3.7: IMPLICIT DIFFERENTIATION (IMP. DIFF.)

(A) Implicit Functions

Note

Ex $y = 2x + 1$ describes y as an explicit function of x.

Form: $y = f(x)$

The equation $20x - 10y = -10$ describes y as an implicit function of x.

Why? Let $f(x) = 2x + 1$
If we plug in $f(x)$ for y...

$$20x - 10f(x) = -10$$
$$20x - 10(2x+1) = -10$$
$$\cancel{20x} - \cancel{20x} - 10 = -10$$
$$-10 = -10$$

We get an identity (true $\forall x \in Dom(f)$).
Then, f (or y) is an implicit function of x.
So, the graph of f is the same as part or all of the graph of $20x - 10y = -10$.

Here, all: $y = 2x + 1$ is equivalent to $20x - 10y = -10$.

Ex $y = \frac{1}{x}$ describes y as an explicit func. of x
Form: $y = f(x)$

y buried in there.

$xy = 1$ " "implicit func." "

<u>Ex</u> (pp. 146-7)

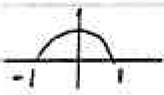
buried in there

$x^2+y^2=1$ determines many imp. funcs.

let's see if we can get some funcs out of here

 fails VLT, but...

$$y^2 = 1-x^2$$
$$y = \pm\sqrt{1-x^2}$$

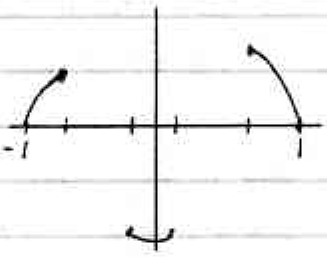
① $f_1(x) = \sqrt{1-x^2}$ 

② $f_2(x) = \sqrt{1-x^2}$, $x \geq 0$ 

③ $f_3(x) = -\sqrt{1-x^2}$ 

④
$$f_4(x) = \begin{cases} \sqrt{1-x^2}, & -1 \leq x \leq -0.7 \\ -\sqrt{1-x^2}, & -0.2 \leq x \leq 0.2 \\ \sqrt{1-x^2}, & 0.6 \leq x \leq 1 \end{cases}$$

Smiley!



For ①-④, [the coords. of] every point (x,y) on the graph satisfies $x^2+y^2=1$.
(Their graphs are pieces of the original unit circle.)

(etc.)

How many? There is literally no point.

<u>Ex</u> $x^2+y^2=-1$ No imp. funcs!

(B) Implicit Diff'n (Imp. Diff.)

No $x^2+y^2=-1$

Assume that equations determine a diff'e' (where needed) implicit func. f.

$$y=f(x)$$

$$D_x y = y' \quad \text{(Notation)}$$

Warmups!

① $D_x(7y) = 7y'$

② $D_x(y^2) = 2y\underbrace{y'}_{\text{tail}}$

f(x) imp.!

③ $D_x(y^3) = 3y^2y'$

Overall we're D_x... what rule?

④ $D_x(x^3y^2) = (3x^2)(y^2) + (x^3)(2yy')$ (Product Rule)

$\wedge$ copy $= 3x^2y^2 + 2x^3yy'$

+ copy $\wedge$

⑤ $D_x(x+y)^4 = 4(x+y)^3 D_x(x+y)$ (Gen. Power Rule)

$= 4(x+y)^3(1+y')$

⑥ $D_x\cos^3(xy) = D_x[\cos(xy)]^3$

$= 3[\cos(xy)]^2 \underbrace{D_x[\cos(xy)]}_{[-\sin(xy)][D_x(xy)]}$ (Gen. Power Rule)

$[-\sin(xy)]\underbrace{[D_x(xy)]}_{[(1)y+xy']}$ (Trig) (Product Rule)

Let's glue e-thing together.

$= -3(y+xy')\cos^2(xy)\sin(xy)$

Put trig stuff at end
- unambig w/()

Ex If $x^2-2x+y^2+6y=15$,

(a) Find y'

Later, we'll see (CTS, etc.) how

Isolate y? (YUK!)
Use Imp. Diff.

① D_x [each term on] both sides.

$$D_x(x^2-2x+y^2+6y)=D_x(15)$$

Many write "15"

$$2x-2+2yy'+6y'=0$$

careful!

Solve for y'...

② Isolate terms with y' on one side.

$$2yy'+6y'=2-2x$$

③ Factor out y'

$$y'(2y+6)=2-2x$$

④ Isolate y' by $\div$

$$y'=\frac{2-2x}{2y+6}$$

what's different about this expr for y'?

$$y'=\frac{1-x}{y+3}$$

OK to have y, but not y'

Up to 17

(b) Find the slope of the tangent line to the graph of the eq. at $(4,1)$.

Don't have to do on HW. I'd have to be a jerk to give you another pt.

Check: $(4,1)$ is on the graph.

Can skip
$$\left(\begin{array}{l} x^2-2x+y^2+6y=15 \\ (4)^2-2(4)+(1)^2+6(1)\overset{?}{=}15 \checkmark \end{array}\right)$$

$$\text{Slope} = y' = \left[\frac{1-x}{y+3}\right]_{\substack{\text{eval at} \\ (4,1)}}$$

$$= \frac{1-4}{1+3}$$

$$= -\frac{3}{4}$$

<u>Another Method</u>

$$x^2-2x+y^2+6y=15 \quad D_x$$
$$2x-2+2yy'+6y'=0$$

Plug in $x=4, y=1$ NOW! (If you just need one deriv. value)

We don't need an expr. for y'. Formula helpful if need to find >1 deriv. value.

$$2(4)-2+2(1)y'+6y'=0$$
$$6+8y'=0$$
$$y'=-\frac{3}{4}$$

ⓒ Same for $(4,-7)$ ← ✓ on graph

$$y' = \left[\frac{1-x}{y+3}\right]_{(4,-7)}$$

$$= \left[\frac{1-4}{-7+3}\right]$$

$$= \frac{3}{4}$$

ⓓ Same for $(1,2)$ ← ✓ on graph

$$y' = \left[\frac{1-x}{y+3}\right]_{(1,2)}$$

$$= 0$$

ⓔ What's going on?

Few remembered!

(I claim this is the eq. of a... circle. What technique can I use to show that?

$$x^2-2x+y^2+6y=15$$

$$(x^2-2x+1)+(y^2+6y+9)=15+1+9$$

$$\underset{\text{CTS}}{\uparrow} \qquad \underset{\text{CTS}}{\uparrow}$$

$$(x-1)^2+(y+3)^2=25$$

und → +→0 → − → und

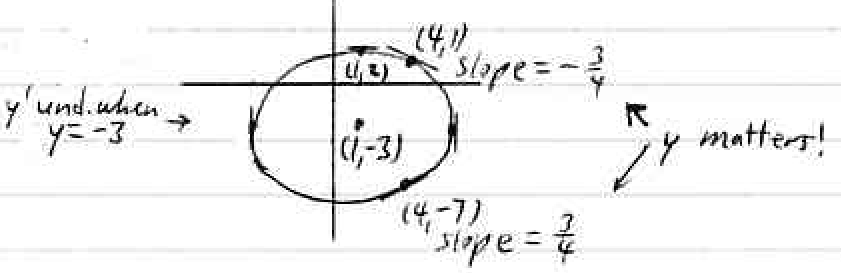
Circle, Center $\underbrace{(1,-3)}_{\text{Makes left side = 0}}$, $r=\sqrt{25}=5$

$(x-h)^2+(y-k)^2=r^2$
Center: (h,k)
radius $=r$

We still "read" slopes left-to-right

y-coord like "tiebreaker"

You can get the center from $y'=\frac{1-x}{y+3}$



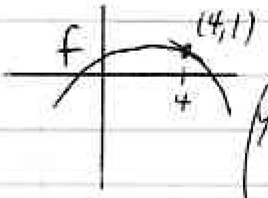
What if you want to use f notation?

Passes VLT

We're not supposed to

slopes left-to-right

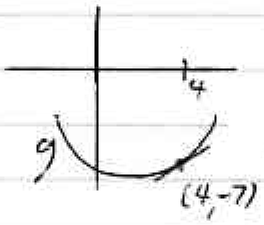
unambig.



($y = f(x)$, where f is an implicit func. determined by the eq.)

$$f'(4) = -\frac{3}{4}$$

Deriv. depends on y-coord. It matters what part of the original graph you're on.



$$g'(4) = \frac{3}{4}$$

SKIP

(f) Find y''

$$y' = \frac{1-x}{y+3}$$

$$y'' = D_x\left(\frac{1-x}{y+3}\right) \quad \text{Quotient Rule}$$

$$= \frac{(y+3)D_x(1-x) - (1-x)D_x(y+3)}{(y+3)^2}$$

$$= \frac{(y+3)(-1) - (1-x)(y')}{(y+3)^2}$$

$$= \frac{-(y+3) - (1-x)y'}{(y+3)^2}$$

Plug into y'

$$= \frac{-(y+3) - (1-x)\left(\frac{1-x}{y+3}\right)}{(y+3)^2} \cdot \frac{(y+3)}{(y+3)}$$

$\cdot$ each term in N, D by $(y+3)$

Can we still cancel? NO!

$$= \frac{-(y+3)^2 - (1-x)^2}{(y+3)^3}$$