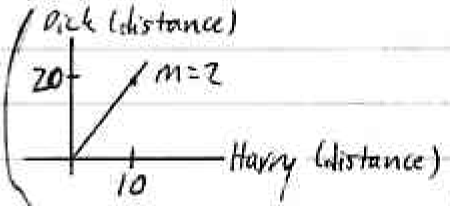
3.6: CHAIN RULE

(A) Idea

Kindergarten

Dick runs twice as fast as Harry.



If Tom and Dick start running at the same time.

Tom runs 3× " " Dick.

Then, Tom runs 6× " " Harry.

$$\frac{d(\text{Tom})}{d(\text{Harry})} = \frac{d(\text{Tom})}{d(\text{Dick})} \cdot \frac{d(\text{Dick})}{d(\text{Harry})}$$

$$6 = 3 \cdot 2$$

Products of D's come from compos. of funcs, not products (121 book)

(B) Forms of the Rule

Be prepared to recognize f of u, please Remem. Ch.1?

Chain: $y \mid u \mid x$ — $\left.\begin{array}{l} y=f(u) \\ u=g(x) \end{array}\right\} y=f(g(x))$

Composition of funcs.

Then,

① $\dfrac{dy}{dx} = \dfrac{dy}{du}\overbrace{\dfrac{du}{dx}}^{\text{tail}}$ — if exist

② $y' = f'(u) \cdot \overbrace{u'}^{\text{tail}}$ — D_x (Deriv. of outside · Deriv. of inside)

Cooler idea "du's" cancel, but not technically true! Larson Deriv. of outside · Deriv of inner guy

(③ $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\overbrace{g'(x)}^{\text{tail}}$)

(④ or $f'(u)\overbrace{g'(x)}^{\text{tail}}$)

When you have a func inside a func.

Ex Find $\frac{d}{dx}\underbrace{(3x^2+4)^7}_{=y}$

composite function where

$u = 3x^2+4$ "inside"
$y = u^7$

What do we do to u to get y?

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Boss gives you? Hey, what's u?

$$= (7u^6)(6x)$$

unwrap

7(blah)^6 · D(blah) D(inside)

$$= 7(3x^2+4)^6(6x)$$
$$= 42x(3x^2+4)^6$$

We study applics of Chain Rule

(C) Generalized Power Rule

$$D_x x^n = nx^{n-1}$$

How to Ace As Mama fox said to Baby fox: don't forget your tail!

Messy $\left(D_x [g(x)]^n = n[g(x)]^{n-1} \cdot \overbrace{D_x g(x)}^{\text{"tail"}}\right)$

Take deriv.

$u = g(x)$

Comes from Chain Rule

$$\boxed{D_x \underset{\text{func. of } x}{u}^n = nu^{n-1} \cdot \underbrace{D_x u}_{\text{tail}}}$$

Basic Power Rule · tail (effect of Chain Rule)

$y = u^n$: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Old Ex $D_x (3x^2+4)^7$

$$= 7(3x^2+4)^6 \cdot \overset{\text{tail}}{(6x)}$$

D_x

$$= 42x(3x^2+4)^6$$

$$\underline{Ex} \quad f(x) = \frac{6x}{\sqrt[3]{(x^4+7x)^2}}$$

We'll do Quot. Rule later

Use Quotient Rule or...

$$\left(= \frac{6x}{(x^4+7x)^{2/3}} \right)$$

$$= 6x(x^4+7x)^{-2/3}$$

You don't need quotient rule unless you want to find altern. forms quickly.

((Can write quotients as products!)
Product Rule

$$\underbrace{6x}_{\wedge \atop + \text{ copy}}\underbrace{(x^4+7x)^{-2/3}}_{\text{copy} \atop \wedge}$$

$$\begin{aligned} f'(x) = &(6)(x^4+7x)^{-2/3} \\ &+ (6x)[-\tfrac{2}{3}(x^4+7x)^{-5/3}\underbrace{(4x^3+7)}_{\text{tail}}] \end{aligned}$$

D_x

(OK if I tell you not to simplify.)

Mathem. gives ≈; just term-matching. Maybe use simplify, together?

$$= \frac{6}{(x^4+7x)^{2/3}} - \frac{4x(4x^3+7)}{(x^4+7x)^{5/3}}$$

LCD

Nice!: $(x^4+7x)^{2/3}(x^4+7x)^{\overset{1 \text{ or } 3/3}{\downarrow}} = (x^4+7x)^{5/3}$

You'd eventually get what you'd get by the Quot. Rule:

$$= \frac{6}{(x^4+7x)^{2/3}} \cdot \frac{(x^4+7x)}{(x^4+7x)} - \frac{4x(4x^3+7)}{(x^4+7x)^{5/3}}$$

$$\frac{(x^4+7x)^{2/3}(6) - (6x)[\frac{2}{3}(x^4+7x)^{-1/3}(4x^3+7)]}{(x^4+7x)^{4/3}}$$ ← factor $(x^4+7x)^{-1/3}$

Up to 27

$$= \frac{6(x^4+7x) - 4x(4x^3+7)}{(x^4+7x)^{5/3}}$$

$$= \frac{14x - 10x^4}{(x^4+7x)^{5/3}}$$

3rd Ex $D_x \sec^5 x$

$$= D_x (\sec x)^5$$
$$= 5 \underbrace{(\sec x)^4}_{\text{tail}} (\sec x \tan x)$$

How many sec x factors?

$$= 5 \sec^5 x \tan x$$

2nd Ex (Reciprocal Rule)

Don't even need the Quotient Rule

$$D_x \left(\frac{1}{g(x)}\right) = D_x [g(x)]^{-1} \leftarrow \text{reciprocal!}$$

vs. $g^{-1}(x)$ ← func. inv.

($\underline{Ex}$ $\csc x \neq \sin^{-1} x$)

$$= (-1)[g(x)]^{-2} \underbrace{[g'(x)]}_{\text{tail}}$$

$$= -\frac{g'(x)}{[g(x)]^2}$$

1st Ex $D_x \left(\frac{x^3}{\sqrt{2x-1}}\right)$

Could → $x^3(2x-1)^{-\frac{1}{2}}$
or Quotient Rule ← :)

Good, bec. $(\sqrt{2x-1})^2$ →

$$= \frac{Lo\,D(Hi) - Hi\,D(Lo)}{(Lo)^2}$$

$$= \frac{(\sqrt{2x-1})D_x(x^3) - (x^3)D_x(\overbrace{\sqrt{2x-1}}^{(2x-1)^{\frac{1}{2}}})}{(\sqrt{2x-1})^2}$$

$$= \frac{(\sqrt{2x-1})(3x^2) - (x^3)\left[\frac{1}{2}(2x-1)^{-\frac{1}{2}}\underbrace{(2)}_{\text{tail}}\right]}{2x-1}$$

Simplifying... Distribute

$$= \frac{\left(3x^2(\sqrt{2x-1}) - \frac{x^3}{\sqrt{2x-1}}\right)}{2x-1} \cdot \frac{\sqrt{2x-1}}{\sqrt{2x-1}}$$

$$= \frac{3x^2(2x-1) - x^3}{(2x-1)(2x-1)^{1/2}}$$

$$= \left(\frac{6x^3 - 3x^2 - x^3}{(2x-1)(2x-1)^{1/2}}\right)$$

$$= \frac{5x^3 - 3x^2}{(2x-1)^{3/2}}$$

Up to 27

Quotient Rule → One term

$\leftarrow 2^{nd}, 3^{rd}$

(D) Generalized Trig Rules

If $u = g(x)$, then
diff'e

Good review!
sin → cos
tan → cot

$D_x(\sin u) = (\cos u)\underbrace{(D_x u)}_{\text{tail}}$ $\qquad D_x(\cos u) = (-\sin u)(D_x u)$

$D_x(\tan u) = (\sec^2 u)(D_x u)$ $\qquad D_x(\cot u) = (-\csc^2 u)(D_x u)$

$D_x(\sec u) = (\sec u \tan u)(D_x u)$ $\qquad D_x(\csc u) = (-\csc u \cot u)(D_x u)$

either tail

sec(blah)
sec u tan u
OK if
sub back

Ex $\quad f(x) = \sec(\overbrace{9x^2-1}^{u})$

$$f'(x) = [\sec(9x^2-1)\tan(9x^2-1)]\underbrace{[D_x(9x^2-1)]}_{\text{tail} = 18x}$$

$$= 18x\sec(9x^2-1)\tan(9x^2-1)$$

$\underline{Ex}$ If $f(x) = \cos^5(7x)$, find $f'(x)$.

Overall, we have a power 1st.

$$f(x) = [\underbrace{\cos(7x)}_{u}]^5 \quad \text{Gen. Power Rule 1st}$$

$$f'(x) = 5[\cos(7x)]^4 \cdot \underbrace{D_x[\cos(7x)]}_{[-\sin(7x)][\underbrace{D_x(7x)}_{=(7)}]} \quad \text{Gen. Trig Rule 2nd}$$

tail inside a tail

$$= [-7\sin(7x)]$$

$$= -35\cos^4(7x)\sin(7x)$$

(You) $\underline{Ex}$ (Similar) If $f(x) = \tan^4(\pi x)$, find $f'(x)$

$$f(x) = [\underbrace{\tan(\pi x)}_{u}]^4$$

$$f'(x) = 4[\tan(\pi x)]^3 \cdot \underbrace{D_x[\tan(\pi x)]}_{[\sec^2(\pi x)][\underbrace{D_x(\pi x)}_{=(\pi)}]}$$

Keep going until you have no more D_x's!!

$$= 4\pi\tan^3(\pi x)\sec^2(\pi x)$$

$$\underline{Ex} \quad \text{If } f(x) = \sin\underbrace{(x^2)}_{u}, \text{ find } f'(x).$$

I like

Overall, sin(blah)

(Gen. Trig Rule 1st)

What's the deriv. of sin?

$$f'(x) = [\cos(x^2)]\underbrace{[D_x(x^2)]}_{\text{tail} = (2x)} \quad \text{(Power Rule 2nd)}$$

$$= 2x\cos(x^2)$$

(You)

$$\underline{Ex} \quad \text{If } f(x) = \csc(x^2+1)^3, \text{ find } f'(x).$$

$$f(x) = \csc\underbrace{[(x^2+1)^3]}_{u}$$

(Gen. Trig Rule 1st)

What's deriv of sec? csc?

$$f'(x) = -\csc[(x^2+1)^3]\cot[(x^2+1)^3]\cdot D_x[(x^2+1)^3]$$

(Gen. Power Rule 2nd)

$$= 3(x^2+1)^2(2x)$$

tail

$$\left(= -\csc[(x^2+1)^3]\cot[(x^2+1)^3]\cdot 3(x^2+1)^2(2x) \right)$$

Up to 59

$$= -6x(x^2+1)^2\csc[(x^2+1)^3]\cot[(x^2+1)^3]$$

(E) Tangent Lines

Ex Find the x-coords of all points with a horizontal tangent line.

(a) $y = (x^2-9)^7$

For what x is $y' = 0$?

$$y' = 7(x^2-9)^6(2x)$$
$$= 14x(x^2-9)^6 \overset{\text{set}}{=} 0$$

$x = 0$ or $\left(\begin{array}{c} x^2-9=0 \\ (x+3)(x-3)=0 \end{array} \right)$

$x=-3$ $\quad$ $x=3$

Why you can't [illegible] by pr. calcs.

(Doesn't look in as much in $(-4,4)$; Leading term $= x^{14}$)

Even func.

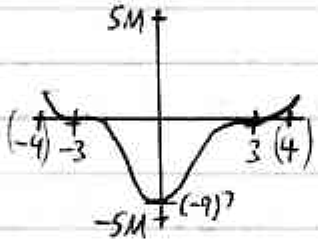
Why do you think $\pm 5M$?

$(\ )^{⑦}$

$(-9)^7 \approx -4.8M$

How do you find y-int?

Turns out



(b) $y = x + \cos 2x$

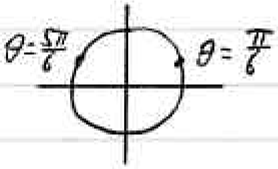
(#68) $y = x + \cos(2x)$

$$y' = 1 - \sin(2x)\cdot 2$$

$$y' = 1 - 2\sin(2x) \overset{\text{set}}{=} 0$$

$$\underbrace{\sin(2x)}_{\theta} = \frac{1}{2}$$

In what other Q is sin>0? Who's the famous bro? 5=6-1



$$\theta = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad \theta = \frac{5\pi}{6} + 2\pi n, \ n \text{ integer}$$

sin θ has period 2π
sin 2x has period π

$$2x = \frac{\pi}{6} + 2\pi n \qquad 2x = \frac{5\pi}{6} + 2\pi n$$

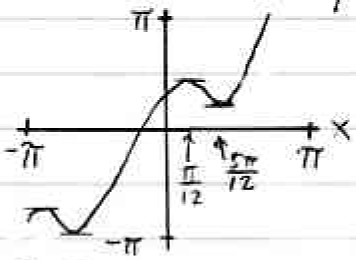
$$x = \frac{\pi}{12} + \pi n \qquad x = \frac{5\pi}{12} + \pi n, \ n \text{ integer}$$

∞ many hilltop, valley bottom pairs
What is this (x)?

Turns out

$y = x + \cos(2x)$

(What does this look like?)
(What effect does this have?)
(Upward drift → ∞ as x → ∞)



$\left(\frac{\pi}{12} - \pi = -\frac{11\pi}{12}\right)\left(\frac{5\pi}{12} - \pi = -\frac{7\pi}{12}\right)$