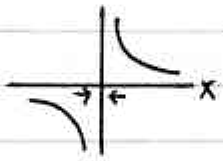
# 2.4: LIMITS INVOLVING INFINITY ($\infty$)

In 3.2, helps for DNE to incl. $\pm\infty$

## (A) $\lim_{x \to a} f(x)$

Ex $f(x) = \frac{1}{x}$



$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

not a real #
Special case of DNE

reciprocal func.

| $x$ | $f(x) = \frac{1}{x}$ |
|---|---|
| $\frac{1}{10}$ | $10$ |
| $\frac{1}{100}$ | $100$ |
| $\vdots$ | $\vdots$ |
| $\downarrow$ $0^+$ | $\downarrow$ $\infty$ |

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

| $x$ | $f(x) = -\frac{1}{x}$ |
|---|---|
| $-\frac{1}{10}$ | $-10$ |
| $-\frac{1}{100}$ | $-100$ |
| $\vdots$ | $\vdots$ |
| $\downarrow$ $0^-$ | $\downarrow$ $-\infty$ |

$\lim_{x \to 0} \frac{1}{x}$ DNE

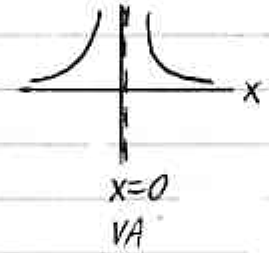
can't even say $\infty$ or $-\infty$

Can you think of Ex where $\lim = \infty$

$\left|\frac{1}{x}\right|$

<u>Ex</u> $f(x) = \frac{1}{x^2}$

$\frac{1}{x^2} > 0$, even




$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

<u>Ex</u> Express each limit as $\infty, -\infty$, or DNE (can't even say $\infty$ or $-\infty$).

Indicate convincingly how you obtained your answer
! They can plug in calculator
forbid calcs?
but how!

ⓐ $\lim_{x \to -4^-} \frac{3}{x^2+4x}$ ← Factor

$$= \lim_{x \to -4^-} \frac{3}{\underbrace{x}_{\to -4}\underbrace{(x+4)}_{\to 0^-}} \quad \begin{matrix} \to 3 \\ \to 0^+ \end{matrix}$$

$x+4 \to 0$ but stays neg

$= \infty$

ⓑ $\lim_{x \to -4^+} \frac{3}{x^2+4x}$

$$= \lim_{x \to -4^+} \frac{3}{\underbrace{x}_{\to -4}\underbrace{(x+4)}_{\to 0^+}} \quad \begin{matrix} \to 3 \\ \to 0^- \end{matrix}$$

$= -\infty$

ⓒ $\lim_{x \to -4} \frac{3}{x^2+4x}$ DNE

can't say $\infty$ or $-\infty$

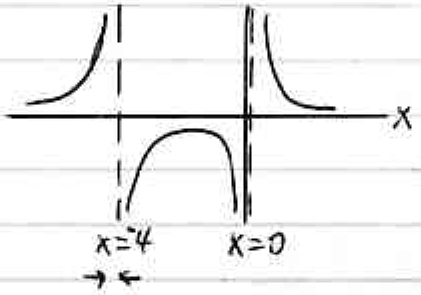
Can make hole at 3: $\cdot\frac{(x-3)}{(x-3)}$

<u>Note</u> $f(x)=\left(\frac{3}{x^2+4x}\text{, or}\right)\underbrace{\frac{3}{x(x+4)}}_{\text{simplified rational expr.}}$

und. at $x=0,-4$

$\Rightarrow$ VAs at $x=0,-4$

<u>Turns out</u>




<u>Ex</u> ⓐ $\lim_{x\to 2^-} \frac{4x}{\underbrace{(3x-6)^2}_{0^-}} \begin{matrix}\to 8 \\ \to 0^+\end{matrix}$ Squares are never "−."

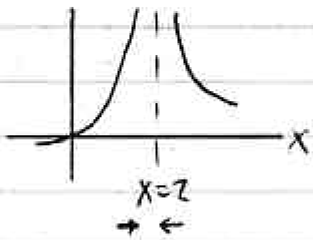
$=\infty$

ⓑ $\lim_{x\to 2^+} (\quad)$

$=\infty$

ⓒ $\lim_{x\to 2} (\quad) = \infty$

<u>Turns out</u>

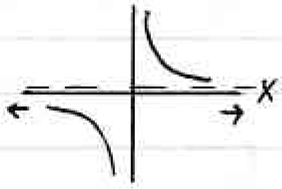



up to 9

## (B) $\lim_{x \to \pm\infty} f(x)$

More interesting Ex in (D)

**Ex** $f(x) = \frac{1}{x}$




← $y=0$ is a Horizontal Asym. (HA), bec.

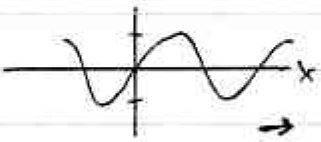
$+ \lim_{x \to \infty} \frac{1}{x} = 0$

| $x$ | $f(x) = \frac{1}{x}$ |
|---|---|
| 10 | $\frac{1}{10}$ |
| 100 | $\frac{1}{100}$ |
| ⋮ | ⋮ |
| ↓ | ↓ |
| $\infty$ | 0 |

$+ \lim_{x \to -\infty} \frac{1}{x} = 0$

ε=1, 1 OK, but what of ε=½, ¼?

**Ex** $\lim_{x \to \infty} \sin x$ DNE $\quad \not\to$ a #




**Ex** $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

Squeeze Thm extends to $x \to \infty$ Trapped in tunnel


Squeeze Thm (modified for $\lim_{x \to \infty}$)

$$-1 \le \sin x \le 1$$

$$(x>0) \quad \frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

As $x \to \infty$: $\frac{-1}{x} \to 0$, $\frac{1}{x} \to 0$, So, $\frac{\sin x}{x} \to 0$

## Definition (Memorize)

f is eventually defined forever thataway

Assume $f$ is defined on some interval $(c, \infty)$. $\longrightarrow$

"$\lim_{x \to \infty} f(x) = L$" means that
↑ a real #

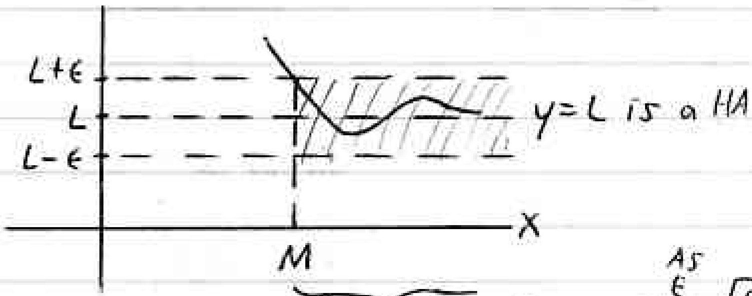
Think M for Million
Point of no return

$\forall \epsilon > 0, \exists M > 0$ such that

if $\underbrace{x > M}_{\text{"you go out far enough"} \rightarrow}$, then $\underbrace{f(x) \text{ is in } (L-\epsilon, L+\epsilon)}_{\text{i.e., } |f(x)-L| < \epsilon}$

"everyone wins"

As I shrink $\epsilon$, I can still find an M (a pt. of no return) after which the graph must stay in bet. the lines.




(or M can be higher)

δ can be smaller than what you have

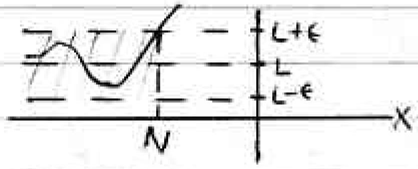
As ϵ shrinks $\boxed{\epsilon}$
M →
Have to move wall farther out to trap the graph

## Modifications for $\lim_{x \to -\infty} f(x) = L$

$f$ defined on $(-\infty, c)$ $\longleftarrow$

Think N for big Negative

$\forall \epsilon > 0, \exists N < 0$ s.t. if $x < N$ ...



## © $\lim\limits_{x \to \pm\infty} \frac{c}{x^k}$ ← c: real # ← k: +, rat'l #

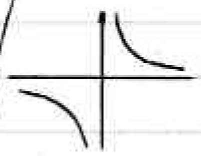
$\frac{\text{real \#}}{\pm\text{-thing blowing up or down}}$

Exs $\frac{1}{x}$, $\frac{3}{x^2}$, $\frac{-5}{x^{3/4}}$

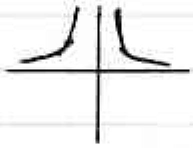
SKIP

What quadrants?

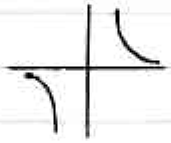
$\frac{1}{x}$



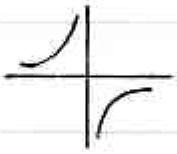
$\frac{1}{x^2}$



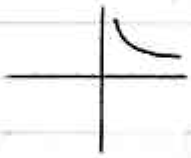
$\frac{1}{x^3}$



$-\frac{1}{x}$



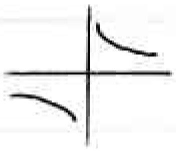
$\frac{1}{x^{1/2}}$, or $\frac{1}{\sqrt{x}}$



$\underbrace{\hphantom{xxxxxx}}$

$\sqrt{x}$, or $x^{1/2}$
und
for $x < 0$

$\frac{1}{x^{1/3}}$, or $\frac{1}{\sqrt[3]{x}}$



For all, $\lim\limits_{x \to \infty} f(x) = 0$

$\lim\limits_{x \to -\infty} f(x) = 0$ if $f(x)$ is defined for $x < 0$

$\lim\limits_{x \to \infty} \frac{c}{x^k} = 0$ and

$\lim\limits_{x \to -\infty} \frac{c}{x^k} = 0$ if $x^k$ is defined for $x < 0$
(otherwise, DNE) Ex $\lim\limits_{x \to -\infty} \frac{1}{x^{1/2}} = \lim\limits_{x \to -\infty} \frac{1}{\sqrt{x}}$ DNE

## (D) Exs

as opposed to factored form

Ex Find $\lim\limits_{x\to\infty} \dfrac{4x^3+x-1}{5x^3-x}$ $\begin{matrix}\leftarrow N(x)\\ \leftarrow D(x)\end{matrix}$ ] polys in standard form

Trick: ÷ each term in $N, D$ by the highest power of $x$ in $D$.

$$= \lim_{x\to\infty} \frac{\frac{4x^3}{x^3}+\frac{x}{x^3}-\frac{1}{x^3}}{\frac{5x^3}{x^3}-\frac{x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4+\frac{1}{x^2}-\frac{1}{x^3}}{5-\frac{1}{x^2}} \quad \begin{matrix}\to 4\\ \to 5\end{matrix}$$

$$= \frac{4}{5}$$

Show!

Why does this make sense?
Use only as VS.

Note 1 For extreme values of $x$,

As $x \to \pm\infty$ $4x^3$ dominates $N$. These terms die out relative to the $N$.

$$\frac{4x^3+x-1}{5x^3-x} \approx \frac{4\cancel{x^3}}{5\cancel{x^3}} \quad \text{(Informally)}$$

Note 2 (Suppose $f(x)$ is rational.)

$$f(x) = \frac{N(x)}{D(x)}$$

L2-13b Used HA

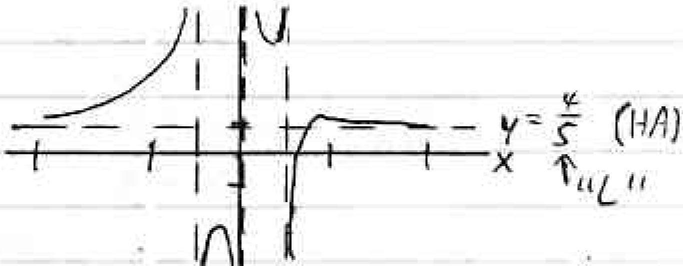
Case 1 If $\deg(N) = \deg(D)$, then the graph of $f$ has a HA

at $y = \underbrace{\text{ratio of leading coeffs}}_{\text{"}L\text{"}}$,

and $\lim_{x\to\infty} f(x) = L$
and $\lim_{x\to -\infty} f(x) = L$ $\Big\} \Rightarrow$ "$y=L$" is a HA.

Turns out here $f(x) = \dfrac{4x^3+x-1}{5x^3-x}$

$5x^3-x=0$
$x(5x^2-1)=0$
$x=0, \pm\sqrt{\frac{1}{5}}$
$\sqrt{\frac{1}{5}} \approx .447$
or $\pm \frac{\sqrt{5}}{5}$




Why shouldn't we be surprised that there are 3 VAs?

The zeros/roots of $D(x) = 5x^3 - x$ are $-\frac{\sqrt{5}}{5}, 0, \frac{\sqrt{5}}{5}$

None make $N(x)=0$, so VAs.
If $\frac{0}{0} \Rightarrow$ hole? VA?

Otherwise, we have a $\frac{0}{0}$ situation and we may have a hole.

Case 2 If $\deg(N) < \deg(D)$, then $L=0$. $\leftrightarrow x$ (HA)
Ex $\frac{1}{x}$


Ex $\frac{x}{x^2+1}$
Denom crushes the num. as $x \to \pm\infty$

Case 3 If $\deg(N) > \deg(D)$, then there is no HA.

If you zoom out...

Ex $\frac{5^{th}\text{-deg}}{3^{rd}\text{-deg}} \xrightarrow{\text{long} \div} \underbrace{(2nd\text{-deg})}_{\text{determines "long-run" behavior as } x\to\infty,-\infty} + \underbrace{(\text{proper})}_{\text{decays} \to 0 \text{ as } x\to\pm\infty}$

Zoom out on graphing calc: $\boxed{\cup}$

Ex Find $\lim\limits_{x\to-\infty} \dfrac{4-2x^3}{3x^2+1} \begin{matrix}\to\infty \\ \to\infty\end{matrix}$  $\frac{\infty}{\infty}$ is an indeterminate form - need more work! like $\frac{0}{0}$

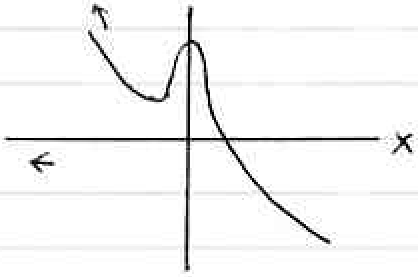
Trick not as good for $\lim\limits_{x\to 4}$ for ex.

$$= \lim_{x\to-\infty} \frac{\frac{4}{x^2} - \frac{2x^3}{x^2}}{\frac{3x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{\overset{\to 0}{\frac{4}{x^2}} - \overset{\to -\infty}{2x}}{3 + \underset{\to 0}{\frac{1}{x^2}}} \begin{matrix}\to\infty \\ \to 3\end{matrix}$$

$$= \infty$$

Turns out




Ex $\dfrac{\to 3}{\to \pm\infty} \to 0$

Ex $\dfrac{\to\infty}{\to(-3)} \to -\infty$