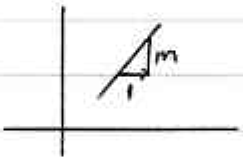
3.5: (INCREMENTS and) DIFFERENTIALS

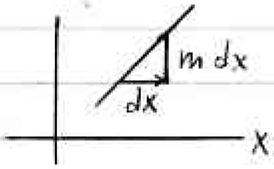
Ⓐ Interpreting Slope

This is something you should've learned in Alg I, but didn't. No one knew.

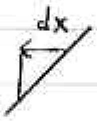
$f(x) = mx + b$

Similar △s





If $dx < 0$



If $x \uparrow$ by $1 \Rightarrow$ y changes by m
$m < 0$ (drop)

run 1
$\Rightarrow$ rise m

run dx
$\Rightarrow$ rise m dx

What do we call slope in calculus, basically?

$$\boxed{\text{slope} = \frac{\text{rise}}{\text{run}} \Rightarrow \text{rise} = (\text{slope})(\text{run}) = f'(x)\,dx}$$

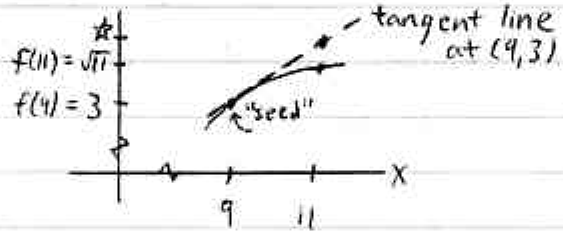
Ⓑ Ex: Approximate $\sqrt{11}$

If you don't have $\boxed{\sqrt{\ }}$ on calc – I hope you do

Let $f(x) = \sqrt{x}$ ← diff'e on $(0, \infty)$ (not C, vertical tan line)
We know $f(9) = 3$.

We don't know $\sqrt{11}$ exactly as a decimal – can't; irrat'l
We know √ someone who's kinda close

We can approx. $f(11)$ by starting at $(9,3)$ and walking along the tangent line.



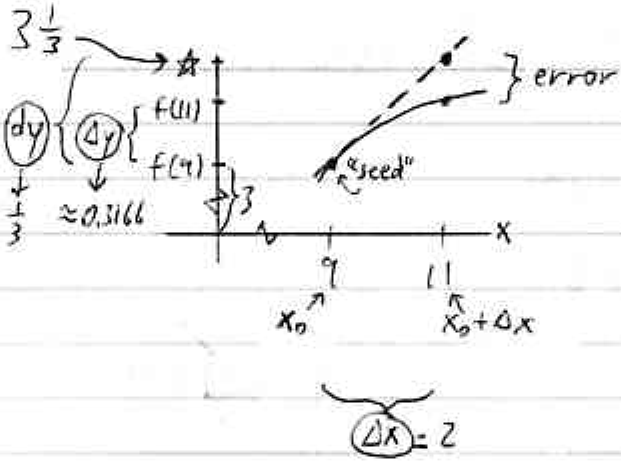
☆ = our linear approximation for $f(11)$ if x changes from 9 to 11.

error = $dy - \Delta y$

Label later, after we introduce

Climb: tangent line approximates the graph best close to $x=9$.

x or x_0



(Save)

① Δx = "DELTA x"
= change in x
= increment of x

Here, $\Delta x = 2$.

② The <u>differential dx</u> = Δx
Here, $dx = 2$.

③ The <u>differential dy</u>

= rise <u>along the tangent line</u> <u>(at $(x_0, f(x_0))$</u> as $x_0 \to x_0 + \Delta x$)

as we move from

= (slope)(run)

rise = (slope)(run) along a line

x_0: old x

more accurate

$$dy = f'(x)\,dx$$
or x_0

explains (the notation) $\frac{dy}{dx} = f'(x)$

$\frac{dy}{dx}$ as $dx \to 0$ (like $h \to 0$)

Here, $dy = \underbrace{f'(9)}\,\underbrace{dx}_{=2}$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$
$$f'(x) = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$
$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$dy = \left(\tfrac{1}{6}\right)(2)$$

what do we do with this to approx. $\sqrt{11}$?

$$dy = \tfrac{1}{3}$$

$\left(\begin{array}{l} f(x+\Delta x) = \\ f(x) + dy \end{array}\right)$

④ Our linear approx:

$$f(11) \approx \overbrace{f(9)}^{\text{old } y} + \overbrace{\tfrac{1}{3}}^{dy}$$
$$= 3 + \tfrac{1}{3}$$
$$= 3\tfrac{1}{3}$$

So $\sqrt{11} \approx 3\tfrac{1}{3}$ $(3.\overline{3})$

Turns out $\sqrt{11} \approx 3.3166$

Let's see how close we were.

$\left(\begin{array}{l} \text{increment of } y \\ f(x+\Delta x) - f(x) \end{array}\right)$

⑤ Δy = "actual" change in y

Here, $f(11) - f(9)$
$= \sqrt{11} - 3$
$\approx$ (turns out) 0.3166 ← We hope $\approx dy$

It's a matter of opinion

(We hope dy is close to Δy.
$\tfrac{1}{3} = 0.\overline{3}$ $\qquad$ 0.3166
i.e., we hope our approx. ($3\tfrac{1}{3}$) is close to $\underbrace{f(11)}_{\approx 3.3166}$.)

Note

(⑥ error $= dy - \Delta y$
$\approx \frac{1}{3} - 0.3166$
≈ 0.0167)

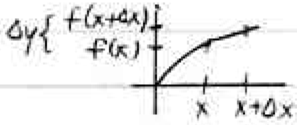
SKIP

Error ≈ 0 $(dy \approx \Delta y)$ if $\Delta x \approx 0$

We can find general formulas for Δy and dy.

$f(x) = \sqrt{x}$

$\Delta y = f(x + \Delta x) - f(x)$
$= \sqrt{x + \Delta x} - \sqrt{x}$

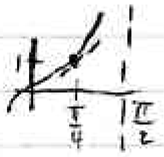


$dy = f'(x)\,dx$
$= \frac{1}{2\sqrt{x}}\,dx$

© Ex: Approximate $\tan(42°)$

Let $f(x) = \tan x$
We know $\tan(45°) = 1$.
$f\left(\frac{\pi}{4}\right) = 1$
$\boxed{RAD}$

There's something off here.. Turns out we don't need to convert 45°, but it's a good reminder!



① $\Delta x =$ new $x -$ old x
$= 42° - 45°$
$= -3°\left(\frac{\pi \text{ rad}}{180°}\right)$
$= -\frac{\pi}{60}$ rad (assumed)

We need to → RAD

② $dx = \Delta x = -\frac{\pi}{60}$

③ $dy = f'(\underset{\text{old } x}{\frac{\pi}{4}})\,dx$

$f(x) = \tan x$
$f'(x) = \sec^2 x$
$f'(\frac{\pi}{4}) = \sec^2(\frac{\pi}{4})$
$= [\sec(\frac{\pi}{4})]^2$
$= (\sqrt{2})^2$
$= 2$

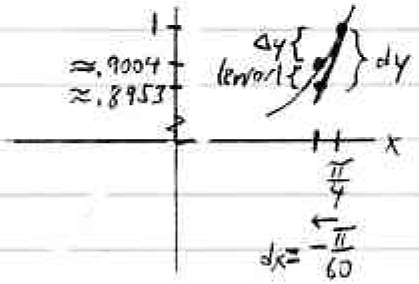
$\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$dy = (2)(-\frac{\pi}{60})$
$= -\frac{\pi}{30}$

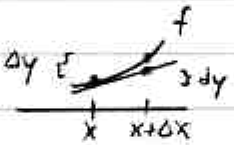
④ $\tan 42° \approx \overbrace{f(\frac{\pi}{4})}^{\text{old } y} + dy$
$= 1 - \frac{\pi}{30}$
≈ 0.8953

Turns out: $\tan 42° \approx 0.9004$ (✓ in $\boxed{DEG}$)

Makes sense $dy < 0$.



① Error



$y = f(x)$

Physicists, engineers use (caution)

Numerical analysis → process

$\underbrace{\Delta y}_{\substack{\text{propagated}\\ \text{error } (y)}} = f(x + \underbrace{\Delta x}_{\substack{\text{measurement}\\ \text{error}}}) - f(x)$ ← measured value

dy estimates Δy.

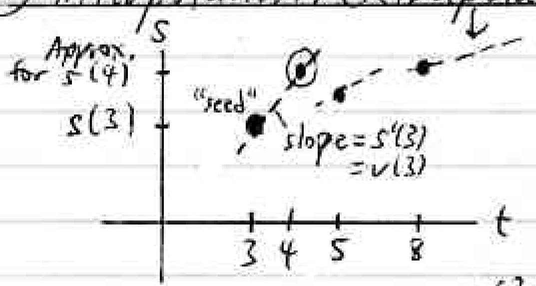
exact meas.

Ex x = radius
y = volume
$f(x)$

hideous, or don't know
Realworld!
In interpolation, conflict
May ~~collide~~ w/other tangent lines

Maybe 1 pt. is more acc than the other. Can weight differently!
Dave's stalker cop tracing her
Gt. for court care

(E) Interpolation/Extrapolation



No nice $s(t)$ rule available.
Finite # of measurements. Table?

t	$s(t)$	$s'(t) = v(t)$
3		
5	position	velocity
8		

Know $s(3)$, $\overbrace{s'(3)}^{=v(3)}$ (= slope of tangent line at $(3, s(3))$)
Linear approx. $\Rightarrow$ approx. for $s(4)$
(Follow the tangent line!)