3.2: DEFINITION OF DERIVATIVE

(A) f' and Its Domain

If f is a function, its <u>derivative</u> is the function f' defined by

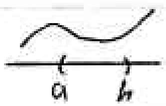
$a \leftarrow x$ now 3.1

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ "Slope function"

if it exists

Then, f is <u>differentiable</u> ("diff'e") at x.

f is diff'e on $(a,b) \Leftrightarrow$
$f'(x)$ exists $\forall x$ there.



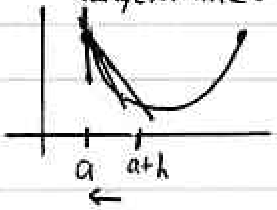
f is diff'e on $[a,b] \Leftrightarrow$

① f is diff'e on (a,b)

② $\displaystyle\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ exists

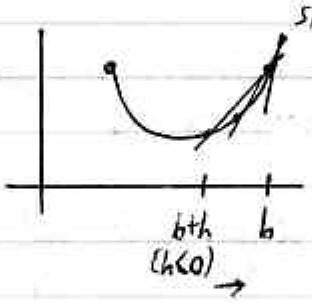
tangent line's slope = <u>right-hand deriv.</u> (m_R) at a

Don't do ½-circle: b/t vertical tan lines



$h \to 0$ how?

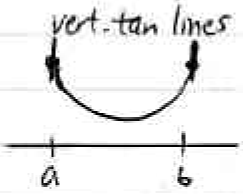
and ③ $\lim\limits_{h \to 0^-} \dfrac{f(b+h)-f(b)}{h}$ exists



f is diff'e at a here ←
and at b here ↗

The domain of f' consists of all #s at which f is diff'e and endpoints (a or b) where the appropriate one-sided limit exists.

NOT diff'e on $[a,b]$:



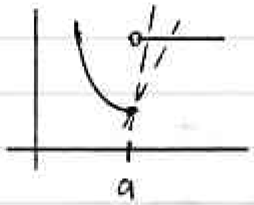
$\frac{1}{2}$-circle

When animals attack...

(B) When $f'(a)$ DNE

includes $\pm\infty$

(1)



$f'(a)$ DNE

Pf. p.103

(If f is diff'e at a, then f is cont. at a.) $\underset{\text{at } a}{D} \Rightarrow \underset{\text{at } a}{C}$

Logically equiv.

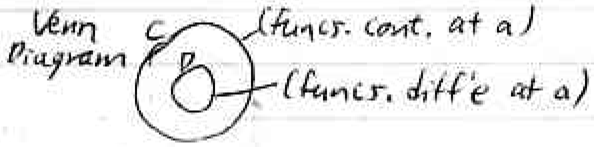
Its contrapositive: $\text{not} \underset{\text{at } a}{C} \Rightarrow \text{not} \underset{\text{at } a}{D}$

If Pat is daddy, then Pat is a man

(If f is not cont. at a, then f is not diff'e at a.)

In (2)-(5), f is cont. at a, but $f'(a)$ DNE.

$\underset{\text{at } a}{C} \not\Rightarrow \underset{\text{at } a}{D}$



Wishbone

secant lines → tangent

Another perspective: tan lines approach

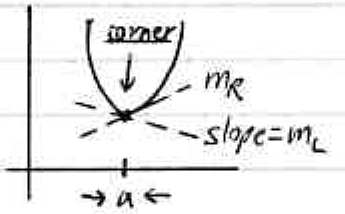
Counterex: Gelbaum 36 Diff'e func w/ discont deriv.

$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

$f'(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Discont. at 0

(2)



$\overbrace{m_L \neq m_R}^{\text{exist}}$

$\Rightarrow$ corner

ℓ_1: left-hand tangent line at a

its slope $= \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$

$=$ left-hand derivative at a

ℓ_2: right-hand " "

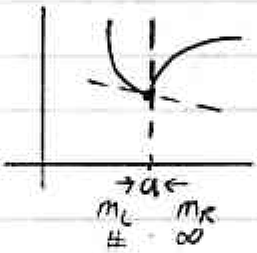
its slope $= \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$

$=$ right-hand derivative at a

The two slopes are different, so $f'(a)$ DNE.

We "read" slopes left-to-right, even if considering m_R

(3)



m_L exists

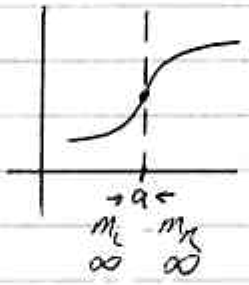
m_R: $\pm\infty$

$\Rightarrow$ corner

} or

(4)

slopes slopes still +
+



m_L: $\pm\infty$

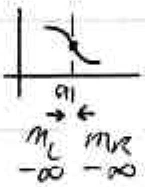
m_R: $\pm\infty$

$\Rightarrow$ vertical tangent line

$\lim_{x\to a} |f'(x)| = \infty$

$f'(x) \to \pm\infty$ as $x\to a^-$, a^+

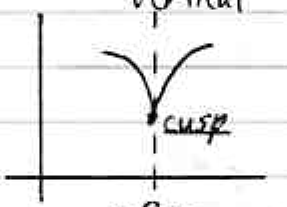
one-sided OK if a is endpt. endpt. for domain



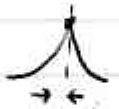
Peak

slopes slopes
$-$ $+$

(5) vertical



$\begin{array}{cc} m_L & m_R \\ -\infty & \infty \end{array}$ $\exists$ cusp

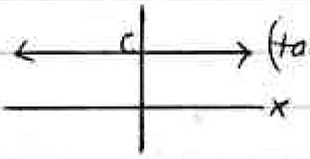


$\begin{array}{cc} m_L & m_R \\ \infty & -\infty \end{array}$

"Different ∞s"

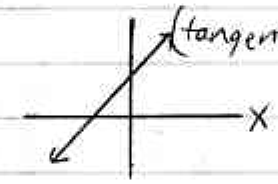
© Shortcuts for Finding $f'(x)$

$f(x) = c,$
$\Rightarrow f'(x) = 0$



If $f(x)$ is const, then $f'(x)$ is 0
If $f(x)$ is linear, then $f'(x)$ is const.
If $f(x)$ is quad, linear
cubic, quad.

$f(x) = mx + b,$
$\Rightarrow f'(x) = m$



<u>Ex</u> If $f(x) = x^3$, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$(x+h)^3$: Use Binomial Theorem

3 ways to hit (3)
3 3
vs

Pascal's Δ

	Row
1	0
1 1	1
1 2 1	2
1 3 3 1	3

coeffs.

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

Dick Cheney taker over

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + \overset{\to 0}{3xh} + \overset{\to 0}{h^2})}{\cancel{h}}$$

$$= 3x^2$$

Man overboard $\quad \cancel{3}x^{\cancel{3}\,2}$

may surprise you

x^0 ≠

Ch. 7 in mahoul (log. diff.)

$f'(x) = nx^{n-1}$ $x \in Dom(f)$ and $Dom(f')$ i.e., $x \neq 0$ when $n \leq 0$

$$\boxed{\underline{\text{Power Rule}} \quad f(x) = x^n \ (n \text{ real}), \ \Rightarrow f'(x) = nx^{n-1}}$$

$f(x)$	$f'(x)$
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
$x^{9/2}$	$\frac{9}{2}x^{7/2}$

$$\frac{1}{x} = x^{-1} \qquad (-1)x^{-2} = -\frac{1}{x^2}$$

$$\frac{1}{x^2} = x^{-2} \qquad (-2)x^{-3} = -\frac{2}{x^3}$$

$$\frac{1}{\sqrt[3]{x^5}} = \frac{1}{x^{5/3}} = x^{-5/3} \qquad -\frac{5}{3} x^{-8/3} = -\frac{5}{3x^{8/3}}$$

$$-\frac{5}{3(\sqrt[3]{x^8})}$$

simplify

$$x^2(\sqrt[3]{x^2})$$

Linearity, like lim

$$\boxed{\begin{gathered} g(x) = cf(x), \\ \Rightarrow g'(x) = cf'(x) \end{gathered}}$$

Constant factors pop out.

<u>Ex</u> If $g(x) = 10x^3$

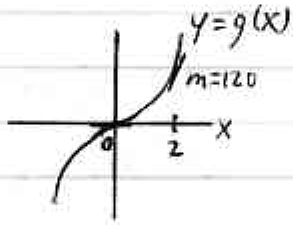
$$\Rightarrow g'(x) = 10(3x^2) = 30x^2$$

$\cancel{10}x^{\cancel{3}2}$

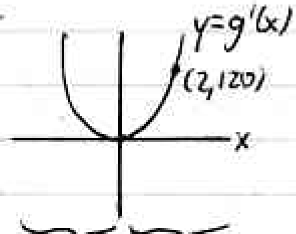
30

$$g'(0) = 0$$

$$g'(2) = 30(2)^2 = 120$$



SKIP



$g'(x) \geq 0$, so $g(x) \to$ or $\nearrow$

$g' \searrow$, so g "slows down"

$g' \nearrow$, so g "speeds up"

Up to 11

we'll use short cuts to help us analyze.

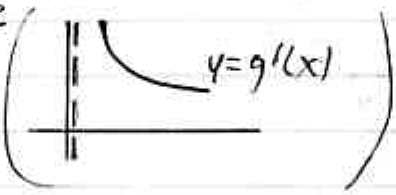
ⓓ When $f'(a)$ DNE (More)

Ex If $g(x) = 7\sqrt{x}$
$$= 7x^{1/2}$$
$$\Rightarrow g'(x) = 7\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= \tfrac{7}{2}x^{-\frac{1}{2}} \text{ or } \frac{7}{2\sqrt{x}}$$

0 an endpt. ⇒ 1-sided one

vertical tan line (1-sided)

Note



g is diff'e on $(0,\infty)$

Precalc ideas $\sqrt[5]{-1} = -1$

Ex If $f(x) = x^{4/5}$ → Dom $=\mathbb{R}$ → $x^{4/5} \geq 0$ (even)

$$\Rightarrow f'(x) = \tfrac{4}{5}x^{-1/5} \text{ or } \frac{4}{5(\sqrt[5]{x})}$$

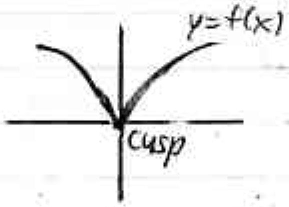
OK if $x<0$

$x \to 0^+$: m_L: ∞

$x \to 0^-$: m_R: $-\infty$

⇒ cusp

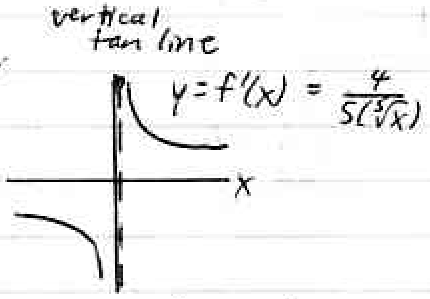
must be a simplified fraction



$f'(0)$ DNE

f is diff'e on $(-\infty,0)\cup(0,\infty)$

Note



As $x\to 0^-$ $f'(x)\to-\infty$ As $x\to 0^+$ $f'(x)\to\infty$

Ex 4 (pp. 102-3)

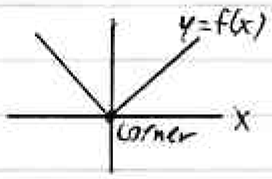
Their analysis is much more complicated

If $f(x) = |x|$

$$= \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

then $f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$ What about $x=0$?

Secants → tangent more appropriate analysis than lim (tan lines)



$\overset{\rightarrow}{m_L} = -1$ $\overset{\leftarrow}{m_R} = 1$

$f'(0)$ DNE

→ 3.2 except 13-14 odd

Ⓔ Notation

$f(x) = x^3 \overset{D_x}{\Rightarrow} f'(x) = 3x^2 \overset{D_x}{\Rightarrow} f''(x) = 6x$

y	y'	y''
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
	$\frac{d}{dx}(x^3)$	$\frac{d^2}{dx^2}(x^3)$
	$D_x(x^3)$	$D_x^2(x^3)$

Differential operator

$\frac{d}{dx}(f(x)) = 2x$

(assumed) $f'(x)$

Later - chain rule, related rates, Calc III - need in beg. too

$f'''(x) = 6$

$f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

order

poly → 0

Need (): $f^4 = f \circ f \circ f \circ f$