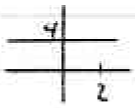
2.3: FINDING LIMITS

a, c real #s

Ⓐ Review 2.1 Ⓐ

Ex Find $\lim_{x \to 2} 4 = 4$



① $\lim_{x \to a} c = c$

As $x \to 7$, $x \to 7$

Ex Find $\lim_{x \to 7} x = 7$

② $\lim_{x \to a} x = a$

Ex Find $\lim_{x \to 2} (3x^2 - 5x + 1)$

need

$$= 3(2)^2 - 5(2) + 1$$
$$(= 12 - 10 + 1)$$
$$= 3$$

③ $\lim_{x \to a} \underbrace{f(x)}_{\text{polynomial}} = f(a)$

Plug in $x=a$ and evaluate!

Ex Find $\lim_{x \to 5} \frac{x^2}{x-3}$

$$= \frac{(5)^2}{5-3} \quad \leftarrow \text{not } 0$$

$$= \frac{25}{2}$$

④ $\lim_{x \to a} \underbrace{f(x)}_{\text{rational}} = f(a)$

if a is in Dom(f)

SKIP

why bother?

Alternate 4

~~If $f(x) = \frac{n(x)}{d(x)}$ ← polys (maybe constants), and $d(a) \neq 0$, then~~

~~$\lim_{x \to a} f(x) = f(a)$~~

④ covers ①-③

(B) Properties of Limits

If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then

(1) $\lim\limits_{x \to a} [f(x)+g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

limit of sum = sum of limits

(2) $\quad - \quad\quad -$

diff. diff.

(3) $\quad \cdot \quad\quad \cdot$

product product

(4) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} \leftarrow$ <u>if</u> $\neq 0$

limit of quotient = quotient of limits

(5) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n, \quad (n=1,2,3,\ldots)$

limit of power = power of limit

Back in alg. what kinds of radicals gave us trouble?

(6) $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\underbrace{\lim\limits_{x \to a} f(x)}_{\text{if } >0 \text{ if } n \text{ is even}}}, \quad (n=2,3,4,\ldots)$

limit of root = root of limit

WARNINGS: $\lim\limits_{x \to -1} \sqrt{x}$ DNE $\quad \sqrt{-1}$ not real!

$\lim\limits_{x \to 0} \sqrt{x}$ DNE, because $\lim\limits_{x \to 0^-} \sqrt{x}$ DNE

can't

$$⑦ \lim_{x\to a} [cf(x)] = c\left[\lim_{x\to a} f(x)\right]$$

constant factors "pop out"

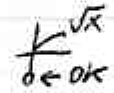
True of P_x, S

①, ②, ⑦ ⇒ $\lim_{x\to a}$ "linear"

Also true for $\lim_{x\to a^-}$, $\lim_{x\to a^+}$ <u>except</u>

If I'm confusing, which of these 7 ~~lists~~ props should I take another look at?

Modify ⑥: <u>Ex</u> $\lim_{x\to 0^+} \sqrt{x} = 0$



Note (<u>Ex</u> $\lim_{x\to 0^-} \sqrt{-x} = 0$

etc.)

Don't say "plus roots" My ex gives wrong idea: + root p.20 rules are "nice"

<u>Algebraic functions</u>: rational funcs., except roots involving x OK.

(f is an <u>algebraic function</u> if $f(x)$ only consists of x, constants, $+, -, \cdot, \div$, and/or constant rational exponents.)

<u>Ex</u> $\sqrt{x}$ $(= x^{1/2})$ OK

<u>Ex</u> 2^x, $\sin x$ NOT

2^x not, though it's also a func. we plug into to find limits

If f is algebraic, then

$\lim_{x\to a} f(x) = f(a)$ <u>if</u> a is in $Dom(f)$,

i.e., $f(a)$ is a real #.

(If "plugging in" $x=a$ gives us a real #, that's the limit!)

WARNING: The limit ~~may~~ may or may not exist if "even roots" are involved.

Beware $\sqrt{0}$, $\sqrt[4]{0}$, etc.!!

Ex Find $\lim_{x\to 2}\underbrace{\left[\frac{(x^2+1)(3x-9)^2}{x}+\sqrt{x+3}\right]}_{\text{algebraic } f(x)}$

Plug in $x=2$

$$f(2)=\frac{((2)^2+1)(3(2)-9)^2}{2}+\sqrt{2+3}$$

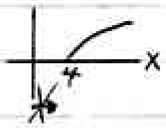
$$=\left(\frac{(5)(-3)^2}{2}+\sqrt{5}\right)$$

$$=\frac{45}{2}+\sqrt{5}$$ >0 $\leftarrow$ real #

not 0 (no $\sqrt{0}$ (issues))

The limit is $\frac{45}{2}+\sqrt{5}$

Ex $\lim_{x\to 4}\sqrt{x-4}$ DNE



The lim is 0 if x→4 how?

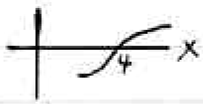
Ex $\lim_{x\to 4^+}\sqrt{x-4}=0$

Ex $\lim_{x\to 1}\sqrt{x-4}$ DNE

$\sqrt{1-4}=\sqrt{-3}$ not real!

Ex $\lim_{x\to 4}\sqrt[3]{x-4}=\sqrt[3]{4-4}=0$

odd, so any real radicand is OK



Ex $\lim\limits_{x\to 1^+} \underbrace{\sqrt{3x^2-3}}_{}$

$f(1)=\sqrt{0}=0\ldots$

BEWARE!

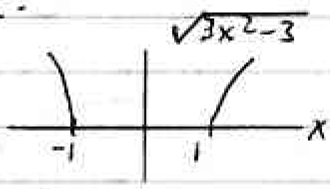
0 or DNE?

Is $\sqrt{3x^2-3}$ defined as $x\to 1^+$?

YES, because $3x^2-3>0$ as $x\to 1^+$ $(x>1)$

Answer $=0$

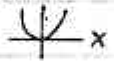
Turns out:



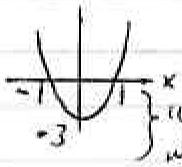
radicand never neg.

(If time)

$3x^2$



$3x^2-3$



"Dies" when $\sqrt{\ }$

what does $|x|$ look like? What happens if $x \leftarrow x-7$?

Ex $\lim\limits_{x\to 7} \underbrace{\sqrt{(x-7)^2}}_{=|x-7|} = \boxed{0}$

Ex $\lim\limits_{x\to 1^+} (7\sqrt{3x^2-3}+5)$

$= \lim\limits_{x\to 1^+} 7\sqrt{3x^2-3} + \lim\limits_{x\to 1^+} 5$

$= 7\underbrace{\left(\lim\limits_{x\to 1^+}\sqrt{3x^2-3}\right)}_{=0} + 5$

Up to 15

$= 5$

I said ①, ②, ⑦ ⇒ linear

("$\lim\limits_{x\to\ldots}$" is linear: ① We can take limits term-by-term.
② Constant factors pop out.
(if the limits exist))

(c) The $\frac{0}{0}$ Problem

(Please do not) Can't cancel the "x"'s

Prelim. Ex $\lim_{x \to 1} \frac{x+1}{x-1} \begin{matrix} \to 2 \\ \to 0 \end{matrix}$ DNE bec. $\not\to$ a real #

Before, "simplified rat'l expr. $x=1 \Rightarrow 0 = 0$" but what about $\lim_{x \to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$ not rational, but still "simp." may be misleading

As we approach 1 from the right, we get $\frac{\approx 2}{.1} \approx 20$, $\frac{\approx 2}{.01} \approx 200$, $\vdots$

Turns out:

x=1 VA (Vertical Asymptote)

$1^+ \leftarrow$

x	$\frac{x+1}{x-1}$
1.1	$\frac{2.1}{.1} = 21$
1.01	$\frac{2.01}{.01} = 201$
1.001	$\frac{2.001}{.001} = 2001$
	$\vdots$

$\frac{\to \text{a non-0 \#}}{\to 0}$ DNE for now (in 2.4, we'll discuss $\infty, -\infty$)

Not necess. DNE!

Ex $\lim_{x \to 1} \frac{x^2-1}{x^2-x} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$ Indeterminate form -need more work!

$x=1$ is a zero (root) of both num., denom. (polys.), so $(x-1)$ is a factor of both.

$= \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}$ cancel ($x \neq 1$ as we take $\lim_{x \to 1}$)

$= \lim_{x \to 1} \underbrace{\frac{x+1}{x}}_{\text{rat'l, 1 in domain, so plug in } x=1}$

$= \frac{1+1}{1}$

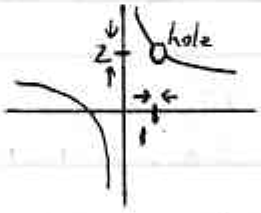
$= 2$

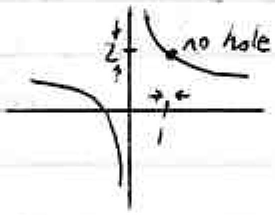
Up to 35

Do right 1st

Note $\dfrac{x^2-1}{x^2-x}$ $\qquad$ $\dfrac{x+1}{x}$

Exact same graph, except we fill in hole (Sprinkle)





We technically aren't supposed to care what happens at 1. Up to 35

$\lim\limits_{x\to 1}$ is 2 for both.

we may as well study the one w/out the hole

Ex $\displaystyle\lim_{x\to 0} \frac{\sqrt{9-x}-3}{x} \to \frac{0}{0}$

Also, rat'lize the den.

Rationalize the num.

$$= \lim_{x\to 0} \frac{(\sqrt{9-x}-3)}{(x)} \cdot \underbrace{\frac{(\sqrt{9-x}+3)}{(\sqrt{9-x}+3)}}_{\left(\begin{array}{l} =1 \text{ if } x\le 9 \\ \text{near } x=0 \text{ (in particular)}\end{array}\right)}$$

We only care what happens around 0, anyway.

$$= \lim_{x\to 0} \frac{(9-x)-9}{x(\sqrt{9-x}+3)} \qquad (A-B)(A+B)=A^2-B^2$$

$$= \lim_{x\to 0} \frac{\cancel{-x}^{-1}}{\cancel{x}(\sqrt{9-x}+3)}$$

$$= \lim_{x\to 0} \underbrace{\frac{-1}{\sqrt{9-x}+3}}_{\text{(algebraic; try plugging in } x=0)}$$

Remove lim when do direct sub

$$= \frac{-1}{\sqrt{9-0}+3}$$

Up to 51

$$= -\frac{1}{6}$$

Fixing "$\frac{0}{0}$"

Factor, Cancel
Rationalize num./den.
L'Hôpital's Rule (Math 151, 10.1)

① Sandwich / Squeeze Theorem

You want to show s-thing about h(x).

Suppose $\underset{(\text{low})}{f(x)} \le \underset{(\text{mid})}{h(x)} \le \underset{(\text{high})}{g(x)}$

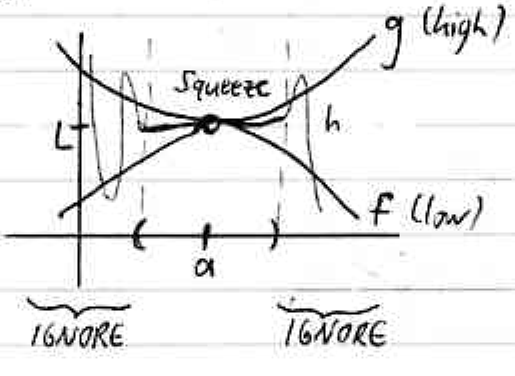
for every x in an open interval containing a, except maybe at a, itself.

If $\lim_{x \to a} \underset{(\text{low})}{f(x)} = L$ ← real #,

and $\lim_{x \to a} \underset{(\text{high})}{g(x)} = L$,

then $\lim_{x \to a} \underset{(\text{mid})}{h(x)} = L$.

Can go wild



<u>Ex</u> Use the Sandwich / Squeeze Thm. to prove that

$$\lim_{x \to 0} x^2 \cos\left(\tfrac{1}{x}\right) = 0.$$

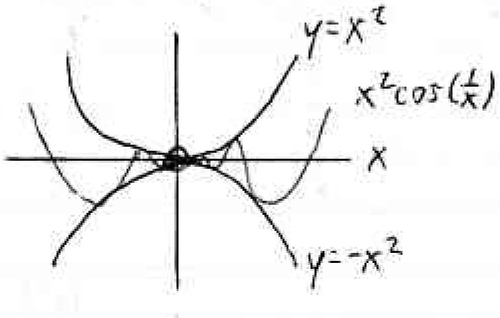
<u>Sol'n</u>

$$-1 \le \cos\left(\tfrac{1}{x}\right) \le 1 \qquad (\forall x \ne 0)$$

What's the problem if we mult. by a neg qty? If we mult. by a qty. that's sometimes +, s-times − ⇒ diff cases?

$x^2 > 0$ $(\forall x \ne 0)$, so we can mult. through by x^2 w/out flipping $\le \to \ge$

$$\underbrace{-x^2}_{\text{As } x \to 0 \,\downarrow\, 0} \le \underbrace{x^2 \cos\left(\tfrac{1}{x}\right)}_{\text{So, } \downarrow\, 0} \le \underbrace{x^2}_{\downarrow\, 0} \qquad (\forall x \ne 0)$$



<u>Ex</u> Prove that $\lim_{x \to 0} x^3 \sin(\frac{1}{x}) = 0$.

<u>Sol'n</u>

$$-1 \le \sin(\tfrac{1}{x}) \le 1 \qquad (\forall x \ne 0)$$

$$|\sin(\tfrac{1}{x})| \le 1 \qquad \text{"} \quad \text{"}$$

Note: $|x^3| > 0 \quad (\forall x \ne 0)$

$$|x^3||\sin(\tfrac{1}{x})| \le |x^3| \qquad \text{"} \quad \text{"}$$

$$|x^3 \sin(\tfrac{1}{x})| \le |x^3|$$

Note: $|☆| \le 4$
$\Leftrightarrow -4 \le ☆ \le 4$

$$\underbrace{-|x^3|}_{\text{As } x \to 0 \;\downarrow\; 0} \le \underbrace{x^3 \sin(\tfrac{1}{x})}_{\text{So, } \downarrow\; 0} \le \underbrace{|x^3|}_{\downarrow\; 0} \qquad (\forall x \ne 0)$$