189 \ \frac{ft.}{min.}$ when the water

⑧ The water level rises at ≈ " " is 3 ft. deep.



<u>Note</u> When $h = 4$, ⑥ $1.337 \approx \frac{1}{4}\pi(4)^2\left(\frac{dh}{dt}\right)$

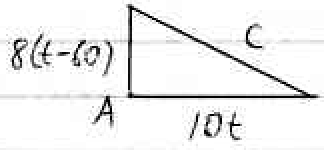
Why is this lower? $\frac{dh}{dt} \approx 0.106 \ \frac{ft.}{min.}$

SKIP

Hard to underst.; not simplified in manual.

<u>Ex</u> (#14) A girl starts at a point A and runs east at $10 \ \frac{ft.}{sec.}$ One minute later, another girl starts at A and runs north at $8 \ \frac{ft.}{sec.}$ At what rate is the distance between them changing 1 minute after the second girl starts?

Let t = #secs since girl A starts

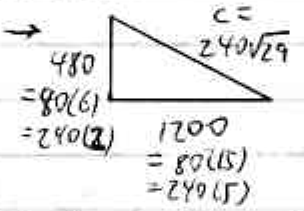


$c^2 = (10t)^2 + [8(t-60)]^2$

$c^2 = 100t^2 + 64(t-60)^2$

$2c\frac{dc}{dt} = 200t + 128(t-60)$

Let $t = 120$ →



$2(240\sqrt{29})\left(\frac{dc}{dt}\right) = 200(120) + 128(120-60)$

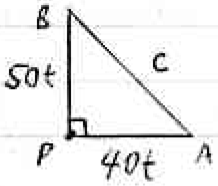
$\frac{dc}{dt} = \frac{66}{\sqrt{29}} \approx 12.3 \ \frac{ft}{sec.}$

Prelim Ex

Cars A and B leave Point P at the same time. Car A moves east at 40mph. Car B moves north at 50mph. Three hours later, how fast are they moving apart?

In case they want to take a shot at each other

Let $t = \#$ hours elapsed $(t > 0)$



We want $\left.\frac{dc}{dt}\right|_{t=3}$

$$\text{Pyth. Thm.} \Rightarrow c^2 = (40t)^2 + (50t)^2$$
$$= 4100t^2$$
$$\Rightarrow c = \pm\sqrt{4100t^2}$$
$$= (10\sqrt{41})t$$

$$\Rightarrow \frac{dc}{dt} = 10\sqrt{41} \text{ mph} \approx 64.03 \text{ mph} \quad \text{for } \underline{\text{all}}\ t > 0$$

indep. of t!! Rate of change of distance bet. cars stays the same.

I'm going to do a variation, we assumed... the cars left...

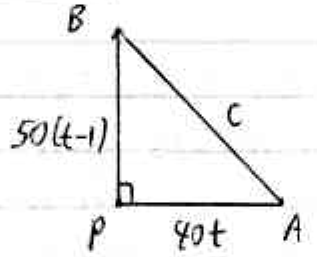
Ex Car B leaves Point P 1 hour later than Car A. Car A → 40mph, Car B ↑ 50mph. Four hours after A leaves P, how fast are A and B moving apart?

We must be precise!

Let $t = \#$ hours after A leaves P

Method 1

Is it $50t$?

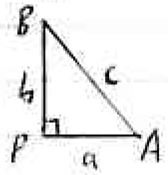


$(t > 1.)$

Pyth. Thm. $\Rightarrow c = \sqrt{(40t)^2 + [50(t-1)]^2}$

Find $\left.\frac{dc}{dt}\right|_{t=4}$ (a bit messy)

Method 2 (Related Rates)



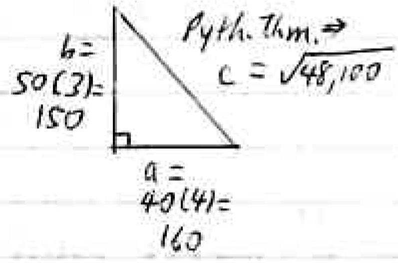
$$c^2 = a^2 + b^2$$

Find $\frac{dc}{dt}$ at $t=4$, if $\frac{da}{dt}=40$, $\frac{db}{dt}=50$

$$D_t(c^2) = D_t(a^2) + D_t(b^2)$$

$$\cancel{2}c\frac{dc}{dt} = \cancel{2}a\frac{da}{dt} + \cancel{2}b\frac{db}{dt}$$

At $t=4$,



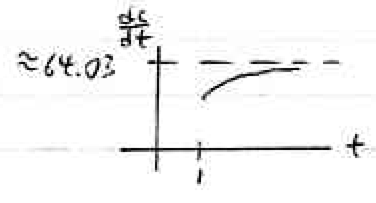
$$\sqrt{48{,}100}\,\frac{dc}{dt} = 160(40) + 150(50)$$

$$\left(\frac{dc}{dt} = \frac{13{,}900}{\sqrt{48{,}100}}\right)$$

$\frac{dc}{dt} \approx$ 63.38 mph (at $t=4$ hrs.)

Note $\frac{dc}{dt} \approx 63.64$ mph (at $t=5$ hrs.)

$\frac{dc}{dt} \nearrow$, but not $\nearrow^{\infty}$. There's a limiting value - do you know what it is? (Not the #; the idea) We approach a situation where, for all intensive purposes, A, B started at same time. Problem: Globe, mts, oceans.



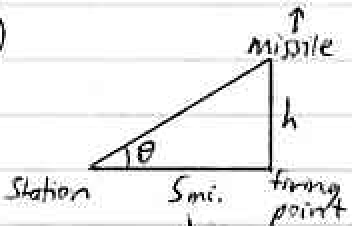
As $t \to \infty$, $\frac{dc}{dt} \to 10\sqrt{41}$

the effect of A's head start dissipates.

Ex (#46) (Don't have to write!)

In book, so don't have to write down. Looked up: weird motion is missile…: Missile going faster and faster – See 3.8 #39 comments. 20–22° vs. 89–91°: covers in 1 sec. The "5 mi." is relevant to the entire process; it stays constant – can put in general diagram!

① A missile is fired vertically from a point that is 5 miles from a tracking station and at the same elevation. [For now,] its angle of elevation θ changes at a constant rate of $2°$ per second. Find the velocity when the angle of elevation is $30°$.

② 

(Doesn't change!)

(Let h = height of missile.)

③ Relate h and θ.

Don't have to

$$\tan\theta = \frac{h}{5}$$
$$h = 5\tan\theta$$

What = 2°/sec.

④ Given: $\frac{d\theta}{dt} = \left(2\ \frac{\text{deg}}{\text{sec}}\right) \times \left(\frac{\pi\ \text{rad}}{180\ \text{deg}}\right) = \frac{\pi}{90}\ \frac{\text{rad}}{\text{sec}}$

Find $\frac{dh}{dt}$ (= velocity)

In case θ shows up later? e.g., $D_t(\theta^2) = 2\theta\frac{d\theta}{dt}$

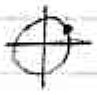
when $\theta = 30° = \frac{\pi}{6}$ rad (Just to be safe?)

Before, Pyth. Thm. What relates $\theta, h, 5$?

⑤
$$h = 5\tan\theta$$
$$D_t(h) = D_t(5\tan\theta)$$
$$\frac{dh}{dt} = 5(\sec^2\theta)\left(\frac{d\theta}{dt}\right)$$

⑥ Plug in $\theta = \frac{\pi}{6}$, $\frac{d\theta}{dt} = \frac{\pi}{90}$

⑦ $\frac{dh}{dt} = 5(\sec^2\theta)\left(\frac{d\theta}{dt}\right)$
$= 5\left[\sec\frac{\pi}{6}\right]^2\left(\frac{\pi}{90}\right)$



$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

can leave as irrat'l
square will kill √

$= 5\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{\pi}{90}\right)$
$= 5\left(\frac{4}{3}\right)\left(\frac{\pi}{90}\right)$
$= \frac{20\pi}{270}$
$= \frac{2\pi}{27}\ \frac{mi.}{sec}$
$\approx 0.233\ \frac{mi.}{sec.}$

may want to convert to...

or $\left(\frac{2\pi}{27}\ \frac{mi.}{sec}\right)\left(\frac{3600\ sec}{1\ hr.}\right) \approx 837.8\ \frac{mi.}{hr.}$

⑧ (Concl.)