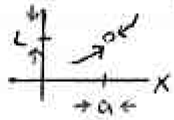
2.2: ϵ-δ DEFINITION OF $\lim_{x \to a} f(x) = L$

func.

real #s

epsilon (lowercase)
delta

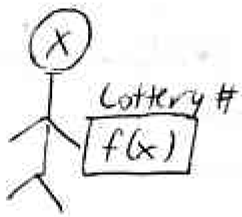
Along the graph, we're approaching (a, L) from the left, right



$f(x) \to L$
as $x \to a$

Ⓐ Lottery Idea

You won't see this in a calc textbook.
Looks drunk



(Exact winning #)
Target

He's the lottery manager's son.
Don't you wish in real life?

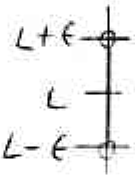
"disqualified"

Player x $(x \neq a)$ wins $\Leftrightarrow$
$f(x)$ is "close enough" to L
within ϵ of

Erdős — oh a cute little ϵ
typically, a small qty.

fixed #, > 0
"tolerance"

Range game



x wins $(x \neq a)$ $\Leftrightarrow$ $f(x)$ is in ⫯

$\Leftrightarrow L - \epsilon < f(x) < L + \epsilon$

$\quad -L \qquad -L \qquad -L$

$\Leftrightarrow -\epsilon < f(x) - L < \epsilon$

Rewrite using part 1 <

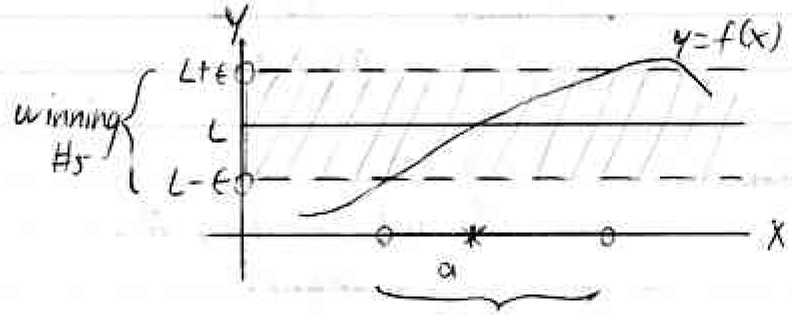
Note $-4 < ☆ < 4$
$\Leftrightarrow |☆| < 4$

distance

$\Leftrightarrow \underbrace{|f(x) - L|}_{\text{how "far" } f(x) \text{ is from } L} < \epsilon$

12.12.01?

Ex Fix $\epsilon > 0$.

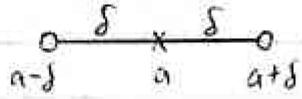


a winning interval (WI); open (excludes endpts.)

all the players in here win bec. their lot #s (func. values) are in this winning band; a disqual

a symmetric WI

has the form:



$\Leftrightarrow$ $(a-\delta, a+\delta)$, $x \neq a$ for some $\delta > 0$

$\Leftrightarrow$ $a-\delta < x < a+\delta$, $x \neq a$ "

$\Leftrightarrow$ $-\delta < x-a < \delta$, $x \neq a$ "

$\Leftrightarrow$ $\underbrace{0 <}_{(x \neq a)} \underbrace{|x-a| < \delta}_{x \text{ is less than } \delta \text{ away from } a.}$ "

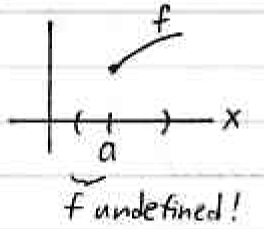
If I take this WI and shrink it, do I still have a WI?

Any smaller value for δ $(\delta > 0)$ also gives us a WI.

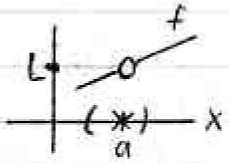
(B) ϵ-δ Def'n

Assume (the function) f is defined on an open interval containing a, except $f(a)$ may be undefined.

NO



OK



"$\lim\limits_{x \to a} f(x) = L$" means that

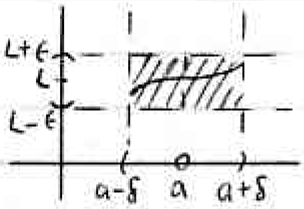
$\underset{\text{for all/any}}{\forall} \epsilon > 0, \ \underset{\text{there exists}}{\exists} \delta > 0$ such that

if $\underbrace{x \text{ is in } (a-\delta, a+\delta) \text{ and } x \neq a}_{\text{i.e., } 0 < |x-a| < \delta}$,

then $\underbrace{f(x) \text{ is in } \underbrace{(L-\epsilon, L+\epsilon)}_{\text{winning \#s}}}_{\text{i.e., } |f(x)-L| < \epsilon}$

Star Wars
Move in walls ⊢ ⊣ so that graph is trapped

Then $f(x)$ is in here

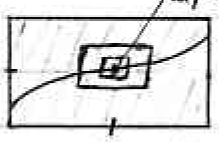


If x is in here

<u>Idea</u> Regardless of how small the tolerance, ϵ, is ($\epsilon > 0$), we can <u>always</u> find a "δ" ($\delta > 0$) that ($x \neq a$) makes $(a-\delta, a+\delta)$ a WI ($x \neq a$). "Trap the graph." (We can always find winners (x) immediately around a.)

1, .1, .01, ...

As ϵ shrinks



(may not be on graph)

We tend to shrink our δ (Move in walls.)

Maybe δ was already pretty small...

<u>(C) One-Sided Limits</u>

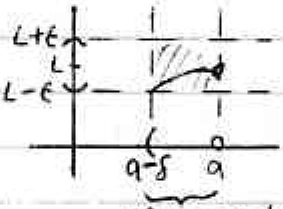
$\lim_{x \to a^-} f(x) = L$ means

$\lim_{x \to a^+} f(x) = L$

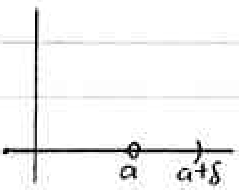
$\forall \epsilon > 0, \exists \delta > 0$ such that

δ = full-width, not $\frac{1}{2}$-width

Then $f(x)$ is in here
$(L-\epsilon, L+\epsilon)$



If x is in here
$(a-\delta, a)$ only modification

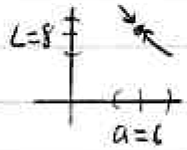


not cooked-up problems! Does work out nicely for linear.

(II) Proving a Limit Statement

Ex (#20) Prove $\lim_{x \to 6} (9 - \frac{1}{6}x) = 8$

① Let $a = 6$, $f(x) = 9 - \frac{1}{6}x$, $L = 8$



Show: $\forall \epsilon > 0, \exists \delta > 0$ such that
if $0 < \underbrace{|x-6|}_{☆☆} < \delta$, then $\underbrace{|f(x) - 8|}_{☆} < \epsilon$

☆☆ How far x is from 6; if a player x is in here, then ☆ (S)he wins

(If x is close enough to 6, then f(x) is close enough to 8.)

② Rewrite ☆ in terms of ☆☆

$$
\begin{aligned}
|f(x) - 8| &= |(9 - \tfrac{1}{6}x) - 8| \\
&= |-\tfrac{1}{6}x + 1| \\
&= |-\tfrac{1}{6}(x-6)| \\
&= |-\tfrac{1}{6}||x-6| \\
&= \tfrac{1}{6}|x-6|
\end{aligned}
$$

$|pq| = |p||q|$

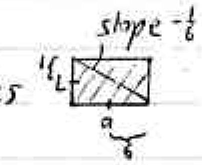
Need $< \epsilon$

③ Assuming ϵ is fixed $(\epsilon > 0)$,
Choose an appropriate δ.

Need
$\frac{1}{6}|x-6| < \epsilon$
$|x-6| < 6\epsilon$

Choose $\delta = 6\epsilon$. (Any smaller δ (>0) also works.)

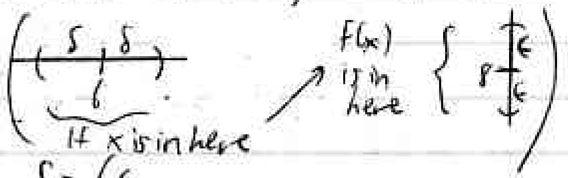
Ex If $\epsilon = 1 \Rightarrow \delta = 6$ works
$.1 \quad .6$



④ Verify that our δ works.

If x is close enough to 6, then f(x) is close enough to 8.

Show: If $0 < |x-6| < \delta$, then $|f(x)-8| < \epsilon$



Whatever ε is, make $\delta = 6\cdot$ that

Choose $\delta = 6\epsilon$

$$0 < |x-6| < \delta$$
$$\Rightarrow 0 < |x-6| < 6\epsilon$$
$$\Rightarrow 0 < \tfrac{1}{6}|x-6| < \epsilon \qquad (\div 6)$$

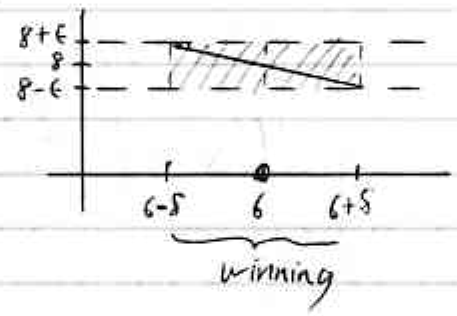
$= \text{by } ②$

$$\Rightarrow 0 < |f(x)-8| < \epsilon$$

quod erat demonstrandum – that which was to be proved (shown)

QED (end of proof)

Note



$\delta = 6\epsilon$ works

Slope argument $= -\frac{1}{6}$ $\quad 6\epsilon$ $\quad -\epsilon$