1.1: ALGEBRA (Back of book)

Assume x, y in $\mathbb{R}$ (the set of all real #)

① $\sqrt{x^2} = |x|$

Ex $\sqrt{(-3)^2} = |-3|$

$\sqrt{9}$ $\quad$ 3

3

|product| = prod(||)
sum is more complicated
= if one is 0
or same sign

② $|xy| = |x||y|$

③ $(x+y)^2 = x^2 + 2xy + y^2$

Assume $x, y > 0$

④ $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$

Ex $\sqrt{5} \neq \sqrt{1} + \sqrt{4}$

$1 + 2$

3

$x^2+4 = (x+2)^2$
$x^2+4 = x^2+4x+4$
$0 = 4x$
$x = 0$
Sometimes wrong
if $x, y < 0$

Ex $\sqrt{x^2+4} \neq x+2$

⑤ $\sqrt{xy} = \sqrt{x}\sqrt{y}$

+y in the way

$\frac{x^2+y}{x^2} = 1+y$
$x^2+y = x^2+x^2y$
$y - x^2y = 0$
$y(1-x^2) = 0$
$\cancel{y=0}$ $\quad$ $x = \pm 1$ only
(Assume $x, y > 0$)

⑥ $\frac{\cancel{x^2}^{\,1} + y}{\cancel{x^2}_{\,1}}$ WRONG! (OK if $x = 1$)

⑦ $\frac{\cancel{x^2}^{\,1}(y+z)}{\cancel{x^2}_{\,1}}$ RIGHT! ← x^2 is a <u>factor</u> of the num.
← denom.

1.2: FUNCTIONS

(A) Basics

A graph in $\overset{y}{+}x$ describes y as a function of x ("$y=f(x)$", g, etc.)

state it precisely

$\Leftrightarrow$ it passes the Vertical Line Test (VLT)

$\Leftrightarrow$ there is no vertical line that cuts the graph more than once

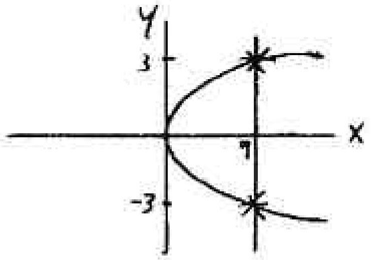
$\Leftrightarrow$ no input (x) yields more than one output (y)

NO $x \Rightarrow \boxed{*} \begin{matrix} \nearrow y \\ \searrow y \end{matrix}$

Then, "y" and "$f(x)$" are interchangeable.

What does $y=x^2$ look like?

Ex $x=y^2$



fails VLT

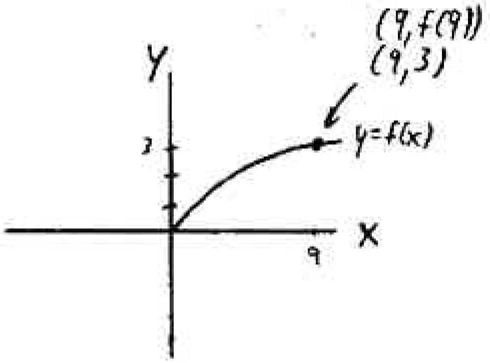
Calc. buttons

$$\begin{array}{ccc} \underline{x} & & \underline{y} \\ 9 \Rightarrow \boxed{*} & \begin{matrix} \nearrow \\ \searrow \end{matrix} & \begin{matrix} 3 \\ -3 \end{matrix} \end{array}$$

$\cancel{y=f(x)}$

Ticks later.
Plot point later.
It's OK that graph disappears

Ex $y=\sqrt{x}$



passes VLT

$$\left.\begin{aligned} y&=\sqrt{x} \\ f(x)&=\sqrt{x}\end{aligned}\right\} \text{"same"}$$

Idea

Inputs x		Outputs (function values) y	Point
⋮		⋮	
9 ⟹	func. f Rule: $f(x)=\sqrt{x}$	⟶ $\underbrace{f(9)}_{\text{"f of 9"}}=\sqrt{9}=3$	$(9,3)$

looks like a cigarette

of/at

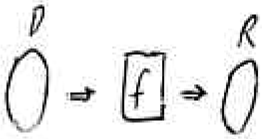
not "f times 9"

x : <u>independent</u> variable
y : <u>dependent</u>

What do you call

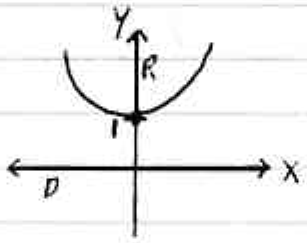
The <u>domain</u> of f = the set of "legal" <u>inputs</u> (<u>x-coords.</u> picked up by the graph)

The <u>range</u> of f = the resulting set of <u>outputs</u> (<u>y-coords.</u> or <u>function values</u>)



Ex $\underbrace{f(x)}_{\text{or } y} = x^2 + 1$

Don't go nuts during a test...



Clap onto axes

1.1

Domain of f
(= the set of all real #s)
$= \mathbb{R}$
$= (-\infty, \infty)$ ← interval form

Range of f
$= \{y : y \geq 1\}$ (set-builder notation)
= the set of all real #s "y" such that $y \geq 1$
$= [1, \infty)$
↑ include 1

Natural: from 121
Assumed: what about applics.?

Ⓑ Finding the [Natural] Domain of f

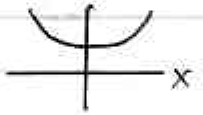
the set of all real #s except those values of x (say) that make $f(x)$ undefined (i.e., not real)

we're not dealing w/ i
Sins?
$\sqrt[3]{x}$ ok $\sqrt{x}$
In applics: neg. values?
$\log_b x$

$\frac{1}{x}$: need $x \neq 0$
even $\sqrt{x}$ $x \geq 0$
$\ln x$ $x > 0$

$\underline{Ex}$ $f(x) = \text{polynomial, like } x^2+1$

$\text{Domain} = \mathbb{R} \text{ or } (-\infty, \infty)$



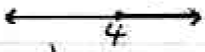
all "x"'s legal

$\underline{Ex}$ $f(x) = \underbrace{\sqrt{x-4}}$

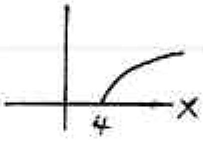
real $\Leftrightarrow$ $x - 4 \geq 0$

$x \geq 4$

$\text{Domain} = \{x : x \geq 4\}$



$= [4, \infty)$



$\underline{Ex}$ $f(x) = \sqrt[3]{x-4}$, Domain $= \mathbb{R}$

$\underline{Ex}$ $f(x) = \frac{1}{x^2-25}$

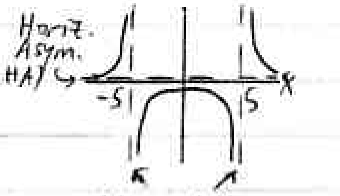
is real $\underline{\text{except}}$ when $x^2 - 25 = 0$

$(x+5)(x-5) = 0$

-5 $\quad$ 5

Ch. 2: We'll talk about this explosive behavior

Turns out:



Vertical Asymptotes (VAs)

$\text{Domain} = \{x : x \neq \pm 5\}$

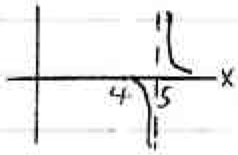
$= (-\infty, -5) \cup (-5, 5) \cup (5, \infty)$

exclude -5 $\quad$ union

(You) Few got!

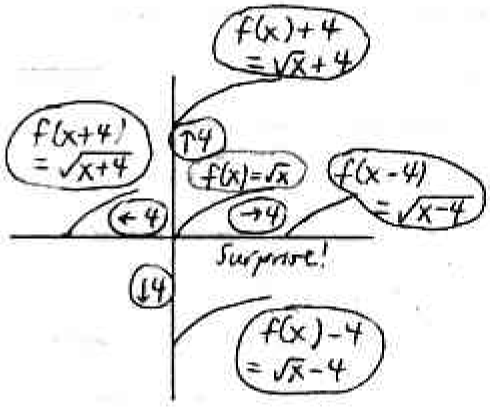
$\underline{Ex}$ $f(x) = \frac{\sqrt{x-4}}{x^2-25}$ $\leftarrow$ Need $x \geq 4$ $\leftarrow$ Need $x \neq \pm 5$

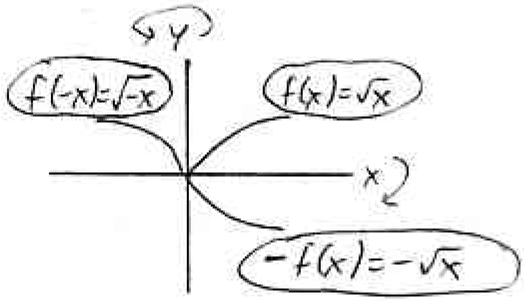
Turns out:



$\text{Domain} = [4, 5) \cup (5, \infty)$

(C) Translations and Reflections





(D) Arithmetic of funcs.

If f, g are funcs., then $f+g$ is a func. defined by $(f+g)(x) = f(x) + g(x)$, x is in Dom(f) and Dom(g)

$f-g, fg, \frac{f}{g} \leftarrow \neq 0$ similar

Also, $(f-g)(x) = f(x) - g(x)$

$(fg)(x) = f(x)g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \leftarrow \neq 0$

Ex If $f(x) = 3x^2$,
$g(x) = x-3$,

(You)

then $(f-g)(x) = f(x) - g(x)$
$= 3x^2 - (x-3)$
Need
$= 3x^2 - x + 3$

$(f-g)(2) = 3(2)^2 - 2 + 3$
$= 13$

$(2 \to \boxed{f-g} \to 13)$

Ⓔ Composition of Funcs.

What's another way of combining funcs?

- applying a <u>sequence</u> of funcs.

If f, g are funcs., the <u>composite function $f \circ g$</u> is defined by $(f \circ g)(x) = f(g(x))$

$\overleftarrow{\text{apply}}$

$$\underbrace{x \Rightarrow \boxed{g} \Rightarrow g(x) \Rightarrow \boxed{f}}_{\boxed{f \circ g}} \Rightarrow f(g(x))$$

1st

and also...

x is in $Dom(g)$, and
$g(x)$ is in $Dom(f)$

<u>Ex</u> Let $f(x) = 3x$ ← tripling func.
$g(x) = x^2$ ← squaring func.

ⓐ
$$(f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2$$

$$x \rightarrow \boxed{g} \Rightarrow x^2 \Rightarrow \boxed{f} \Rightarrow 3x^2$$

ⓑ
$$(g \circ f)(x) = g(f(x)) = g(3x) = (3x)^2 = 9x^2$$

like $-, \div$
Comp. is not comm.

$\circ$ not comm.

Often, $f \circ g \neq g \circ f$

(order often matters!)

$$x \rightarrow \boxed{f} \rightarrow 3x \Rightarrow \boxed{g} \Rightarrow 9x^2$$

If f: doubling
g: tripling
$\Rightarrow f \circ g = g \circ f$

$\underline{Ex}$ If $f(x) = \dfrac{3x}{x^2-2}$,

$g(x) = x+3$,

(You)
A ___few___ got

find $(f \circ g)(x)$

$$(f \circ g)(x) = f(g(x))$$

$$= f(x+3)$$

$$f(\text{☆}) = \frac{3(\text{☆})}{(\text{☆})^2 - 2}$$

Some $x \to (x+3)$, but can be confusing.
Some forgot ( )'s →

Need

$$= \frac{3(x+3)}{\underbrace{(x+3)^2}_{\text{careful!}} - 2}$$

$$= \frac{3x+9}{x^2+6x+9-2}$$

prime →

$$= \frac{3x+9}{x^2+6x+7}$$

Ex Express $y = \sqrt{3x+4}$ in a composite func. form.

Calc III: bushy tree

$$y = f(\underbrace{g(x)}_{u=g(x)})$$

$$\underbrace{\qquad\qquad}_{y=f(u)}$$

f of u

$$\begin{array}{c} y \\ |\,f \\ u \\ |\,g \\ x \end{array}$$

Book: Let u = the part you'd eval on a calc 1st if x were a #.

Let u = the "inside part"
y = "what we do to u"

One way

calc. buttons

$$x \Rightarrow \boxed{g} \Rightarrow \underset{u}{3x+4} \Rightarrow \boxed{f\ \sqrt{\ }} \Rightarrow \underset{y}{\sqrt{3x+4}}$$

what do we do to u?

$$\boxed{\text{If } u = \underbrace{3x+4}_{g(x)}, \text{ then } y = \underbrace{\sqrt{u}}_{f(u)}}$$

Also

$$\boxed{\begin{array}{l} u = 3x \\ \Rightarrow y = \sqrt{u+4} \end{array}}$$

Don't do this!

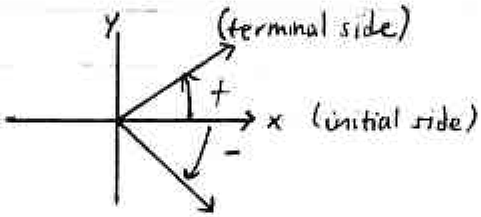
or $u = \sqrt{3x+4}$
$y = u$
simple symbol-swap so I leave you w/something dumb

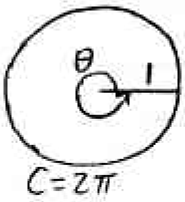
$u = x$ ← identity func. $g(x) = x$
$\Rightarrow y = \sqrt{3u+4}$

1,3: TRIG (Back of book)

Ⓐ Angles (∠s)

Standard position





$\theta = 360^\circ$
$= 2\pi$ radians
assumed

DEG vs. RAD

$\theta = 180^\circ$
$= \pi$ rad

(Conversion factor)

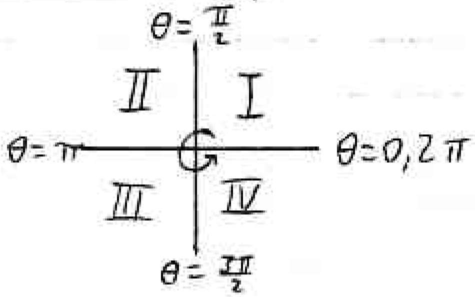
Ex $45^\circ \times \frac{\pi\text{ rad}}{\cancel{180^\circ}_{4}} = 1$ $= \boxed{\frac{\pi}{4}\text{ rad}}$

Up to 3

Ex $\frac{\pi}{6}\text{ rad}\ {}^{\nearrow 180^\circ} = \boxed{30^\circ}$

Quadrants (Qs)

Where's QI?



"Twins"

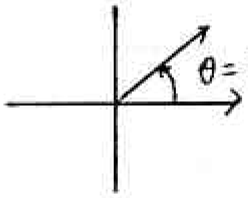
what if I do 2 full revols.?

$\frac{\pi}{4} + 4\pi$

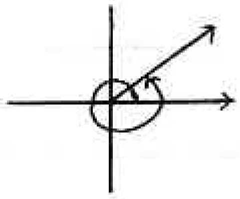
up to 3

twins (actually, ∞ many!

if you place these angles in standard position, they look alike - they have same terminal side



$\theta = \frac{\pi}{4}$



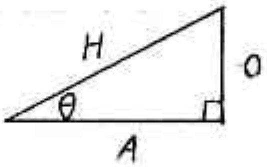
$\theta = \frac{\pi}{4} + 2\pi$
$= \frac{9\pi}{4}$

$$\left.\begin{array}{l}\theta = \frac{\pi}{4} + 2\pi n, \\ n \text{ integer}\end{array}\right\}$$ coterminal ∠s

-have the same trig values

(B) Right $\triangle$s $\longrightarrow$ 6 Trig Funcs

Relative to θ



Pyth. Thm →
$O^2 + A^2 = H^2$

Ancient Curse

"SOH-CAH-TOA"

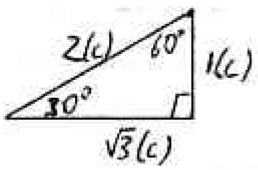
$\sin\theta = \frac{O}{H}$ $\quad$ $\cos\theta = \frac{A}{H}$ $\quad$ $\tan\theta = \frac{O}{A}$

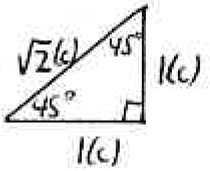
$$\left(\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \csc\theta = \frac{H}{O} & \sec\theta = \frac{H}{A} & \cot\theta = \frac{A}{O} \end{array} \right)$$

Family of Similar $\triangle$s

They forget 30-60-90 sides in terms of ∞, there are two very well known types of rt. triangles. Someone once died for this...

Special $\triangle$s



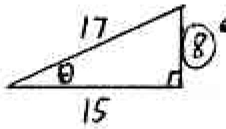


$(c>0)$

$\sin 30° = \frac{1}{2}$, etc. $\qquad$ $\tan 45° = 1$, etc.

Ex If $\cos\theta = \frac{15}{17}$, and θ is acute, find $\tan\theta$.

3-4-5, 5-12-13 in HW
only a few knew 8-15-17
Also 7-24-25
9-40-41



← from Pyth. Thm.

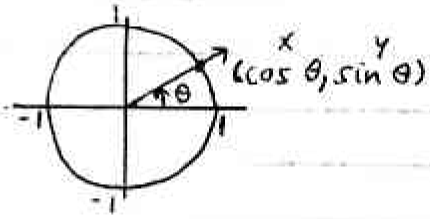
$\tan\theta = \boxed{\frac{8}{15}}$

Up to 11

3-4-5, 6-8-10, ...
5-12-13, ...

$\approx$ 1/3 saw
~ 2/3

© Unit Circle ⇒ 6 Trig Funcs



$-1 \le \cos\theta \le 1$

$-1 \le \sin\theta \le 1$

QI

No one knew slope

1/5 saw this

How many times does 30° go into 180°?

	Special θ	$\sin\theta$ (↗)	$\cos\theta$ (↘)	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ slope!
0°	0	$\frac{\sqrt{0}}{2} = 0$	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	
90°	$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2} = 1$ SIGN +	0 COS +	undefined Asymptote TAN +

flip

Reciprocals: $\underbrace{\csc\theta \qquad \sec\theta}_{\text{Careful!}}$ $\qquad \cot\theta$

$\csc\theta = \frac{1}{\sin\theta}$

"Brothers"

People didn't know →

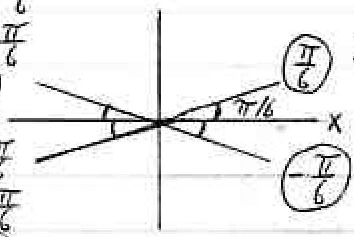
have the same reference $\angle$

acute $\angle$ bet. terminal side, x-axis

Famous bros

5 is 1 less than 6. $\left\{ \begin{array}{l} \pi - \frac{\pi}{6} \\ \frac{6\pi}{6} - \frac{\pi}{6} \\ \frac{5\pi}{6} \end{array} \right.$

$\frac{\pi}{6}$ Special ref. $\angle$



7 is 1 more than 6. $\left\{ \begin{array}{l} \pi + \frac{\pi}{6} \\ \frac{6\pi}{6} + \frac{\pi}{6} \\ \frac{7\pi}{6} \end{array} \right.$

$-\frac{\pi}{6}$ or $\left. \begin{array}{l} 2\pi - \frac{\pi}{6} \\ \frac{12\pi}{6} - \frac{\pi}{6} \\ \frac{11\pi}{6} \end{array} \right\}$ 11 is 1 less than twice 6.

How do we get 11 from 6?

Brothers have the same trig values, except maybe for signs.

CAST

sin, csc	S	A	All are "+"
tan, cot	T	C	cos, sec

All Students Take Calculus (Suckers)

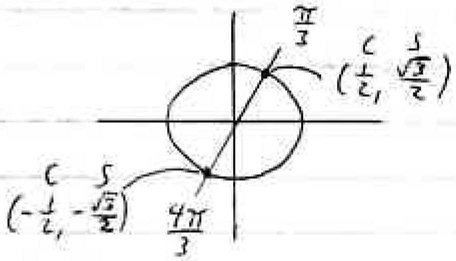
Simplified Special $\angle$

Ex Find the exact value of $\cot\left(\frac{4\pi}{3}\right)$
(Only use a calc to ✓.)

$\frac{4\pi}{3}$: Ref. $\angle = \frac{\pi}{3}$
Q III

what Q?

x-coord is cos



4 is 1 more than 3

tan = ?

$$\cot\left(\frac{4\pi}{3}\right) = \frac{\cos\left(\frac{4\pi}{3}\right)}{\sin\left(\frac{4\pi}{3}\right)}$$

$$= \frac{\cancel{-}\frac{1}{2}}{\cancel{-}\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ Rationalizing the denom.

$$= \boxed{\frac{\sqrt{3}}{3}}$$

Up to 41, except 21

✓: $\boxed{\text{RAD}}$
(⇒ Dec. Approx)

S	A
T	C

cot "+"

(I) Trig Identities

(1) Reciprocal $\quad \csc x = \frac{1}{\sin x}$, etc.

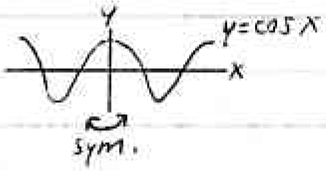
(2) Quotient $\quad \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$

(3) Pythagorean

$$\sin^2 x + \cos^2 x = 1 \quad \div \sin^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\div \cos^2 x: \quad \tan^2 x + 1 = \sec^2 x$$

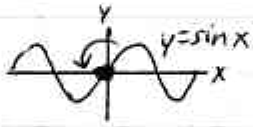
(4) Even/Odd $\quad \cos x, \sec x$ are <u>even</u>

$$\cos(-x) = \cos x$$



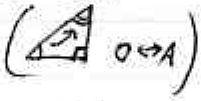
the others are <u>odd</u>

$$\sin(-x) = -\sin x$$



(5) Cofunction

$\sin x = \cos(\frac{\pi}{2}-x)$
cos $\quad$ sin



tan $\quad$ cot
sec $\quad$ csc

(4) If you have sym about $-x$ you probably don't have a func. of x. Why not? Such a graph would fail! What test?

(5) The two acute ∠s of a right tri. are complementary - they add up to what?

Do you agree that the opp side for this ∠ is the adj. side for that

I mention cofuncs when ⊙x

(5) →

Stewart 4th 7.1.13

<u>Ex</u> Verify $\frac{\sec^2 x - 1}{\sec^2 x} = \underbrace{\sin^2 x}_{\text{TARGET}}$.

top of jigsaw puzzle box

<u>Method I</u>

$$\frac{\sec^2 x - 1}{\sec^2 x} = \frac{\sec^2 x}{\sec^2 x} - \frac{1}{\sec^2 x}$$
$$= 1 - \cos^2 x$$
$$= \sin^2 x \ \checkmark$$

Pyth: $\sin^2 x + \cos^2 x = 1$
$\rightarrow \sin^2 x = 1 - \cos^2 x$

Method 2

$$\frac{\sec^2 x - 1}{\sec^2 x} = \frac{\tan^2 x}{\sec^2 x} \leftarrow \text{Pyth.}$$

$\rightarrow$ sin, cos

$$= \frac{\frac{\sin^2 x}{\cancel{\cos^2 x}}}{\frac{1}{\cancel{\cos^2 x}}} \left(\cdot \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}} \quad \text{or} \quad \frac{\sin^2 x}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos^2 x}}{1} \right)$$

all but SS

$$= \sin^2 x \checkmark$$

Very few were asked to memorize in 141

(C) More Trig IDs (Handout - Know!)

SKIP - on Handout

Sum Formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
= "sum of mixed-up products"

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$
= "cosines (product) - sines (product)"

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$= \text{"}\frac{\text{Sum}}{1 - \text{Product}}\text{"}$$

Difference Formulas

Change all the signs you see ↑

Double-Angle Formulas "Angle-Reducing"

$$\sin(2u) = \sin(u+u)$$
$$= \sin u \cos u + \cos u \sin u$$
$$= 2\sin u \cos u$$

$$\cos(2u) = \cos(u+u)$$
$$= \cos u \cos u - \sin u \sin u$$
$$= \cos^2 u - \sin^2 u$$

and Variations

$\cos^2 u$ $=1-\sin^2 u$ Solve for $\sin^2 u$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$

or $\frac{1}{2} - \frac{1}{2}\cos(2u)$

SIN is BAD

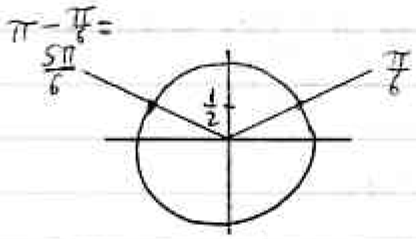
$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$

or $\frac{1}{2} + \frac{1}{2}\cos(2u)$

(F) Trig Eqs.

Ex Solve $\sin x = \frac{1}{2}$

In QI, the sin of what is 1/2? $\frac{\pi}{6}$ and his coterminals. Are there other ∠s whose sin is $\frac{1}{2}$? Which Q?



all of his coterminal angles

$$x = \frac{\pi}{6} + 2\pi n$$
$$x = \frac{5\pi}{6} + 2\pi n$$
n integer

Ex Solve $\sin(\underbrace{3x}_{=\theta,\text{ say}}) = \frac{1}{2}$ Solve $\sin\theta = \frac{1}{2}$ (like 5); $\theta = 3x$.

$$3x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad 3x = \frac{5\pi}{6} + 2\pi n$$
$$x = \frac{\pi}{18} + \frac{2\pi n}{3} \qquad x = \frac{5\pi}{18} + \frac{2\pi n}{3}, \ n \text{ int.}$$