2.5: CONTINUOUS FUNCTIONS

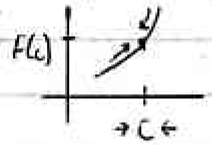
 (Don't lift pencil).

Ⓐ Continuity at a Point (or "at $x=c$")

③ meaningless w/out ①②

f is continuous at c $\Leftrightarrow$ ① $f(c)$ is defined, ("There is literally a point.")

Ex



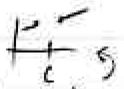
② $\lim_{x \to c} f(x)$ exists,

and ③ $\lim_{x \to c} f(x) = f(c)$ (They're equal.)

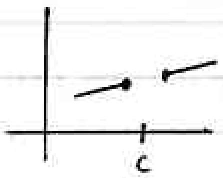
Plug in $x=c$

Otherwise, discontinuous at c.

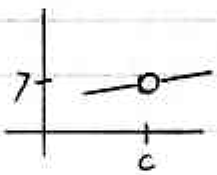
Exs



Some books don't say discont!
Speed-bar
Some don't even talk about discont.
we have A GAP!!

① 

$f(c)$ is undefined

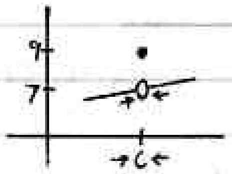
② 

Hole! $f(c)$ is und.

removable discontinuity

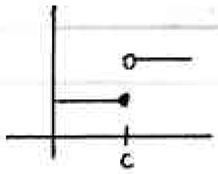
can remove by defining $f(c)=7$.

Of course, you can't change a func. bec. you feel like it!

(3) 

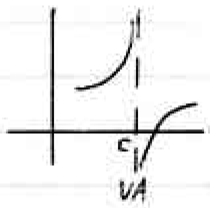
$\underbrace{\lim_{x\to c} f(x)}_{7} \neq \underbrace{f(c)}_{9}$

<u>Removable</u> D. at c
if redefine
f(c) to be 7

(4) 

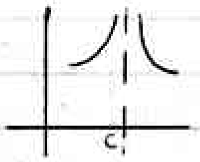
$\lim_{x\to c} f(x)$ DNE

<u>Jump</u> D. at c

(5) 

f(c) und.

<u>Infinite</u> D. at c

(6) 

f(c) und.

Also, $\lim_{x\to c} f(x) = \infty$ not real #
(special case of DNE)

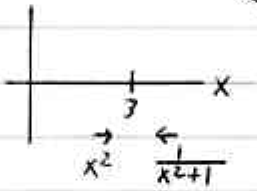
<u>Infinite</u> D. at c

at every real #c (special care — A poly. func is everywhere cont. (on $\mathbb{R}$)

$\lim\limits_{x\to c} f(x)=f(c)$ — A rational func. is cont. on its domain.

Ex $f(x)=\begin{cases} x^2 & ,\text{if } x\le 3 \\ \frac{1}{x^2+1} & ,\text{if } x>3\end{cases}$ $\left(\begin{array}{l}\leftarrow x^2 \text{ is cont. on } (-\infty,3) \\ \leftarrow \frac{1}{x^2+1} \text{ is cont. on } (3,\infty)\end{array}\right)$

Is f cont. at 3? NO

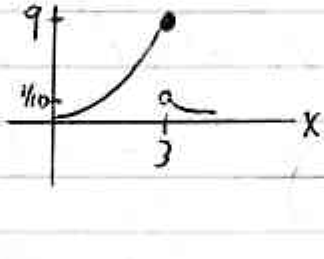


$\left\{\begin{array}{l}\lim\limits_{x\to 3^-} f(x) = (3)^2 = 9 \\ \lim\limits_{x\to 3^+} f(x) = \frac{1}{(3)^2+1} = \frac{1}{10}\end{array}\right.$

3.1, 3.01, ... $\frac{1}{x^2+1}$ rule applies. Not the rule at 3, but so what?

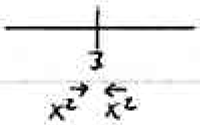
$\Rightarrow \lim\limits_{x\to 3} f(x)$ DNE

Jump D.



What's the problem here? People unsure!

<u>Ex</u> $f(x)=\begin{cases} x^2, & \text{if } x\neq 3 \\ 10, & \text{if } x=3 \end{cases}$ ($\leftarrow x^2$ is cont. on $(-\infty,3)\cup(3,\infty)$)



lim ≠ func. value

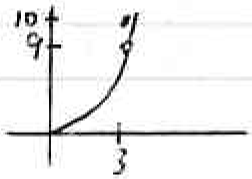
$\lim\limits_{x\to 3} f(x)=(3)^2=9$

<u>BUT</u> $f(3)=10$

$\neq$

f is discont. at 3.

Removable D.



Up to 33

<u>(B) When is f cont. on a closed interval $[a,b]$?</u>

If ① f is defined on $[a,b]$

② f is cont. on (a,b) (i.e, cont. at all x in (a,b))

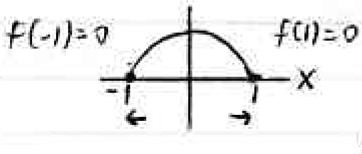
③ $\lim\limits_{x\to a^+} f(x)=f(a)$

④ $\lim\limits_{x\to b^-} f(x)=f(b)$

$x^2+y^2=1$
$y^2=1-x^2$
$y=\pm\sqrt{1-x^2}$

<u>Ex</u> If $f(x)=\sqrt{1-x^2}$, then f is cont. on $[-1,1]$.

left vs. right
I can't dance, play together for chi



① f is defined on $[-1,1]$.

② f is cont. on $(-1,1)$.

③ $\lim\limits_{x\to -1^+} f(x)=f(-1)$ $(=0)$, ④ $\lim\limits_{x\to 1^-} f(x)=f(1)$ $(=0)$

electroder   Ex ③,④ fail:

© Continuity Thms.

If f and g are cont. at c, then so are $f \pm g$ and $\frac{f}{g}$ ← if $g(c) \neq 0$

Don't just say 10 and go home
Find all #s where f is cont.
but -4 strange

Ex $f(x) = \dfrac{x\sqrt{x+4}}{x-5}$ Where is f cont?

Don't say intersection
$\sqrt{x-4} + \sqrt{4-x}$
$x \geq 4 \quad x \leq 4$

x is cont. on $\mathbb{R}$
$\sqrt{x+4}$ is cont. on $[-4,\infty)$
$x-5$ is cont. on $\mathbb{R}$
Need $x-5 \neq 0$
$x \neq 5$

-4 5 x

[because we can

$$[-4,5) \cup (5,\infty)$$

$\lim\limits_{x\to c} f(g(x)) = f(\lim\limits_{x\to c} g(x))$ if ∃ and f cont. here
$= f(g(c))$ if g cont. at c

If g is cont. at c, and f ... $g(c)$,
then $f \circ g$ is cont. at c.

Really a domain?

$\underline{Ex}$ $f(x) = \sec(\frac{1}{x})$. Where is f cont.?

$\frac{1}{x}$ is cont. at all $x \neq 0$.

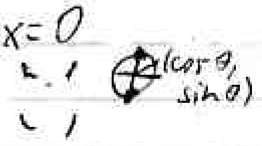
For what x is $\sec(\frac{1}{x})$ undefined? (Cont. elsewhere)

In trig, "0 and und. are recips." – dangerous but usu. works!

can take recips. $\neq 0$

$$\sec(\tfrac{1}{x}) \text{ is und.} \iff \cos(\tfrac{1}{x}) = 0 \text{ or } x = 0$$

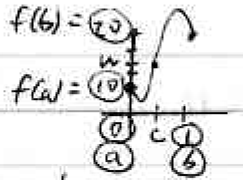
$$\iff \frac{1}{x} = \frac{\pi}{2} + \pi n,\ n \text{ int.}$$



$$\iff x = \frac{1}{\frac{\pi}{2} + \pi n}$$

$$\iff x = \frac{2}{\pi + 2\pi n},\ n \text{ int. or } x = 0$$

$\{x : x \neq 0,\ x \neq \frac{2}{\pi + 2\pi n}\ (n \text{ int.})\}$



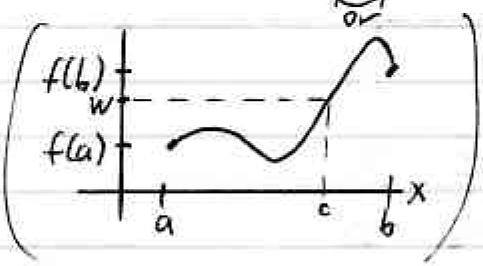
① Intermediate Value Thm. (IVT)

If I have a cont. func. $f(0)=10$, $f(1)=20$. Does f have to be 15? What does IVT say?

If f is cont. on $[a,b]$, then f takes on every value between $f(a)$ and $f(b)$.
(i.e., $\forall w$ in $[f(a), f(b)]$, $\exists c$ in $[a,b]$ such that $f(c)=w$)

We'll see thms like this in 4
"Don't have to write"

or



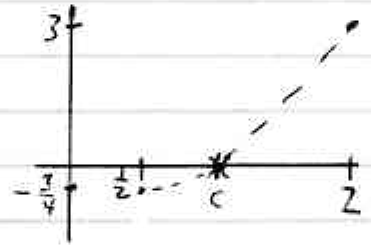
What conditions can be placed on f(a), f(b) that guarantee ...

Special Case If f is cont. on $[a,b]$, and $f(a)$ and $f(b)$ have opposite signs, then f has a zero in (a,b).

Ex $f(x) = x^2 - 1$ $\quad \left(\begin{array}{l} x^2-1=0 \\ x=\pm 1 \leftarrow \text{zeros/roots} \end{array} \right)$

$$f(\tfrac{1}{2}) = (\tfrac{1}{2})^2 - 1 = -\tfrac{3}{4}$$
$$f(2) = (2)^2 - 1 = 3$$

So, $\exists c$ in $(\frac{1}{2}, 2)$ such that $f(c) = 0$.



In real world, you don't use factoring you use approx. algors.

seeds

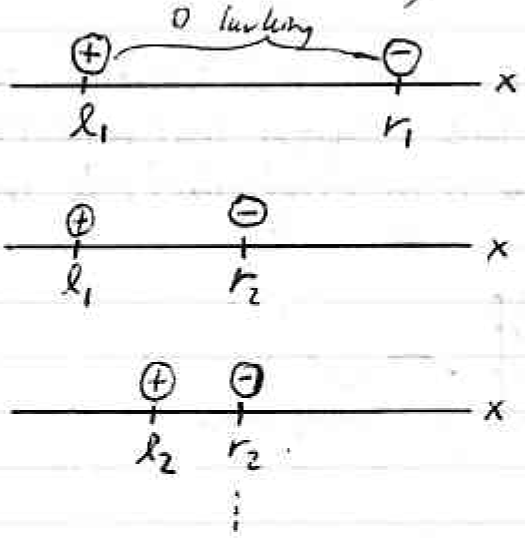
hide-and-seek

I have a # (int) ve bet. 0,100
You guess
You say 50
I say lower
You say 25

You're outta luck

root is bracketed in (a, b)
Recipes 350

Bisection Method for finding a zero of a cont. f



Heck w/ it - take the midpt.

(roundoff error?)

Golub 151 could give wrong sign!

164 $|f(x_i)| < \epsilon$
vs. $|x_{i+1} - x_i| < \epsilon$
No definitive ans.

bad if flat near root
e.g., multiple root

STOP