3.8: RELATED RATES

Ⓐ Implicit Diff'n (Imp. Diff.)

In 3.7, $y = f(x)$.
implicit

Your book tends to combine

factor out y'

May need to be more descriptive in this section

Review Ex Find $D_x(4x^3 + xy - 3y^2)$

$$= 12x^2 + (1)(y) + (x)(y') - 6y\underbrace{y'}_{\text{tail}}$$

$$= 12x^2 + y + (x-6y)y'$$

or $\frac{dy}{dx}$

What's going to be the geom. interp of our answer?

Review Ex If $x^2+y^2=25$, find $\frac{dy}{dx}$ when $x=3$ and $y=4$.

$$x^2+y^2=25$$

If you plug in, you're just verifying that the pt. lies on the graph

Don't plug in YET!
D_x both sides implicitly

$$D_x(x^2) + D_x(y^2) = D_x(25)$$

$$2x + 2y\left(\frac{dy}{dx}\right) = 0$$

$$x + y\left(\frac{dy}{dx}\right) = 0$$

Method 1: $\frac{dy}{dx} = -\frac{x}{y}$ } Plug in!

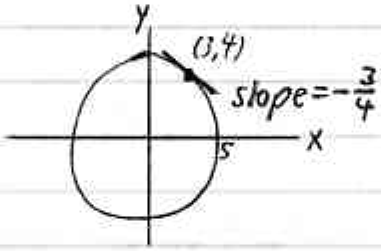
$$= -\frac{3}{4}$$

Method 2: Plug in NOW! (After D_x)

$$(3) + (4)\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = -\frac{3}{4}$$

What's the graph of $x^2+y^2=25$?



In 252, trees!

$z \to x, y$; $x \to r, s$; $y \to r, s$

func from 1st Rev. Ex.

$12x^2 \ldots$ (or is there more?)

Now, $x = f(t)$ — t — $y = g(t)$

Ex find $D_t(4x^3 + xy - 3y^2)$

$$= \underbrace{12x^2\left(\frac{dx}{dt}\right)}_{\text{tail}} + \left(\frac{dx}{dt}\right)(y) + (x)\left(\frac{dy}{dt}\right) - 6y\left(\frac{dy}{dt}\right)$$

NASCAR track

Tank!

Why I'm not an engineer... Just care in terms of horiz. motion W-E

What's a natural?

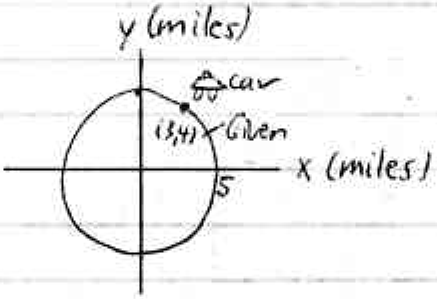
(Given: horiz. comp.)

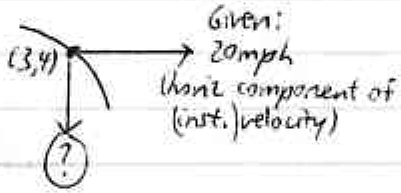
Maybe a couple

Translate!

Interpret using notation!

Ex





If $x^2+y^2=25$, find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 20$ mph when $x=3$ and $y=4$.

$$x^2+y^2=25$$

Diff not wrt x but

D_t both sides implicitly

Helps! Esp. w/ 25 → 0.

$$D_t(x^2)+D_t(y^2)=D_t(25)$$

$$2x\left(\frac{dx}{dt}\right)+2y\left(\frac{dy}{dt}\right)=0$$

$$x\left(\frac{dx}{dt}\right)+y\left(\frac{dy}{dt}\right)=0$$

They're related by this Eq. Useful to find formula for dy/dt, o.w. from Chain Rule

$\frac{dx}{dt}, \frac{dy}{dt}$ are <u>related rates</u>

Plug in.

$$(3)(20)+(4)\left(\frac{dy}{dt}\right)=0$$

$$60+4\left(\frac{dy}{dt}\right)=0$$

Why −?

$$\frac{dy}{dt}=-15 \text{ mph}$$

vertical comp.

(S at a speed of 15 mph) $\left(\begin{matrix}+\\ \uparrow\\ \downarrow\\ -\end{matrix}\right)$

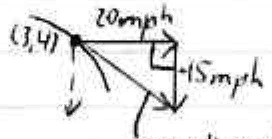
<u>Note</u>

At (3,4),

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\begin{matrix}\leftarrow -15\\ \leftarrow 20\end{matrix}$$

$$=-\frac{3}{4} \quad \text{AGAIN!}$$

From $\frac{dy}{dx}\frac{dx}{dt}=\frac{dy}{dt}$



resultant (slope $=-\frac{3}{4}$)

velocity along circle is 25 mph

↘ Do you know what this vector is called in physics? vel. along circle is what? 3-4-5 in proportion

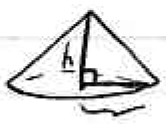
Ⓑ <u>Word Problems</u>

<u>Ex</u> An object expands as a right circular cone at a rate of 20 ft.³/day. If the radius grows at a rate of 36 in./day, how fast does the height change at the instant that the radius is 10 ft. and the height is $\frac{6}{\pi}$ ft.?

slime, goo, toxic waste shape may change

① Read!

② Diagram

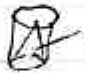


General - nothing specific to our "instant of interest." (Relevant to entire model/process.)

You have a rt. circ. cone, what could I ask you about the cone?

What's area(cyl)? Says can know! Look inside front cover.

③ Key Formula:

$$V = \frac{1}{3}\pi r^2 h \quad \text{(Know!)}$$  $\frac{1}{3}$

Need to know what I'm diff'ing w.r.t. Other info helps. What's another issue that may come up? Then, What is 20 ft.³/day?

④ Given info at the instant of interest (and in general)
Find what?

$$\frac{dV}{dt} = 20 \left(\frac{ft.^3}{day}\right)$$

$$\frac{dr}{dt} = 36 \left(\frac{in.}{day}\right) \cdot \frac{1 \text{ ft.}}{12 \text{ in.}} = 3 \left(\frac{ft.}{day}\right)$$

Unit issues compatibility, RAD

What do we want to find!

Find $\frac{dh}{dt}$ when

$r = 10$ (ft.)
$h = \frac{6}{\pi}$ (ft.)

⑤ D_t both sides of ③

$$D_t(V) = D_t\left(\underbrace{\tfrac{1}{3}\pi r^2 h}_{\text{Use Product Rule}}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[\left(2r\frac{dr}{dt}\right)(h) + (r^2)\left(\frac{dh}{dt}\right)\right]$$

You don't want to solve for $\frac{dh}{dt}$ as is! Don't need general formula.

⑥ Plug in values from ④, and
⑦ Solve for $\frac{dh}{dt}$ (units!)

$$20 = \frac{1}{3}\pi\left[2(10)(3)\left(\frac{6}{\pi}\right) + (10)^2\left(\frac{dh}{dt}\right)\right]$$

$$20 = 120 + \frac{100\pi}{3}\left(\frac{dh}{dt}\right)$$

$\frac{3}{4}x = \ldots$
Solve for x: $\cdot\frac{4}{3}$

$$-100 = \frac{100\pi}{3}\frac{dh}{dt}$$

$$-100\left(\frac{3}{100\pi}\right) = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{3}{\pi} \approx -0.9549 \ \frac{ft.}{day}$$

What does this mean? Why does this make sense? Even though $V\nearrow$, r is also $\nearrow$

slippery goo height shrinking as expands along floor

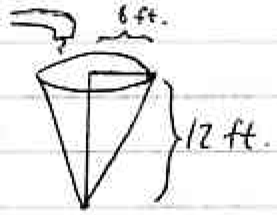
⑧ Conclusion (in English)

At the instant of interest, the height shrinks at a rate of $\frac{3}{\pi}$ $\frac{ft.}{day}$.

(Recap!) Interpret "–"

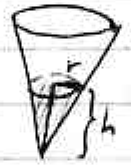
Ex 4 (p.156)

①,② Water tank



Water is pumped in at 10 $\frac{\text{gal.}}{\text{min.}}$ ($\approx 1.337 \frac{\text{ft.}^3}{\text{min.}}$).
Approximate the rate at which the water level rises when the water is 3 ft. deep.

This is the general diagram.

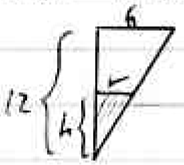


similar cones/triangles

You must incorporate this relationship into your formula! You can't let r, h roam freely, independently. How is r related to h?

③ $V = \frac{1}{3}\pi r^2 h$ ← want in terms of h, alone
(We can do this, because regardless of how much water is in here, r and h have the same relationship)

Similar triangles



$\frac{r}{h} = \frac{6}{12}$

$r = \frac{1}{2}h$ (We must incorporate this dependency relationship into our formula.)

$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V = \frac{1}{12}\pi h^3$

If water is pumped out, $\frac{dV}{dt} \approx -$

④ Given: $\frac{dV}{dt} \approx 1.337$
Find $\frac{dh}{dt}$ when $h = 3$.

(5) $V = \frac{1}{12}\pi h^3$

$\frac{dV}{dt} = \frac{1}{12}\pi\left(3h^2 \frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

(6) $1.337 \approx \frac{1}{4}\pi(3)^2\left(\frac{dh}{dt}\right)$

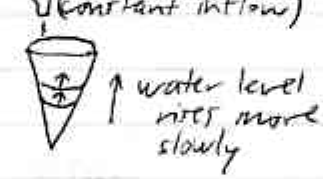
(7) $\frac{dh}{dt} \approx 0.189\ \frac{ft.}{min.}$

(8) The water level rises at ≈ 〃 when the water is 3 ft. deep.

Note When $h=4$, (6) $1.337 \approx \frac{1}{4}\pi(4)^2\left(\frac{dh}{dt}\right)$

$\frac{dh}{dt} \approx 0.106\ \frac{ft.}{min.}$

Why is this lower?

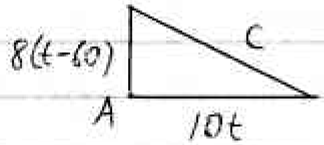


SKIP

Hard to underst.; √ not simplified in manual.

Ex (#14) A girl starts at a point A and runs east at $10\ \frac{ft.}{sec.}$ One minute later, another girl starts at A and runs north at $8\ \frac{ft.}{sec.}$ At what rate is the distance between them changing 1 minute after the second girl starts?

Let t = #secs since girl A starts

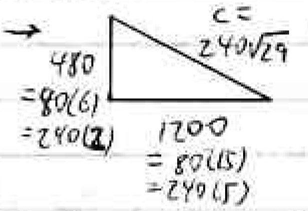


$c^2 = (10t)^2 + [8(t-60)]^2$

$c^2 = 100t^2 + 64(t-60)^2$

$2c\frac{dc}{dt} = 200t + 128(t-60)$

Let $t = 120$ →



$2(240\sqrt{29})\left(\frac{dc}{dt}\right) = 200(120) + 128(120-60)$

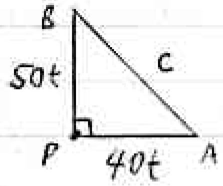
$\frac{dc}{dt} = \frac{66}{\sqrt{29}} \approx 12.3\ \frac{ft}{sec.}$

Prelim Ex

In case they want to take a shot at each others

Cars A and B leave Point P at the same time. Car A moves east at 40mph. Car B moves north at 50mph. Three hours later, how fast are they moving apart?

Let $t = \#$ hours elapsed $(t > 0)$



We want $\left.\frac{dc}{dt}\right|_{t=3}$

$$\text{Pyth. Thm.} \Rightarrow c^2 = (40t)^2 + (50t)^2$$
$$= 4100t^2$$
$$\Rightarrow c = \pm\sqrt{4100t^2}$$
$$= (10\sqrt{41})t$$

$$\Rightarrow \frac{dc}{dt} = 10\sqrt{41} \text{ mph} \approx 64.03 \text{ mph}$$ for __all__ $t > 0$

indep. of t!! Rate of change of distance bet. cars stays the same.

I'm going to do a variation, we assumed... the cars left...

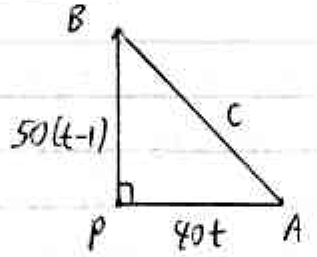
Ex Car B leaves Point P 1 hour later than Car A. Car A $\rightarrow$ 40mph, Car B $\uparrow$ 50mph. Four hours after A leaves P, how fast are A and B moving apart?

We must be precise!

Let $t = \#$ hours after A leaves P

Method 1

Is it $50t$?

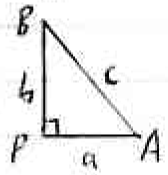


$(t > 1.)$

Pyth. Thm. $\Rightarrow c = \sqrt{(40t)^2 + [50(t-1)]^2}$

Find $\left.\frac{dc}{dt}\right|_{t=4}$ (a bit messy)

Method 2 (Related Rates)



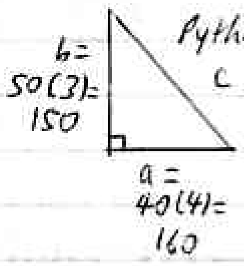
$$c^2 = a^2 + b^2$$

Find $\frac{dc}{dt}$ at $t=4$, if $\frac{da}{dt} = 40$, $\frac{db}{dt} = 50$

$$D_t(c^2) = D_t(a^2) + D_t(b^2)$$

$$\cancel{2}c\,\frac{dc}{dt} = \cancel{2}a\,\frac{da}{dt} + \cancel{2}b\,\frac{db}{dt}$$

At $t=4$,



Pyth. Thm. $\Rightarrow$ $c = \sqrt{48{,}100}$

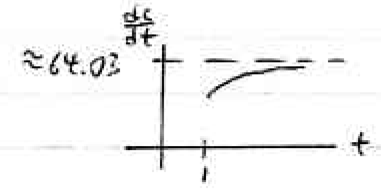
$$\sqrt{48{,}100}\,\frac{dc}{dt} = 160(40) + 150(50)$$

$$\left(\frac{dc}{dt} = \frac{13{,}900}{\sqrt{48{,}100}}\right)$$

$\frac{dc}{dt} \approx$ 63.38 mph (at $t=4$ hrs.)

Note $\frac{dc}{dt} \approx 63.64$ mph (at $t=5$ hrs.)

$\frac{dc}{dt}$ ↗, but not ↗∞. There's a limiting value - do you know what it is? (Not the #; the idea) We approach a situation where, for all intensive purposes, A, B started at same time. Problem: Globe not, ocean r.



As $t \to \infty$, $\frac{dc}{dt} \to 10\sqrt{41}$

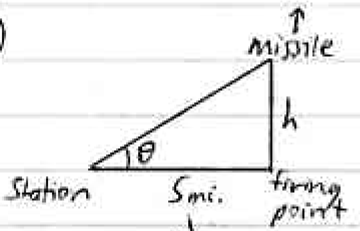
the effect of A's head start dissipates.

Ex (#46) (Don't have to write!)

In book, so don't have to write down. Looked up: weird motion is missile...: Missile going faster and faster - See 3.8 #39 comments 20-22° vs. 80-82°!! covered in 1 sec. The "5 mi." is relevant to the entire process; it stays constant - can put in general diagram!

① A missile is fired vertically from a point that is 5 miles from a tracking station and at the same elevation. [For now,] its angle of elevation θ changes at a constant rate of 2° per second. Find the velocity when the angle of elevation is 30°.

②



(Doesn't change!)

(Let h = height of missile.)

③ Relate h and θ.

Don't have to

$$\tan\theta = \frac{h}{5}$$
$$h = 5\tan\theta$$

What = 2°/sec.

④ Given: $\frac{d\theta}{dt} = \left(2\ \frac{\text{deg}}{\text{sec}}\right) \times \left(\frac{\pi\ \text{rad}}{180\ \text{deg}}\right) = \left(\frac{\pi}{90}\ \frac{\text{rad}}{\text{sec}}\right)$

Find $\frac{dh}{dt}$ (= velocity)

In case θ shows up later? e.g. $D_t(\theta^2) = 2\theta\frac{d\theta}{dt}$

when $\theta = 30^\circ = \left(\frac{\pi}{6}\ \text{rad}\right)$ (Just to be safe?)

Before, Pyth. Thm. What relates $\theta, h, 5$?

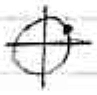
⑤ $h = 5\tan\theta$

$$D_t(h) = D_t(5\tan\theta)$$
$$\frac{dh}{dt} = 5(\sec^2\theta)\left(\frac{d\theta}{dt}\right)$$

⑥ Plug in $\theta = \frac{\pi}{6}$, $\frac{d\theta}{dt} = \frac{\pi}{90}$

(7) $\frac{dh}{dt} = 5(\sec^2\theta)\left(\frac{d\theta}{dt}\right)$

$= 5\left[\sec\frac{\pi}{6}\right]^2\left(\frac{\pi}{90}\right)$



$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

can leave as irrat'l
square will kill √

$= 5\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{\pi}{90}\right)$

$= 5\left(\frac{4}{3}\right)\left(\frac{\pi}{90}\right)$

$= \frac{20\pi}{270}$

$= \frac{2\pi}{27}\ \frac{mi.}{sec}$

$\approx 0.233\ \frac{mi.}{sec.}$

may want to convert to...

or $\left(\frac{2\pi}{27}\ \frac{mi.}{sec}\right)\left(\frac{3600\,sec}{1\ hr.}\right) \approx 837.8\ \frac{mi.}{hr.}$

(8) (Concl.)