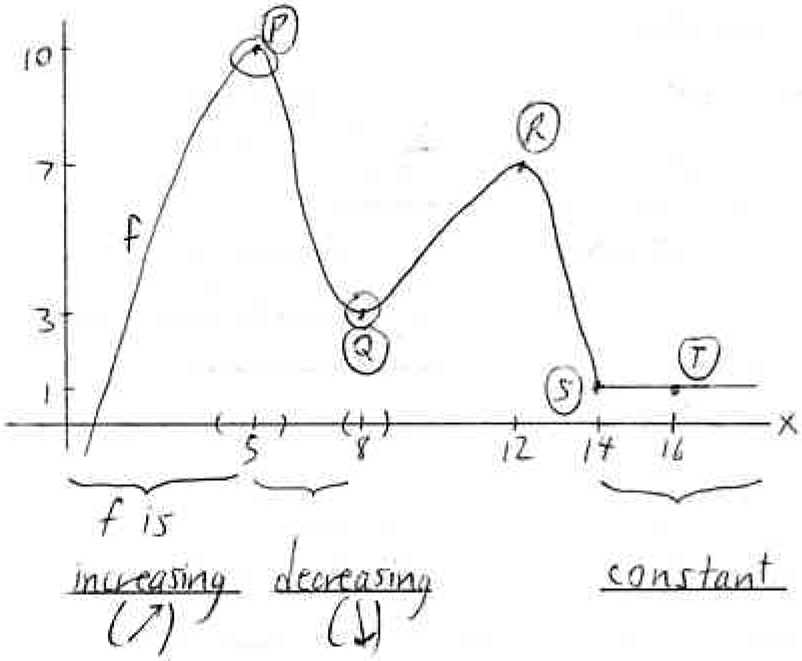
CH.4: APPLICS OF f'

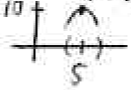
4.1: EXTREMA OF f

EXTREMA = max, min

Ⓐ Terminology



A = Absolute / Global
L = Local / Relative

Ⓟ is a L. Max Pt.  P: King of the Hill (highest pt. in some neighborhood around it)

Book doesn't specify cont.! p.164

bec. 10 is the max value of f on some open interval containing x=5

like Jim Cameron

and an A. Max Pt.  P: King of the World (highest pt. anywhere)

bec. 10 is the max value on the whole domain of f.

(Q) L. Min Pt.

not A. Min Pt.
(S lower)

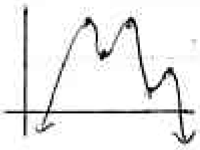
(R) L. Max Pt.
not A. Max Pt.

(S) L. Min Pt.
"Ties are OK"

Simpsons
we don't
take away
hot sash
Guys just as
low as you
are - in a bar

(T) L. Min Pt.
and L. Max Pt.

Ex



2 A. Max Pts.
0 A. Min Pts.

3 L. Max Pts.
2 L. Min Pts.

We go to hell
in both
directions.

Ex

Don't need "open
int" stuff for A. Max.
Larson agrees

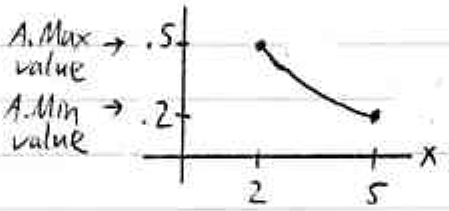
(P) is an A. Min Pt.
but not a L. Min Pt. (there is no open interval containing 6 on which f is defined)
2 is an endpoint extremum.

(B) Extreme Value Thm. (EVT)

Outside this interval, it can go nuts, like Pres. Bush in his bad days.

If f is <u>cont.</u> on a <u>closed</u> interval $[a,b]$, then we are <u>guaranteed</u> an A. Max. value <u>and</u> an A. Min. value on $[a,b]$.

<u>Ex</u> $f(x) = \frac{1}{x}$ on (the interval) $\underbrace{[2,5]}_{\text{Closed}}$ ← Think: "Restricted domain"

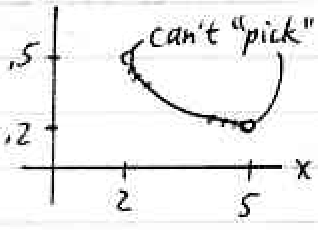


$\frac{1}{x}$ cont. here
So, EVT applies

It doesn't mean we <u>can't</u> have A. Min/Max, but no guarantee. Anything goes
Adversary argument
If we're competing to get the highest y-coord., we'd never stop
Nelson - ha ha!
If you pick this pt...
We're not allowed to take (2,.5)

<u>Ex</u> $f(x) = \frac{1}{x}$ on $(2,5)$

<u>not</u> closed
EVT does not apply (no <u>guarantee</u> of A. Max/Min values)



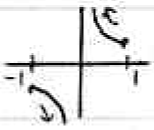
No A. Max. or A. Min. value.

.49
.499
.4999
⋮

Does the EVT apply?

<u>Ex</u> $f(x) = \frac{1}{x}$ on $[-1,1]$

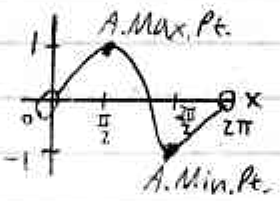
$\frac{1}{x}$ is not cont. at 0
EVT does not apply



No A. Max. or A. Min. value.

Does this mean we can't have abs. extrema? No- We don't have that guarantee.

Ex $f(x) = \sin x$ on $\underbrace{(0, 2\pi)}_{\text{EVT does not apply, yet}}$



(We get abs. extrema, anyway.)

Calculus more directly deals w/local behavior (lim as $x \to \infty$ notwithstanding)

What kinds of tangent lines can we have there?

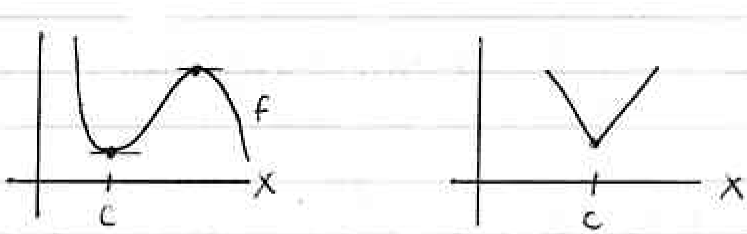
What's the deal w/ f'? (slope)

Think $|x|$

Ⓒ Critical #s (CNs)

If f has a L.Max. or L.Min. at $x = c$, then...

Exs

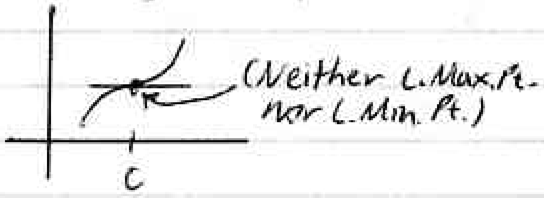


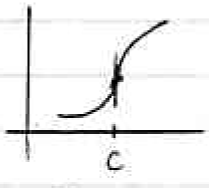
$f'(c) = 0$ or $f'(c)$ DNE

Can you think of a graph w/ a horiz. tan line at a pt. that's neither? (x^3)

If f' is 0, do we have to have a local min/max? No- the graph could "snake in"

Converse is not generally true.





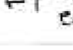
$f'(c)$ DNE, but...

$f'(c) = 0$, but not local ext.

The $\overset{\text{(CNs)}}{\underline{\text{critical \#s}}}$ of f are the #s in $\overbrace{\text{the domain of } f}^{\text{Dom}(f)}$ where f' is 0 or DNE.

L. Min., L. Max. can <u>only</u> appear at CNs.

A reason why we need "open interval"



$f' = 10$, e.g. not even DNE

EVT. why find CNs in (a,b)

cont. on

(D) Finding A.Max., A.Min. of f on $[a,b]$

exist by EVT

"Critical pts." used in multivar calc. M121 - Ch. 7, Swok 862
Where can we have abs. max/min? At local ext and..

Candidates: $\underbrace{\text{Pts. at CNs}}_{\text{L.Max? L.Min?}}$, or endpoints

a b

Steps

Can't have CN at a,b if treat $[a,b]$ as restricted domain.

can't be L.Max Pt.

Setting up the lineup
One by one we ask what's your func. value? (y-coord.?)

(1) Find all CNs in (a,b).

(2) Evaluate $f(c)$ for each critical "c"

(3) Evaluate $\underbrace{f(a)}$ and $\underbrace{f(b)}$

endpoint values

(4) Among the candidates from (2) and (3),
the highest is the A.Max value on $[a,b]$
the lowest " " A. Min. "

I'll specify on test abs./local

Ex Find the [absolute] extrema of
$f(x) = x^{1/3} - x + 4$ on $[-1, 8]$.

① Find all critical #s (CNs) in $(-1, 8)$

Where is $f'(x) = 0$ or DNE?

Right now, we can get a critical #.

$$f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}} - 1$$
$$= \frac{1}{3x^{2/3}} - 1$$

in Dom(f), in $(-1, 8)$

$f'(0)$ DNE, so
0 is a CN.

$$\text{Set } \frac{1}{3x^{2/3}} - 1 = 0$$

$\cdot 3x^{2/3}$

or $\frac{1}{3x^{2/3}} = 1$ Take recips. (if ≠0)
$3x^{2/3} = 1$
$x^{2/3} = \frac{1}{3}$

$$1 - 3x^{2/3} = 0$$
$$x^{2/3} = \tfrac{1}{3}$$
$$(x^{2/3})^3 = (\tfrac{1}{3})^3$$
$$x^2 = \tfrac{1}{27}$$
$$x = \pm\sqrt{\tfrac{1}{27}} \approx \pm 0.192$$

If $x^2 = 4$...
$x = \pm 3$
If we get 4, throw it out

CNs in $(-1, 8)$, in Dom(f)

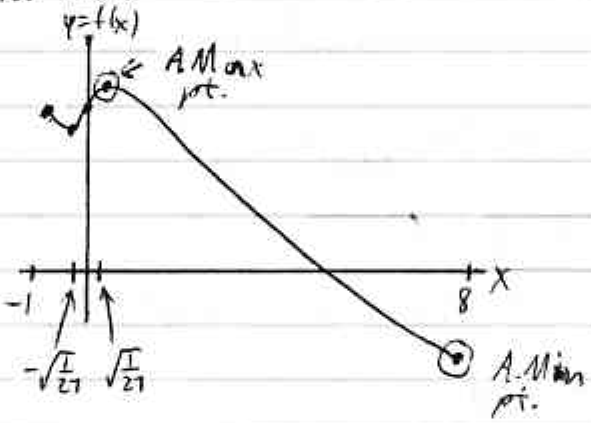
Not simplified

②-④ Table

Also look here for abs. ext. We may find L.Max/Min. Pts. would be A.Max/Min. Pts. Conv Condit

	x	not f' $f(x)$	Candidates	
	$a = -1$	$f(-1) = (-1)^{1/3} - (-1) + 4$	$= 4$	
critical #s	$-\sqrt{\frac{1}{27}}$	$f(-\sqrt{\frac{1}{27}})$	≈ 3.615	
critical #s	0	$f(0)$	$= 4$	on $[-1, 8]$
critical #s	$\sqrt{\frac{1}{27}}$	$f(\sqrt{\frac{1}{27}})$	≈ 4.385	← A. Max. value
	$b = 8$	$f(8)$	$= -2$	← A. Min.

Turns out



f' at $\pm\sqrt{\frac{1}{27}}=0$
$f(0)$ undef

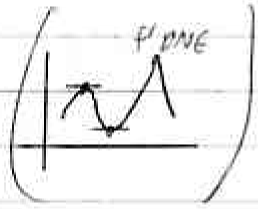
Up to 15

Can do 4.1 Hw, but we'll do some problems next time - algebraic tricks

Prereq

(E) Finding Critical #s of f (CNs)

in Dom(f) where $f'(x)=0$ or DNE



Me: $(3x+1)\sqrt{2x-1}$
$f' = \frac{9x-2}{\sqrt{2x-1}}$

Ex $f(x) = (3x+1)\sqrt{x^2-9}$

① Find Dom(f) - optional (but can help!)

If I know the sign of $x+3$ ...

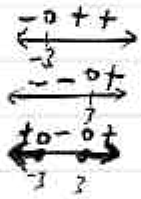
Me: $a \ge b$ ($a,b \ge 0$) $\sqrt{a} \ge \sqrt{b}$
$|x| \ge 3$

$$x^2-9 \ge 0$$
$$(x+3)(x-3) \ge 0$$

"Parabola Method"

OR Sign chart

$x+3$
$x-3$
$(x+3)(x-3)$



$$Dom(f) = (-\infty,-3] \cup [3,\infty)$$

② Find $f'(x)$

$$f(x) = (3x+1)\sqrt{x^2-9}$$

Use Product Rule

$$f'(x) = (3)(\sqrt{x^2-9}) + (3x+1)\cdot D_x (x^2-9)^{\frac{1}{2}}$$
$$= 3\sqrt{x^2-9} + (3x+1)\left[\tfrac{1}{2}(x^2-9)^{-\frac{1}{2}}(2x)\right]$$

Exercise in algebra
Or factor: $(x^2-9)^{-\frac{1}{2}}[3(x^2-9)+(3x+1)(x)]$

$$= 3\sqrt{x^2-9} + \frac{x(3x+1)}{\sqrt{x^2-9}} \leftarrow \text{LCD}$$

$\rightarrow$ One fraction

$$\left(= \frac{3\sqrt{x^2-9}}{1} \cdot \frac{\sqrt{x^2-9}}{\sqrt{x^2-9}} + \frac{3x^2+x}{\sqrt{x^2-9}} \right)$$

$$= \frac{3(x^2-9) + 3x^2+x}{\sqrt{x^2-9}}$$

$$= \frac{6x^2+x-27}{\sqrt{x^2-9}}$$

Yech, bad grammar. Mathematicians have bad grammar, anyway.

③ Where is $f'(x)$ DNE?

In Dom(f)

$\text{Dom}(f) = (-\infty, -3] \cup [3, \infty)$

Of these #s, -3 and 3 make $f'(x)$ DNE

CNs

④ Where is $f'(x)=0$?

$$f'(x) = \frac{6x^2+x-27}{\sqrt{x^2-9}}$$

Set numer. $= 0$ (hope denom. will be OK)

Solve $6x^2+x-27=0$ using QF: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$649 = 11 \cdot 59$ (Don't be foolish!)

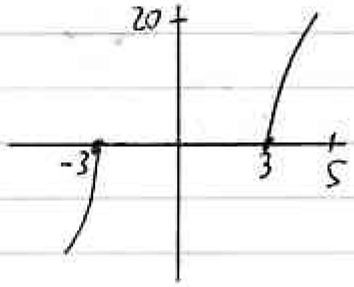
$$x = \frac{-1 \pm \sqrt{649}}{12}$$

$$x \approx -2.21, \; 2.04 \leftarrow$$ Not in Dom(f), so not CNs

Are there CN's where f' DNE but literally. What's the pt.?

The only CNs are $3, -3$.

Turns out



-2, 2i, 2,040 not in Dom(f)

Up to 2.3

This confused them!
If just
abs. min. pt.
Here, f' DNE at $-3, 3$
but if
$f'=0$
one sided
That would disobey "$f'=0$, DNE" need to have local ext.
Here, one-sided derivs happen to be DNE.

No A. Max, Min
No L. Max, Min
↳ Endpoints disqualified

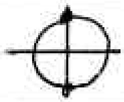
need $(\;)$

Ex $f(x) = \sec x$

① Find Dom(f)

$$f(x) = \frac{1}{\cos x}$$

Dom(f) = all reals except $\underbrace{\text{where } \cos x = 0}_{x = \frac{\pi}{2} + \pi n,\ n \text{ int.}}$



$$= \{x : x \neq \tfrac{\pi}{2} + \pi n,\ n \text{ int.}\}$$

② Find $f'(x)$

$$f'(x) = \sec x \tan x$$

③ Where is $f'(x)$ DNE?

$$f'(x) = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$ DNE when $\cos x = 0$

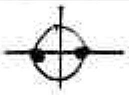
not in Dom(f); can't be CNs!

Are there CNs?
But what...?

④ Where is $f'(x)=0$?

$$f'(x)=\frac{\sin x}{\cos^2 x}$$

Where is $\sin x = 0$?



$x = n\pi$, n integer — CNs

in Dom(f) ✓

sin, cos shifted by $\frac{\pi}{2}$, so diff. zeros

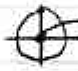
Note

$\sin x, \cos x$ are never simultaneously 0 (for the same x)

Reasons:

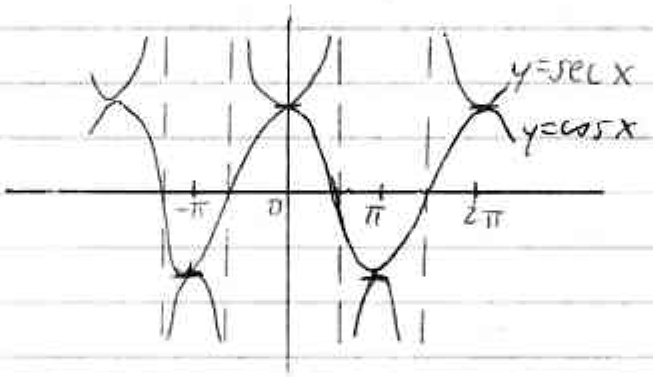
① Graphs shifted by $\frac{\pi}{2}$; different x-ints.

② $\sin^2 x + \cos^2 x = 1$ Not 0

③ 

Should Know

I might want to graph cos x 1st.



it makes sense that $f'(x)$ is DNE only at the VAs. Other than the VAs, graph is very smooth! No corner pts, no pts. w/ vertical tangents.

Note: sec x is never 0
csc x

Ex $f(x)=\tan x$ $\left(=\frac{\sin x}{\cos x}\right)$
$f'(x)=\sec^2 x$ $\left(=\frac{1}{\cos^2 x}\right)$ f' DNE $\Leftrightarrow$ f DNE

(Anything that makes f' DNE is not in Dom(f), so can't be a CN.)

$\sec^2 x = 0$ NEVER!

No CNs

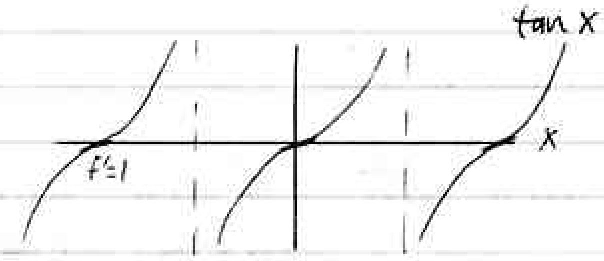
slope = $\sec^2(0) = 1$

Not like x^3!!

Note: $\sec^2 x \geq 1 > 0$!! $\forall x \in$ Dom(f)

Between VAs, snakes always go up

We'll see in 4.4



Most Valuable Thm? Rarely "see", but used in proofs. FTC! e.g., developing arc length. Larson (ES: some think most imp. thm.

4.2: MEAN VALUE THEOREM (MVT)

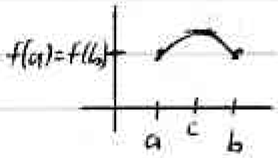
pron. rawls

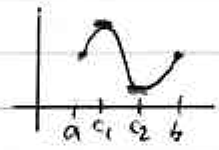
(A) Special Case: Rolle's Thm.

Assuming f is cont., What goes up... must come down.

When is f *guaranteed* to have a critical # (CN)?

Look at all those CNs! (shake cable) (I've looked at your tests... Hope I don't hurt myself.)





My cable!

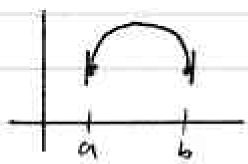
Our CNs can't have f' be DNE.

Thm If (1) f is cont. on $[a,b]$,
(2) f is diff'e on (a,b), *and*
(3) $f(a)=f(b)$
} *hypotheses*

Then, $f'(c)=0$ for some c in (a,b). ← *conclusion*
(Maybe more than one!)
(i.e., $\exists$ horiz. tangent line somewhere in there)

Why not $[a,b]$? (a,b) more powerful. At a, b, only one-sided derivs.

Note Why not "diff'e on $[a,b]$"?



Rolle OK!
(We want Rolle's Thm. to be powerful enough to apply to these cases, too.)

would exclude this ex. (Do you remember a bounded curve where we had vertical tan lines at the ends?)

(*Note 2* If (2) fails, we get critical #s (CNs) where f' is DNE. $\nleftarrow$)

If we have DNE somewhere, FINE! (1) ensures that that # is in Dom(f). If (2) fails, we get crit #s where f' DNE $\nleftarrow$

Ex Consider $f(x)=x^4-18x^2$ on $[-4,4]$

(1),(2) $f(x)$ = poly. → f is con't $[-4,4]$ + diff'e $(-4,4)$ (everywhere)

(3) $f(-4) = (-4)^4 - 18(-4)^2 = -32$
$f(4) = (4)^4 - 18(4)^2 = -32$ } or f is even (-4, 0, 4 sym.)

So, Rolle's Thm. applies.

Find all #s "c" in $(-4,4)$ such that $f'(c)=0$.

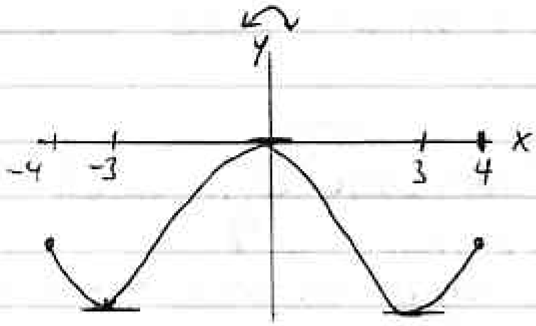
Back of book uses c

$$f(x)=x^4-18x^2$$
$$f'(x)=4x^3-36x$$
$$=4x(x^2-9)$$
$$=4x(x+3)(x-3)\overset{\text{set}}{=}0$$

$0,-3,3$ ← all in $(-4,4)$ ✓

Turns out

f even, diff'e → We have to have either a turnaround pt. or a flat graph at 0 corresp. to a constant func. around there



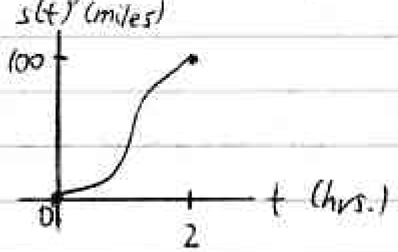
f even

Could do 3, 5, 7, not 1 one it say "is this"

(B) MVT in General

Christine
It drives itself
What can you say about the vel.?
No Back to the Future stunts
[illegible] disappears
continuity

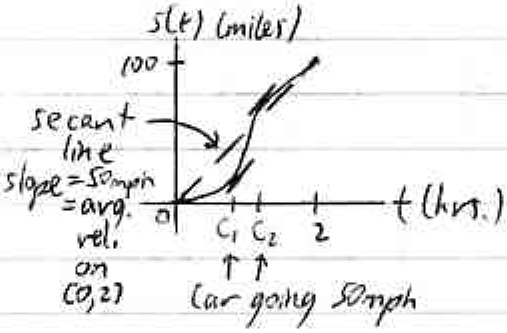
Ex A car goes 100 miles in 2 hours. → (In one direction)



s is cont. on $[0,2]$
diff'e on $(0,2)$

What's the slope of secant line?
What do you think the MVT allows us to conclude? If the avg...
Speedometer must hit 50.
If tell Keanu "go 100 mi. in 2 hrs. Don't hit 50mph, die if don't make it in time" Goodbye...
At how many times is

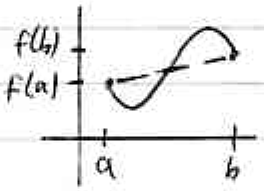
Average (mean) velocity on $[0,2]$ is 50 mph.
MVT ⇒ The car must go 50mph at some instant in $(0,2)$.



Where is tangent line ∥ secant line?
parallel to
(same slope)

$f'(c_1) = 50$ mph
c_2

(MVT)
Thm



If f is
① cont. on $[a,b]$, and
② diff'e on (a,b)
} (Often satisfied!)

Then $\exists\, c$ in (a,b) such that...

familiar

$$f'(c) = \underbrace{\text{slope of secant line}}_{\dfrac{f(b)-f(a)}{b-a}}$$

Bec. looks like 1-liners!

$$\text{or} \quad f(b)-f(a) = [f'(c)][b-a]$$

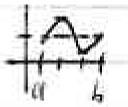
Special Case: If $f(b)=f(a)$, $\exists c$ in (a,b):

$$f'(c) = \frac{\overset{0}{\cancel{f(b)-f(a)}}}{\underbrace{b-a}_{\neq 0}} = 0$$

Rolle's Thm is used in proof

Rotating/tilting my perspective

tilt a function

Rolle  (Again, we look for where tangent line $\parallel$ secant line.) horiz.

Ex Consider $f(x) = x^4 - x$ on $[2,5]$

f is cont. on $[2,5]$, diff'e on $(2,5)$ $\Rightarrow$ MVT applies.

Find a "c" in $(2,5)$ such that ...

$$f(b)-f(a) = [f'(c)][b-a]$$

$$f(5)-f(2) = [f'(c)]\underbrace{[5-2]}_{=3}$$

$$f(5) = (5)^4 - (5) = 620$$

$$f(2) = (2)^4 - (2) = 14$$

$$620 - 14 = [f'(c)][3]$$

$$f'(c) = 202$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(2)}{5-2} = \frac{620-14}{3} = 202$$

For what c is $f'(c)=202$?

$$f(x)=x^4-x$$

$$f'(x)=4x^3-1 \overset{set}{=} 202$$

$$x^3=\frac{203}{4}$$

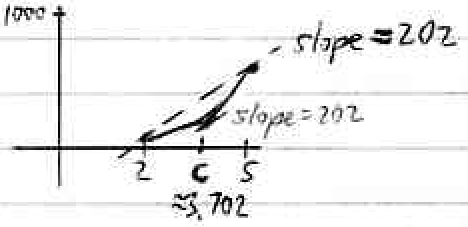
203 ÷ 4 =
$\boxed{A}$ .333... $\boxed{=}$
1 (a b/c) 3

$$x \text{ or } c = \sqrt[3]{\frac{203}{4}} \approx 3.702 \quad \text{in } (2,5) ✓$$

Turns out

Ain't so interesting on this interval!

Draw tan line 1st

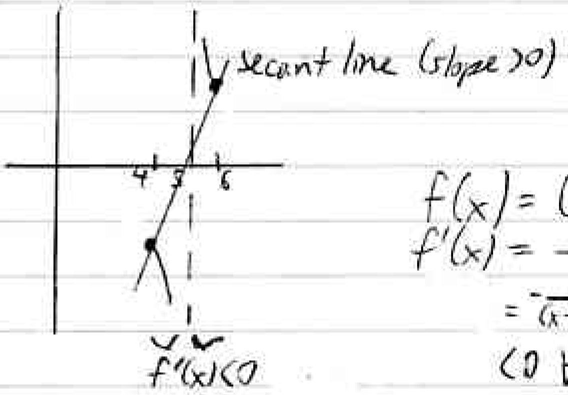


Ex Consider $f(x)=\frac{1}{x-5}$ on $[4,6]$

MVT might not apply but concl. may work out.

f is discont. at $5 \Rightarrow$ MVT does not apply

Are there any pts in here w/a ∥ tangent line?



$$f(x)=(x-5)^{-1}$$

$$f'(x)=-(x-5)^{-2}(1)$$

$$=-\frac{1}{(x-5)^2}$$

$$<0 \ \forall x \neq 5$$

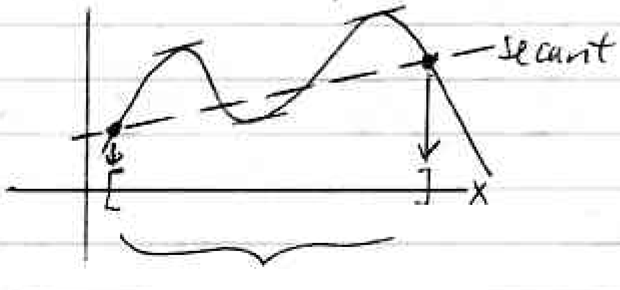
∥ tangent line is possible, but not guaranteed!

That's not the neat part.

Corollary of MVT (Neat!)

what's all this junk about closed intervals? Cooked up?

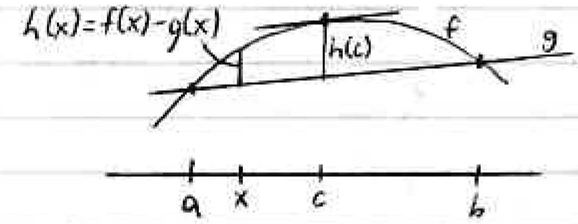
If f is $\underbrace{\text{diff'e everywhere}}_{}$ (implies cont. everywhere), pick <u>any</u> 2 distinct points on graph



Here's your closed interval.

must have a $\parallel$ tangent line in here! (MVT applies)

MVT Proof Idea



$$\left.\begin{array}{l} h(a)=0 \\ h(b)=0 \\ h \text{ cont. on } [a,b] \\ \quad \text{diff'e on } (a,b) \\ (\text{bec. } f,g \text{ are}) \end{array}\right\} \Rightarrow \text{Apply Rolle's Thm. to } h \text{ on } [a,b]$$

$$\Rightarrow \exists c_{\text{in }(a,b)} : h'(c)=0$$

$$\text{i.e., } f'(c)-g'(c)=0$$

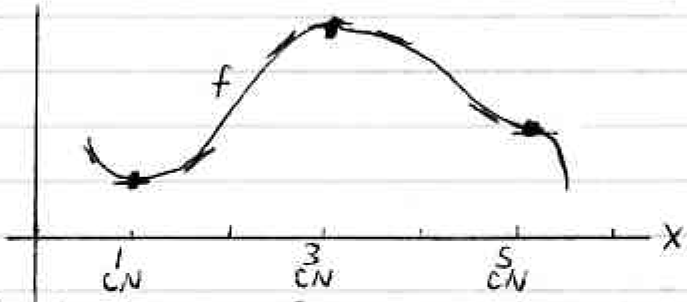
$$f'(c)=\underbrace{g'(c)}_{\text{slope of secant line}}$$

Note: c is where there's max. dist. bet f, g (in the leaf)

4.3: THE FIRST DERIVATIVE TEST (1st DT)

Ⓐ Where is f Increasing ($\nearrow$), Decreasing ($\downarrow$)?

Ex



Gotta be defined on (1, 3)
No

This break it decreasing

$f'(x)$: $-$ 0 $+$ 0 $-$ 0 $-$ (Slopes of tangent lines)

f can ↗ anyway; maybe ↗ ... f' DNE

Pt. on graph when describing $f \nearrow$.

$f'(x) > 0$ (+) on (1, 3) (interval), so $f \nearrow$ on $(1,3)$ or $[1,3]$ (f cont.) (Pts. to the right are higher than pts. to the left.)

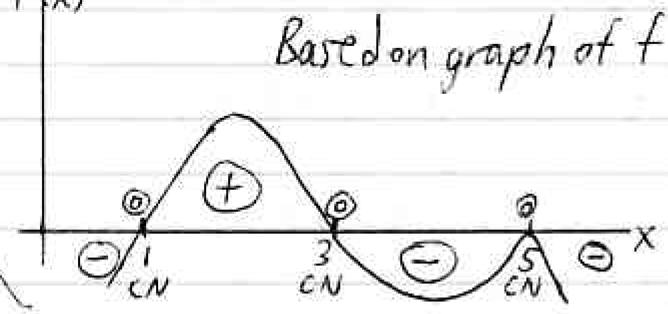
What do we mean by "increasing"?
(In $[1,3]$, if $d > c$, then $f(d) > f(c)$.)

$f'(x) < 0$ (−) on $(3,5)$, so $f \downarrow$ on $(3,5)$

"Test Value" Idea

Based on $f \nearrow$

Note ($f'(x)$ Based on graph of f above)

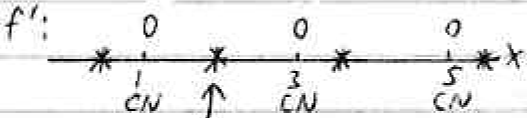


Gelbaum 36
f' ∃ but discont. at 0
$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
We sort of reverse IVT.
IVT ⇒ if f' cont., change sign bet. a, b, then 0 in between.

The CNs (and where f' discont.) are the only places where f' can change sign.

Does it have to change sign? Look at 5.

Let's say I didn't have any graphs, but I knew the 0s are at 1,3,5. Book calls f'(c) the test values



The sign of f' here is the same throughout (1,3).
Can test x=2, 1.1, 2.5, ...

To completely analyze the sign of f' we just have to sample the sign at these 4 places. ∞ problem → finite problem

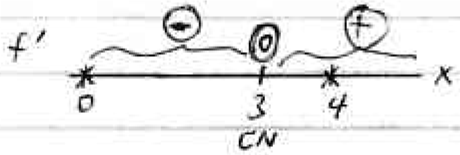
What's the graph of that? Don't use that! (Precalc knowledge) Use calc where possible.)

<u>Ex</u> $f(x) = x^2 - 6x + 8$

$f'(x) = \underbrace{2x-6}_{\text{never DNE}} \overset{\text{set}}{=} 0$

$x = 3$

can also pick 2



Book calls −6, 2 test values

$f'(0) = 2(0) - 6 = -6$ ⊖

$f'(4) = 2(4) - 6 = 2$ ⊕

<u>Graph f</u>

What is the easiest pt. to find on the graph?

$y\text{-int.} = f(0)$
$= 8$

Find x-ints: $f(x) = x^2 - 6x + 8 \overset{\text{set}}{=} 0$

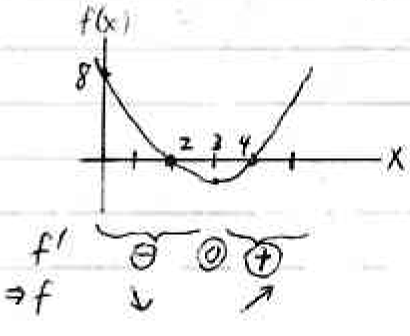
$(x-4)(x-2) = 0$

$x = 2, 4$

y-int = 8, but we don't show that

Vertex: (3,−1)

What kind of pt. do we have here?



(B) 1st DT

to classify pts. at CNs as L. Max. Pts., L. Min. Pts., or Neither

Assume f cont. at c, a CN (NO )

Same order as in book

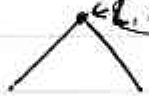
Don't say if local max pt. → ⊕ → ⊖

 Counterex.

(1)

L. Max Pt.

If f': ⊕ 0 ⊖ then
c
f: ↗ ↘



L. Max Pt.

⊕ DNE ⊖
c

(Doesn't matter if $f'(c) = 0$ or DNE)

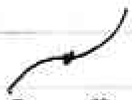
(2)

L. Min Pt

If f': ⊖ ⊕ then
c
f: ↘ ↗

When do we have Neither?

(3)



Neither

If f': ⊕ ⊕ then
c
f ↗ ↗

Neither

If f': ⊖ ⊖ then
c
f ↘ ↘

Ⓒ Exs

Ex $f(x) = 2x^3 - 9x^2 - 24x$

What can f' tell you? What secrets?

ⓐ Find the intervals on which $f \nearrow$ or $\searrow$.
ⓑ Find the local extrema.

$$f'(x) = 6x^2 - 18x - 24 \overset{\text{set}}{=} 0 \quad (f' \text{ never DNE})$$

or $6(x^2 - 3x - 4) = 0$

(Avoid ÷6; you won't have $f'(x)$ anymore.)

$$6(x-4)(x+1) = 0$$

$$\underbrace{x = 4, -1}_{\text{CNs}} \quad \text{in Dom}(f) \checkmark$$

What might be convenient?

$-2 \quad -1 \quad 0 \quad 4 \quad 5 \quad x$

-1: CN, 4: CN

If 2 terms "+" / "−", do I know sign of sum? No!

$$f'(x) = (6)(x-4)(x+1) \leftarrow \text{Factored form, better for sign analysis.}$$

$$f'(-2) = (6)(-2-4)(-2+1) \leftarrow \text{Can skip.}$$
$$= (+)(-)(-)$$
$$= \oplus$$

$$f'(0) = (+)(-)(+) = \ominus$$
$$f'(5) = (+)(+)(+) = \oplus$$

f': $\oplus$ $\quad -1 \quad$ $\ominus$ $\quad 4 \quad$ $\oplus$ $\quad x$

$f \nearrow$ on $(-\infty, -1]$ $\quad$ $f \searrow$ on $[-1, 4]$ $\quad$ $f \nearrow$ on $[4, \infty)$

Apply 1st DT: (f cont.)

L. Max ($\frown$) at -1 $\quad$ L. Min ($\smile$) at 4

L. Max Pt.

← not f'

$(-1, \underbrace{f(-1)})$

Be careful! Don't plug into f'. What get? If you get 0, be suspicious.

$$\left(\begin{aligned} f(-1) &= 2(-1)^3 - 9(-1)^2 - 24(-1) \\ &= 13 \end{aligned} \right)$$

↑ NOT f'!!

$(-1, 13)$

L. Min Pt.

$(4, \underbrace{f(4)})$

$$\left(\begin{aligned} f(4) &= 2(4)^3 - 9(4)^2 - 24(4) \\ &= -112 \end{aligned} \right)$$

$(4, -112)$

ⓒ Sketch the graph of f.

Find x-ints (if you can) ← I'll tell you if I want the x-ints.

$$f(x) = 2x^3 - 9x^2 - 24x \overset{\text{set}}{=} 0$$

$$x\underbrace{(2x^2 - 9x - 24)} = 0$$

$x = 0$ QF: $\left(x = \dfrac{9 \pm \sqrt{273}}{4} \right)$ $x \approx -1.9, 6.4$

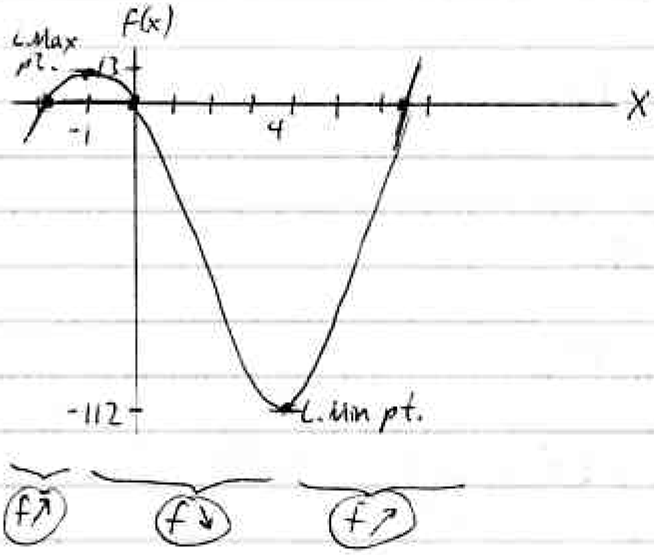
$$\left(\begin{aligned} &\text{Sol'ns of } ax^2 + bx + c = 0 \quad (a \neq 0) \\ &\text{are } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &\text{Factoring works nicely if } b^2 - 4ac \text{ (discriminant) is a nice square.} \end{aligned} \right)$$

y-int. $= f(0) = 0$

Using basic observations regarding f, f'.

Noman's land

Later, we'll find out more about graphs, we'll better interpolate, extrapolate.



What does $\cos x$ look like? What does $+x$ do? Upward drift! Do we still get hills, valleys?

<u>Ex</u> Consider $f(x) = \cos x + x$ on $[-2\pi, 2\pi]$.

Find the local extrema.

$$f'(x) = -\sin x + 1 \overset{\text{set}}{=} 0 \quad (f'(x) \text{ never DNE})$$

$$1 = \sin x$$

$$x = \frac{\pi}{2} + 2\pi n, \ n \text{ int.}$$

$$\boxed{x = \frac{\pi}{2}, -\frac{3\pi}{2}} \text{ in } [-2\pi, 2\pi]$$

f': $+$ 0 $+$ 0 $+$ on -2π, $-\frac{3\pi}{2}$ (CN), $\frac{\pi}{2}$ (CN), 2π $\Rightarrow$ No local ext.

Could do $\pm 2\pi$

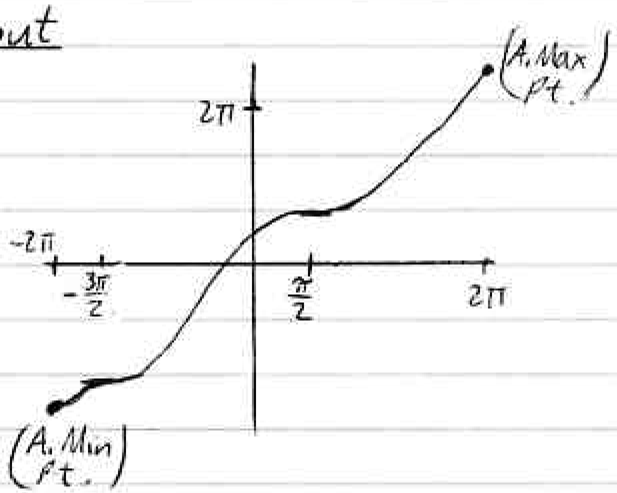
$-\sin\left(-\frac{2\pi}{4}\right) = \sin\left(\frac{2\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ $\oplus$

$$f'(x) = 1 - \underbrace{\sin x}_{\text{in } [-1,1]} \quad \text{never } \ominus \text{, only } 0 \text{ at CNs}$$

(Don't have to do test values!)

By EVT...
$(2\pi, 1+2\pi)$
$(-2\pi, 1-2\pi)$
$\exists$ A. Max, Min.

<u>Turns out</u>



(No local ext.)

$f \nearrow$ on $[-2\pi, 2\pi]$, even though $f' = 0$ sometimes

You'll get a hor. tan line every 2π units if you extend the graph

4.4: f''

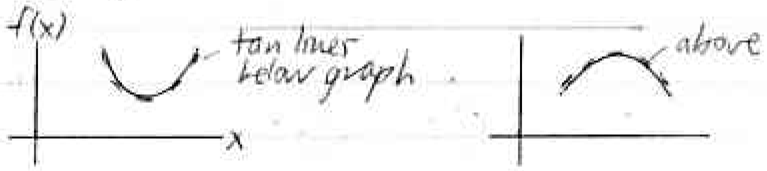
SAT analogies

Ⓐ f is to f' as
f' f''

f ↗ if f' +
f' f''

What does f' tell me about the graph of f? What about f''?
(locally)

Ⓐ Concavity



f graph is
concave up (CU) concave down (CD)

On which graph are the slopes ↗?

what does that tell me about f'?

Can't write confusing

not ⇔
If f' ↗ maybe f'' DNE

Ⓐ If f'' +
⇒ f' ↗
⇒ f graph (CU)
(Key)

"Go together" word play

If f'' −
⇒ f' ↘
⇒ f graph (CD)
(Key)

How to Ace it

Up like a cup

Down like a frown

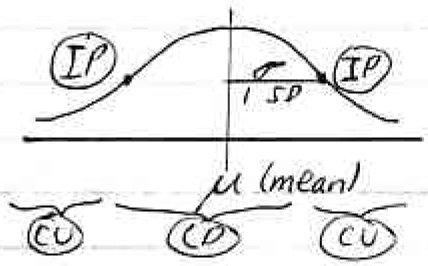
Ⓑ Inflection Pts. (IPs)

where concavity changes (CU ⇄ CD
f cont.

Exs

Berkeley div bio upper 2000 students

Bell curve (Normal)

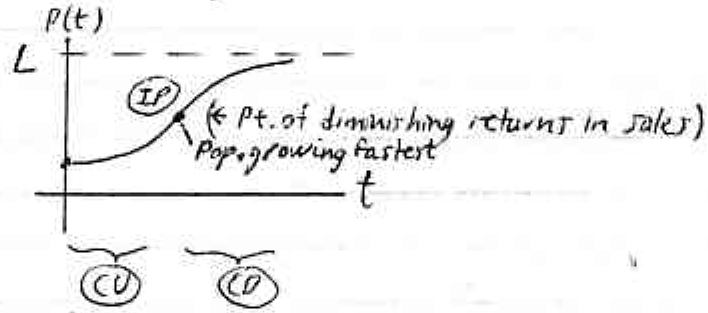


$\approx$68% of data w/in 1 SD
$\approx$95% 2
$\approx$99.7% 3

Malthus
Pop. exp. ↗
Food supply linearly ↗
carrying capacity
1960s said
$L = 10^8$

Don't get bang for buck you used to

Pop. growth (Logistic Model)



f' ↗ "India" f ↗ at ↗ rate

f' ↘ "China" f ↗ at ↘ rate

(2nd deriv.: rate of change of rate of change of f)

Extra credit if you give me your ATM PIN #

A CN is a #
in Dom(f) where
$f' = 0$ or DNE

A possible inflection #
PIN ' ;
f'' ' ;

Special Pts. for f, f'

CNs are the ~~only~~ places where
(L. Max/Min. Pts.) can appear.

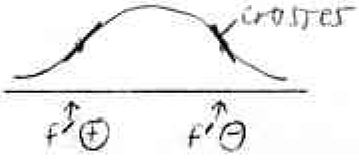
PINs ' ;
IPs ' ;

f' can be anything at an IP.

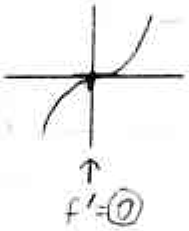
Skim

Tan line crosses bet. f'' + 0 − / − 0 + above → below / below → above

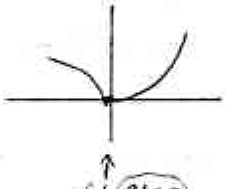
Exs



$f(x) = x^3$



f can't be DNE at IP, though



(No tan line at 0)

from 4.3©

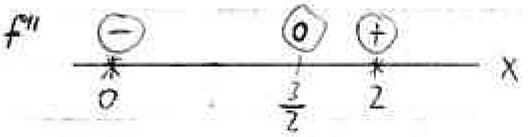
4.3 Ex $f(x) = 2x^3 - 9x^2 - 24x$

What does f'' tell me?

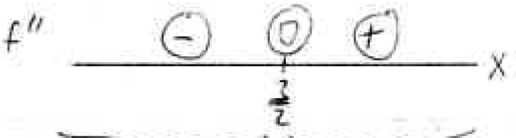
ⓐ Where is the f graph (CU), (CD)?
ⓑ Find any IPs.

just need f' to find f''

$$f'(x) = 6x^2 - 18x - 24$$
$$f''(x) = 12x - 18 \overset{\text{set}}{=} 0 \quad (f'' \text{ never DNE})$$
$$x = \tfrac{3}{2}$$
only PIN in Dom(f) ✓



Test
$$f''(0) = 12(0) - 18 = -18 \quad \ominus$$
$$f''(2) = 12(2) - 18 = 6 \quad \oplus$$

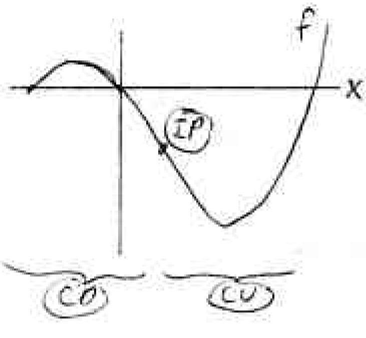


f graph: CD $\xrightarrow{\text{change}}$ CU
$(-\infty, \tfrac{3}{2})$ IP $(\tfrac{3}{2}, \infty)$ } answer to ⓐ

IP at $(\tfrac{3}{2}, f(\tfrac{3}{2}))$
$(\tfrac{3}{2}, -49.5)$ } ⓑ

If I tell you here $f''=0$ and f graph CU, do you believe

Before, we saw a No Man's Land
Can better inter/extrapolate graph



ⓒ 2^{nd} DT

2^{nd} DT may be easier to use, no need to do steps to find graph, but not as applicable. If $f'(c)=0$ ⇒ f defined at c ∴ c in Domain(f), c is CN

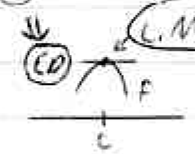
Alternative to 1^{st} DT for classifying pts. at CNs.
(2^{nd} DT may be easier to use, but it may not apply or work sometimes when 1^{st} DT does.)

Assume $\underbrace{f'(c)=0}_{\text{horiz. tan line}}$ (if DNE ⇒ Use 1^{st} DT)

1^{st} DT: Use ↗, ↘
2^{nd} DT: Use concavity + ∃ horiz. tan line

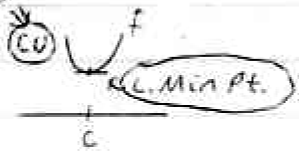
What's $f''(c)$?

① If ⊖



(Wordplay fails. Think: Concavity.)

② If ⊕



③ If ⓪ or DNE ⇒
2^{nd} DT is <u>useless</u>!
Use 1^{st} DT.

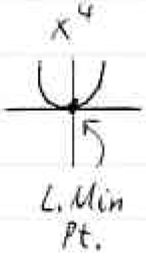
Much like Spears.

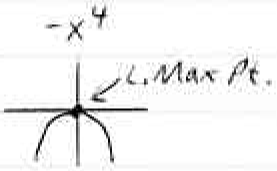
<u>Exs</u> $(f'(0)=0,\ f''(0)=0)$

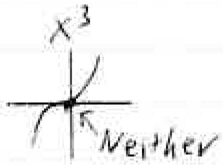
Anything goes!

If odd func cont. at 0, has crit. pt there → Neither unless ✚?
e.g., ✚

If even, L. Max or Min pt.







Anything goes!

4.4/4.5: GRAPHING EXS

Ⓐ Checklist

Old Issues

Where do we literally have f?

What property do we lack? Same if f rat'l, alg. Discrepancy if 2

What kind of discont.? Rat'l Helps w/ graphing

A graph can cross a HA.

Formal VA: $\exists c$ s.t.

What if

① Find Dom(f)
(Maybe find Range(f) later.)

② Where is f cont.?

} Same if f algebraic (rational, plus roots OK)

(VAs? Holes? What kind of discont.?)

③ VAs? HAs?

$\left(\lim_{\substack{x\to c^- \\ \text{or } c^+}} |f(x)| = \infty\right)$ Investigate $\lim_{\substack{x\to\infty \\ -\infty}} f(x)$. Do we get real #s?

④ x-ints? y-int?

Solve $f(x) = 0$ or y (Can be hard)

$f(0)$ (Usually easy)

⑤ Symmetry?

f even f odd

y

New Issues.

Find $f'(x)$	Find $f''(x)$
(1) Find <u>critical #s</u> (CNs) "c"	$\xrightarrow[\text{Need } f'(c)=0 \text{ (horiz. tan line)}]{2^{nd} \text{ DT?}}$ (2) Classify pts. at CNs ↳ Max/Min pts.? (Use 1^{st} DT if you have to graph f, anyway.)
(3) Where is f ↗, ↘?	If $f'(c)$ DNE, If $f''(c)=0$ or DNE, or If f'' hard
(4) Use 1^{st} DT to classify pts. at CNs.	
(Also) Code for ∨, ∧.	
	(5) Find PINs.
	(6) Where is f graph CU, CD? IPs?

$f'(c)$ DNE ⇒ can't apply 2^{nd} DT
When was 2^{nd} DT inconclusive?

(7) Graph
Range?

(B) Exs

Don't skim; need big picture. recall

We can use the old stuff

Constant → HA

Ex Graph $f(x) = 4x^5 - 25x^4$

Old

1. $Dom(f) = \mathbb{R}$
2. f cont. on $\mathbb{R}$
3. No VAs, HAs

$\Big\}$ $f(x)$ = nonconstant poly.

4. Find x-ints (if feasible)

$$\text{Set } 4x^5 - 25x^4 = 0$$
$$x^4(4x - 25) = 0$$
$$\Downarrow \qquad \Downarrow$$
$$x = 0 \qquad x = \frac{25}{4} = 6.25$$

(0,0)

$$\text{y-int} = f(0) = 0$$

5. f not even, odd

New

polys → nice derivs, not even vertical tangent lines!

$$f(x) = 4x^5 - 25x^4$$
$$f'(x) = 20x^4 - 100x^3$$
$$f''(x) = 80x^3 - 300x^2$$

1. Find critical #s (CNs)

$$f'(x) = \underbrace{20x^4 - 100x^3}_{\text{never ONE. Factor!}} \overset{\text{set}}{=} 0$$
$$\| $$
$$20x^3(x - 5) = 0$$

(Do not $\div x^3$. Just factor the left side; you want a re-expression of $f'(x)$. Don't even $\div 20$!!)

$$\Downarrow \qquad \Downarrow$$
$$0 \qquad 5 \quad \text{in } Dom(f) \checkmark$$

CNs

(Since we want to graph f, anyway, we can skip ②.)

② <u>Use 2^{nd} DT to classify pts. at CNs.</u> (or 1^{st} DT ③)
(Fair for a "shorter" question.)

Factoring a little weird

Jack squat

$$f''(x) = 80x^3 - 300x^2$$

$f'(0)=0$, but $f''(0) = 0 \Rightarrow$ Use 1^{st} DT for $x=0$.
Inconclusive!

Sign of f'' $\Rightarrow$ Think concavity!

$f'(5)=0$, $f''(5) = 2500$

$\oplus \Rightarrow$ CU

$f \cup$

L. Min. pt. at $(5, f(5))$
$(5, -3125)$

Need 1^{st} DT for $x=0$.

③ <u>Where is $f \nearrow, \searrow$?</u>

④ <u>Use 1^{st} DT</u> (to classify pts. at CNs)

Break at any disconts., CNs.

Factored form better for sign analysis.

Assume cont.

f' $\oplus$ $\ominus$

$f''(c)$ not positive

See f vs. f'

$f'=0$ $f'<0$

f' DNE

f': (cont.)	$\oplus$		0	$\ominus$		0		$\oplus$	
x	-1		0	1		5		6	
f:	$\nearrow$			$\searrow$				$\nearrow$	
1^{st} DT $\Rightarrow$			L. Max Pt.			L. Min. Pt.			
			$\frown$			$\smile$			
			$(0, f(0))$			$(5, f(5))$			
			$(0,0)$			$(5, -3125)$			
			also y-int.						

$$f'(x) = (20)(x^3)(x-5)$$
$$f'(-1) = (+)(-)(-) = \oplus$$
$$f'(1) = (+)(+)(-) = \ominus$$
$$f'(6) = (+)(+)(+) = \oplus$$

$f'(-1) = 120$
$f'(1) = -80$

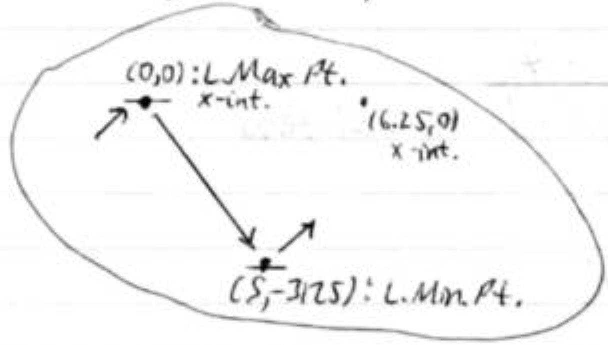
If f defined, discont. at c

$\Rightarrow f'(c)$ DNE, anyway, c a CN bec. in Dom(f)

(Make sure you plug into $f(x)$, not $f'(x)$ or $f''(x)$.)

$f \nearrow$ on $(-\infty, 0] \cup [5, \infty)$
$f \searrow$ on $[0, 5]$

Skeleton graph (optional, but helpful):



(If $f'(0)$ were DNE, don't put horiz. tan line —.)

⑤ Find P.I.Ns

$$f''(x) = \underbrace{80x^3 - 300x^2}_{\text{never DNE}} \overset{\text{set}}{=} 0$$

$$\| $$

$$20x^2(4x-15) = 0$$

$x = 0 \quad x = \frac{15}{4}$ in Dom(f) ✓ PINs

(0 happens to be both CN, PIN.)

Can look at our guess graph?

⑥ Where CU, CD? IPs?

Break at any disconts, PINs.

f'': (cont.) ⊖ at -1, ⊖ at 1, ⊕ at 5 ← knew from ②

x: $-1 \quad 0 \quad 1 \quad \frac{15}{4} \quad 5$

$= 3.75$

f: CD CD CU

no IP!

IP at $\left(\frac{15}{4}, f\left(\frac{15}{4}\right)\right)$

$(3.75, \approx -1977.54)$

$f''(x) = 20x^2(4x-15)$

$f''(-1) = (+)(+)(-) = ⊖$

$f''(1) = (+)(+)(-) = ⊖$

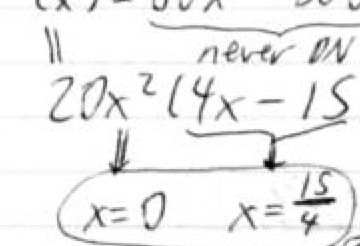
$f''(5) = (+)(+)(+) = ⊕$

$\frac{15}{4} = 3\frac{3}{4}$

If f defined, discont. at $c \Rightarrow c$ CN anyway, but not IP. corner

CD CU possible for LMax but for poly f' DNE? Do 4.4

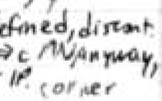
⑦ Graph



Note: Range $= \mathbb{R}$ $(-\infty, \infty)$

(Windows! Between, beyond CNs, IPs, disconts.)

Equidist from 5
3.75, 6.25



Unlike Microsoft Windows, these work.

Ex (4.5#22)

Old

Sketch $f(x) = \dfrac{-4}{x^2+1}$ ← no real zeros

nice cont'd func, no weird breaks

① $Dom(f) = \mathbb{R}$

② f cont. on $\mathbb{R}$

③ No VAs

HA: $y = 0$ ($f(x)$ = proper rational)

f' always < 0

④ No x-ints ($f(x)$ never 0)

y-int $= f(0)$
$= -4$

⑤ f is even

New

$$f(x) = \frac{-4}{x^2+1}$$

$$f'(x) = \left(\frac{\cancel{(x^2+1)(0)}^{0} - (-4)(2x)}{(x^2+1)^2} \right)$$

$$= \frac{8x}{(x^2+1)^2}$$

Skip in class

$$f''(x) = \frac{(x^2+1)^2(8) - 8x[2(x^2+1)(2x)]}{(x^2+1)^4}$$

$$= \frac{8(x^2+1)^2 - 32x^2(x^2+1)}{(x^2+1)^4} \quad \leftarrow \div (x^2+1) \neq 0$$

$$= \frac{8(x^2+1) - 32x^2}{(x^2+1)}$$

$$= \frac{-24x^2+8}{(x^2+1)^3}$$

If just sticking to local ext. 1st DT
Granted, didn't have to simplify

$$f''(x) = \frac{-8(3x^2-1)}{(x^2+1)^3}$$

① Find CNs

$$f'(x) = \frac{8x}{\underbrace{(x^2+1)^2}_{\substack{\text{never DNE} \\ (x^2+1 \text{ never } 0)}}} \overset{\text{set}}{=} 0$$

Don't mess w/
When does a fraction $= 0$?
Num $= 0$, Den $\neq 0$

$$8x = 0$$

$$\underbrace{x = 0}_{CN} \quad \text{in Dom}(f) \checkmark$$

② 2nd DT (can skip)

$f'(0) = 0$, so we can apply 2nd DT.
$f''(0) = 8$ (+)

(CU) f

L. Min. pt. at $(0, f(0))$
$(0, -4)$ also y-int.

③ Where is f ↗, ↘?
④ 1st DT

f': (−) 0 (+)

x: −1, 0, 1

f: ↘ ↗

1st DT ⇒ L. Min pt.

$(0, f(0))$
$(0, -4)$

$$f'(x) = \frac{(8)(x)}{(x^2+1)^2}$$

$$f'(-1) = \frac{(+)(-)}{(+)} = (-)$$

$$f'(1) = \frac{(+)(+)}{(+)} = (+)$$

f ↘ on $(-\infty, 0]$
f ↗ on $[0, \infty)$

↘.↗
$(0, -4)$

⑤ Find PINs

$$f''(x) = \underbrace{\frac{-8(3x^2-1)}{(x^2+1)^3}}_{\text{never DNE}} \overset{\text{set}}{=} 0$$

$$3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3}$$

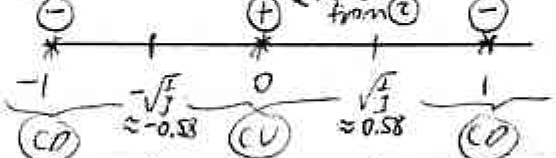
$$x = \pm\sqrt{\frac{1}{3}} \approx \pm 0.58 \quad \text{or } \pm\frac{\sqrt{3}}{3}$$

in Dom(f) ✓

PINs

⑥ Where CU, CD? IPs?

Graph sym.



$$f''(-1) = \frac{(-)(+)}{(+)} = \ominus$$

$$f''(0) = \frac{(-)(-)}{(+)} = \oplus$$

$$f''(1) = \frac{(-)(+)}{(+)} = \ominus$$

IPs at $\left(-\sqrt{\frac{1}{3}}, f\left(-\sqrt{\frac{1}{3}}\right)\right), \left(\sqrt{\frac{1}{3}}, f\left(\sqrt{\frac{1}{3}}\right)\right)$

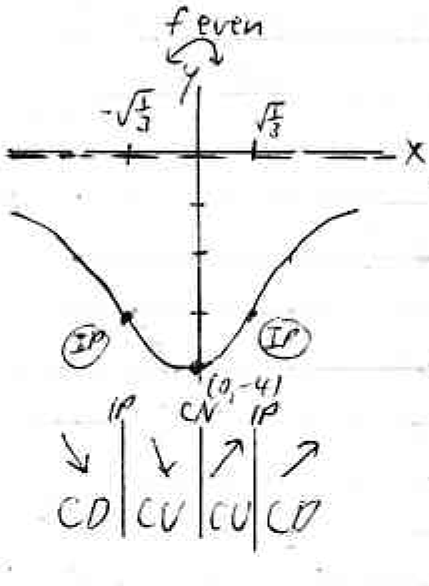
$\left(-\sqrt{\frac{1}{3}}, -3\right), \left(\sqrt{\frac{1}{3}}, -3\right)$

⑦ Graph

f always < 0

If f even, cont. at 0, 0 a CN ∃ either L.Max/Min. Pt. there.

Windows



no x-ints

$f(x) < 0$ ∀ real x

$\text{Range} = [-4, 0)$

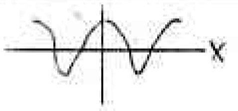
y-coords. picked up

$D_x(\cos x) = -\sin x$

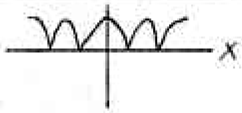
corner, not cusp

$f_x \to \pm\infty$

Ex $\cos x$



$|\cos x|$



4.6: OPTIMIZATION

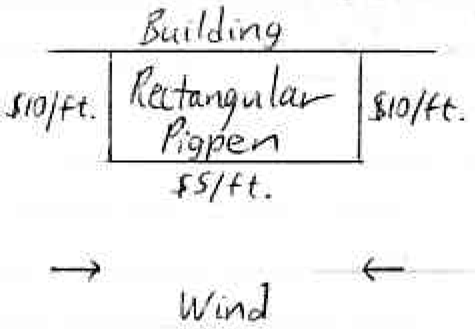
(Hand out 4.6 Handout)

Ex

So let me give you all \$300...

①,② You have \$300 to spend on fencing.

or river

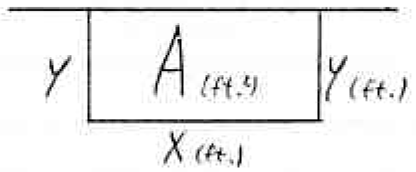


Usually, cost structure.

Wind tends to blow !! to this side of the bldg

Find the dimensions that will maximize the area (of the pigpen), and give that area.

②



③

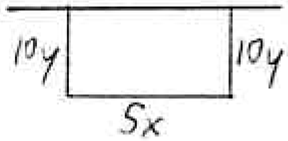
Maximize A (area).

PE $A = xy$

④

Cost (\$)

Constraint



$5x + 20y = 300$

$x + 4y = 60$ SE

⑤

(PE) $A = xy$

(SE) $x + 4y = 60$

$x = 60 - 4y$ (★)

(PE) → $A = (60 - 4y)y$

$A = \underbrace{60y - 4y^2}_{f(y)}$

Sometimes, an optional step but good habit.

⑥ Find Feasible Domain

You could argue $x > 0, y > 0$. It's more convenient to have a closed interval. If $x = 0$ or $y = 0$ what's the area? 1 or —

Need $x \geq 0$ and $y \geq 0$

(★) $60 - 4y \geq 0$

$60 \geq 4y$

$15 \geq y$

$y \leq 15$ → y $[0, 15]$

MAXIMIZE f

Technically can't have CNs at endpts. EVT √s, anyway.

Find all critical #s (CNs) in $(0, 15)$

$f(y) = 60y - 4y^2$

$f'(y) = \underbrace{60 - 8y}_{\text{never DNE}} \overset{\text{set}}{=} 0$

$y = \left(\frac{15}{2} \text{ or}\right) 7.5$ ← CN

in $(0, 15)$ ✓

<u>Verify that we have an A.Max.Pt. there</u> (Don't appeal to "common sense".)

<u>Method 1</u> (4.1) EVT applies

Sometimes it's better to keep the factored form.

y	$A = f(y) = (60-4y)y$
$a = 0$	0
7.5	225
$b = 15$	0

$\Leftarrow$ A.Max.pt. on $[0,15]$

or $(0,15)$, if you wish/prefer

<u>Method 2</u> (2nd DT)

$f'(7.5) = 0$, so can apply.

$f'(y) = 60 - 8y$

$f''(y) = -8$

$f''(7.5) = -8$ $\ominus \Rightarrow$ CD, f L.Max.Pt.

never DNE
CD everywhere ∩
(2): Only CN in town. No chance to turn around again (down!)

(on $[0,15]$,)
f cont.,
7.5 is the only CN
(no other opportunity for f to turn around)
$\Rightarrow$ also A.Max.Pt.

$f(7.5) = 225$

CD everywhere excludes

CD assumes cont.
$f'\ \exists$
(f' DNE $\Rightarrow f''$ DNE)

<u>Method 3</u> (Where is $f \nearrow, \searrow$?)

$$\begin{array}{c|ccccc} f' & & \oplus & & \ominus & \\ \hline y & 0 & & 7.5 & & 15 \\ f & & \nearrow & & \searrow & \end{array}$$

A.Max. Pt.

<u>ANSWERS</u>

7.5 what?

$y = \left(\frac{15}{2} \text{ or}\right) 7.5$ ft.

$A = 225$ ft.^{2}

<u>Find x</u>

(PE) $A = xy \Rightarrow 225 = x(7.5) \Rightarrow x = 30$ ft.

or (A) $x = 60 - 4y$
$= 60 - 4(7.5)$
$= 30$ ft.

Area $= 225$ ft.^{2} $y = 7.5$ ft.
$x = 30$ ft. by

Why so squished?
1 1 more expensive, bldg like matching funds
30 ft. by 7.5 ft.
Can do HW, but next Ex helps w/ 7, 23

More basic/common version.

VARIATIONS

Instead of talking about total cost, we can talk about this partial perimeter "fence side"

This 2-to-1 ratio works for similar probs.

① \$5/ft. \$5/ft. \$5/ft. — \$300 (Partial $P = 60$ ft.)

Optimal

450 ft.^{2} — 15 ft. — 30 ft.

2-to-1

What shape is optimal? Given a fixed P, max A. You're going to prove this in #35.

② \$5/ft. — \$300, $P = 60$ ft.

Optimal

225 ft.^{2} — 15 ft. — 15 ft. Square!

If we go beyond rects., what is better than a square? (circle)

Better: ○ $C = 60$ ft. ≈ 286 ft.^{2} Circle!

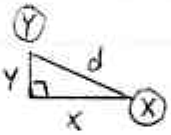
Can do this, but next Ex helps w/ ?.

2-car prob.
A → a
I use for endpt.
I'm me
You're you
What?
Opt. problem.
I'll define t later.

<u>Ex</u> ① At noon, Car X is 50 miles due east of Car Y.
Car X moves west at 30 mph.
Car Y moves north at 10 mph.
What is the closest the cars get?

② <u>Noon</u>

<u>Later</u>



How do we relate d, x, y? Later, we'll differentiate. Do you agree d^2 is min ⟺ d is? As d↑, d^2↑.

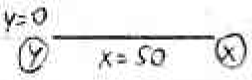
③ Minimize d.

(PE) $d^2 = x^2 + y^2$ ← Easier to min d^2 for now.

$d = \pm\sqrt{x^2+y^2}$

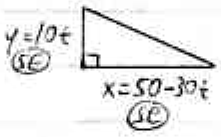
What haven't we used yet? We need a new var.

④ <u>Noon</u>



$t=0$

<u>Let t = # hours after noon</u>



$t>0$

shrinking at a rate of 30mph. eventually, x goes negative. d can't be neg, but x can.

⑤ (PE) $d^2 = x^2 + y^2$

Use (SE)s

d or d^2 — x, y — t

(In Calc III, you draw bushy trees.)

$$d^2 = (50-30t)^2 + (10t)^2$$
$$= \underbrace{1000t^2 - 3000t + 2500}_{g(t)}$$

Why g, not f? It's a reminder

⑥ <u>Feasible Domain</u>: $t \geq 0$
$[0, \infty)$

What must be true of t? What values make no sense? Earth flat, Jason/robot driving, Sun never burns out.

<u>MINIMIZE g, then f</u> $\leftarrow \sqrt{g}$

<u>Find all critical #s in $(0, \infty)$</u> (CNs) - Don't use $-\frac{b}{2a}$ formula; use calculus

$$g'(t) = \underbrace{2000t - 3000}_{\text{never DNE}} \overset{\text{set}}{=} 0$$

$t = \frac{3}{2}$ or 1.5 in dom of g ✓

<u>Note</u> Cars are closest at 1:30 p.m. (We think.)

<u>Verify we have an A. Min. there</u> (2 arguments you could use:)

$g''(t) = 2000 > 0 \quad \forall t$ in $(0, \infty)$

$\Rightarrow g \smile$ (CU) on $(0, \infty)$

or $\Rightarrow g''(1.5) = 2000 > 0 \xRightarrow{\text{2nd DT}}$ L. Min. at 1.5 $\Longrightarrow$ A. Min. Pt.

$g'(1.5) = 0$ needed for 2nd DT

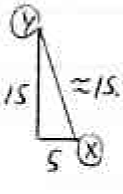
g cont. on $[0, \infty)$
1.5 only CN there
"Only CN in town"

$g'' > 0$ ⇒ concave what? Only CN in town: 1.5! Of course, graph of g is a parabola opening up. Is 250 mi the ans.?

<u>ANSWER</u> $g(1.5) = 250$
$\Rightarrow f(1.5) = \sqrt{250} \; (= 5\sqrt{10}) \approx$ 15.8 mi.

Easier for some people than g → f. See if your ans. seems to work.

<u>Note</u> At 1:30pm



Looks good! :)

(250?! No! <u>Triangle Inequality</u>: the sum of any two side lengths of a triangle must be greater than the third. ▷ Detour)

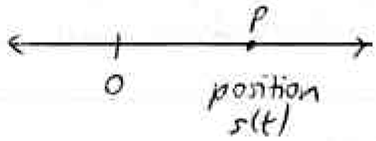
4.7: MORE APPLICS.

(A) Rectilinear Motion

along a line

Stalker

We use p for polynomial, demand



what's deriv. of s

can't be −

what's deriv. of v

Velocity $v(t) = s'(t)$

Speed $= |v(t)|$

Acceleration $a(t) = v'(t) = s''(t)$

$s(t)$	$v(t)$	$a(t)$
hrs. → miles	$\frac{mi.}{hr.}$ (mph)	$\frac{mi.}{hr.^2}$ (mph per hour)

(Acc. idea: go from 0 to 60 mph in 5 secs.)

Ex $s(t) = t^3 - 9t^2 - 48t + 1$

Describe the motion of P during $[-5,5]$.

What might happen if $v=0$?

$$v(t) = \underset{\text{never DNE}}{3t^2 - 18t - 48} \overset{\text{set}}{=} 0$$

$$3(t^2 - 6t - 16) = 0$$

$$3(t-8)(t+2) = 0$$

$\cancel{t=8}$, $t = -2$ (← Direction of P *may* switch left↔right then.)

outside $[-5,5]$

This is not the axis of motion!!

v: $\oplus$ on $(-5,-2)$, $\ominus$ on $(-2,5)$; axis t with marks -5, -2, 5

Direction: right, left

v could be DNE

hit wall

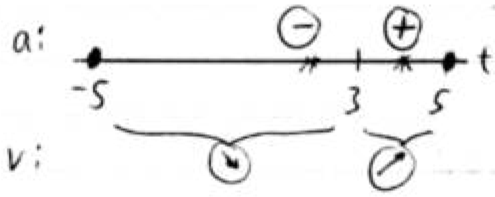
washterf

what can happen then?

$v\nearrow \leftrightarrow v\searrow$

$$a(t) = 6t - 18 \stackrel{\text{set}}{=} 0$$
$$6(t-3) = 0$$
$$t = 3$$

← P may switch speeding up ↔ slowing down then. (in [-5,5])



If v ↘, does that mean the particle is nec. slowing down?

WARNING:

	$v>0$	$v<0$
$v\nearrow$	speed up	slow down
$v\searrow$	slow down	speed up

$|v|\nearrow$
speed ↗
moving away from 0

$v = -10$
$v = -20$
$v\searrow$
speed ↗

← speeding up

Endpts.
Crit. #s
PIN #s
– Key values

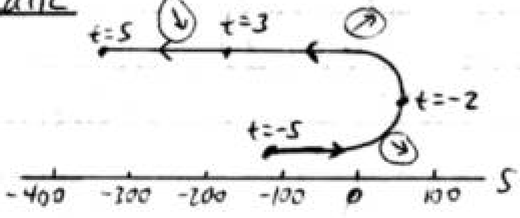
of t	$s(t)$
-5	-109
-2	53
3	-197
5	-339

$s(t)$? t

Judge order: what's $s(t)$ vs. t?

Schematic

Speed



not the actual path of P: it's motion along the line

Note: S t=3 F
$v \to 0$ as $t \to 8$
S

S = Slower
F = Faster

Schematic more "humble" than $s(t)$ vs. t ← project onto s-axis.
"You're interpolating the graph." "Guessing at" $t=3$.
Even w/ 4.5 → concavity, etc.

If x units are produced and sold,

(B) Econ

$R(x)$ = total revenue ($ in)
$C(x)$ = cost ($ out)
$P(x)$ = profit
$= R(x) - C(x)$

What is the avg. cost func.?

Let $C(x)$ = cost of producing x units
(Then, $\frac{C(x)}{x}$ or $c(x)$ = average cost of 1 unit)

What do we want to max., really? P is what?

$R(x)$ = revenue from selling x units
$P(x) = R(x) - C(x)$
= profit if x units are produced and sold

Kil may not like 4 of a ~~toll~~ negative #s of units don't make sense. We're ignoring the common sense restr. that x should be an integer.

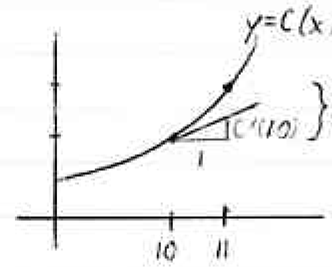
Domain: $[0, \infty)$

Read Ex 4 (pp. 225-6)

C', R', P'

$C'(x)$ = marginal cost function, etc.

Idea



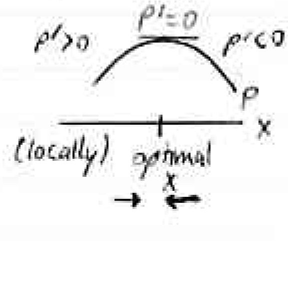
} actual cost of producing 1 more unit
$= C(11) - C(10)$
Hope $\approx C'(10)$

Δ Does this remind you of something? (differentials)

(Read Ex 5 (p. 227))
$(C'(1000) \approx C(1001) - C(1000))$

$P'(x)$ = marginal profit func.

What's P' here?



(What if  ?)

$P'<0$ If you're over here, P', the marginal profit is +, so you want to produce more

underprod. overprod.

SKIP.

If I sell 5 doohickeys at \$10 apiece, my revenue is \$50.

<u>Break down $R(x)$</u>

$$R(x) = (\#\text{units sold})\left(\frac{\text{price}}{\text{unit}}\right)$$

If you have x units, you can sell them at $p(x)$ $\left(\frac{\$}{\text{unit}}\right)$.

$$= x\,p(x)$$

demand

demand function

4.8: NEWTON'S METHOD

(A) Review

Solve an eq.
Can - RHS

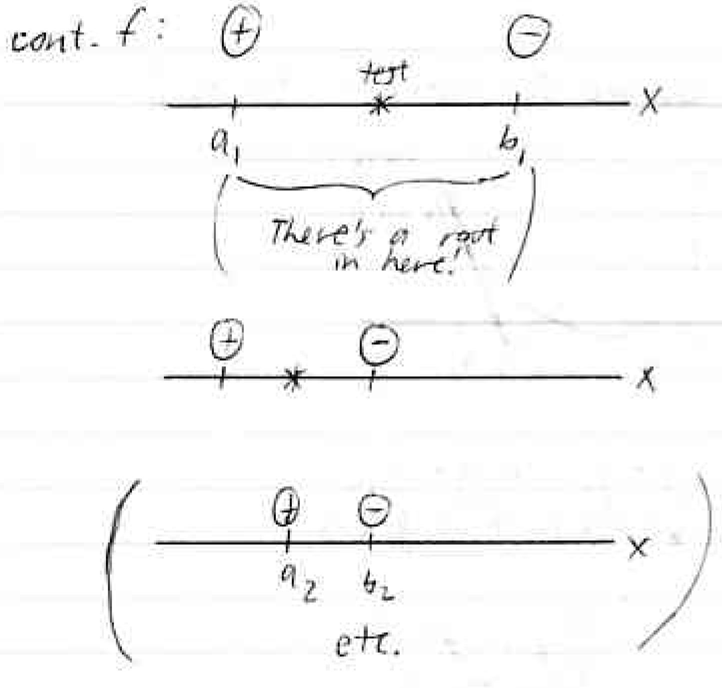
Find a sol'n of $f(x)=0$ (f: x-ints, f': crit. #s, f'': PIN #s)
i.e., Find a zero (or "root") of f.

10^{th}-degree; factoring, rational zero test ($\pm\frac{p}{q}$), etc. fail?
Trig? $\sqrt{\ }$?

Approximate a root!

2.5: Bisection Method (based on IVT)

When are we guaranteed to have a root inside here?
seeds a_1, b_1



rounding error
IT uh ray tiv
Ms
secant method

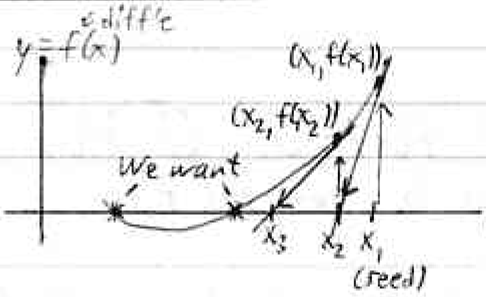
Iterative

Form or known really by Simpson (CMC^3-J '03)

Really Simpson's?

(B) Newton's Method: Idea

We want the 0s of the f so we want the what on the graph? (x-ints.) We just need one.



(Ski down tangent line to new approx.)

This graph might be invisible to us. Ideally, we could ski down / x-int. Instead, we'll ski down? Tan line approxs. the graph near this pt.

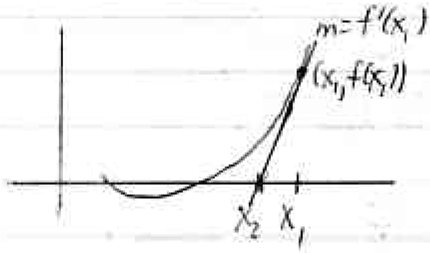
Seed (initial guess) $= x_1$ (maybe based on a graph on a graphing calc.)

Let $x_2 =$ x-int. of tangent line at $(x_1, f(x_1))$

$x_3 =$ " " " $(x_2, f(x_2))$

etc.

(C) How do we get from x_1 to x_2?



This is the same x_1

$$y - y_1 = m(x - x_1)$$

$$y - f(x_1) = [f'(x_1)](x - x_1)$$

When we find x-ints

Plug in $y = 0$
Solve for x

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x$$

if $\neq 0$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$ (prev. approx. − ratio of func. value to deriv. value there)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ ← provided $\neq 0$ $n = 1, 2, 3, \ldots$

Stuck ⇒ New seed!

① Ex

what does this have to do w/ finding 0s? roots

Can use differentials, but one step. Let's say you have a cheapy calc.

Approx. $\sqrt{5}$ to 4 dec. places.

i.e., Approx. the "+" root of $f(x) = x^2 - 5$.

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + 5}{2x_n}$$

$$= \frac{x_n^2 + 5}{2x_n}$$

$\sqrt{5}$ has to be between what and what?

Seed: $x_1 = 2$

$$x_2 = \frac{(2)^2 + 5}{2(2)} = 2.2500$$

Try to avoid rounding off

$$x_3 = \frac{(2.25)^2 + 5}{2(2.25)} \approx 2.2361$$

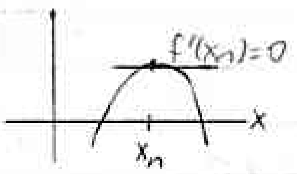
$$x_4 \overset{\text{turns out}}{\approx} 2.2361$$

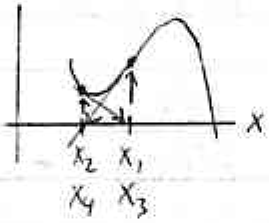
same, so STOP!

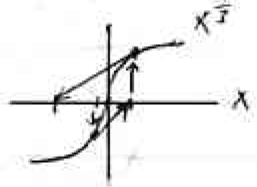
(If the method works, when you get "close" to a root, the # of correct decimal places doubles w/ each iteration. Quadratic convergence)

When Animals Attack

Ⓔ When Newton Fails!







Why does the method fail? ①
Stuck in a valley ②
$x^{\frac{1}{3}}$: diverges ③ $\forall x_1 \neq 0$

Flat tangent line intervals – approx. often not good far away

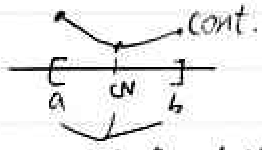
Larson 218:
Suff. for conv.: $\left|\dfrac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$ on an open interval containing the root

(Why are we talking about this now?
Applic. of derivs. and $f'(x)=0$ – find critical #s
$f''(x)=0$ – find PINs)

(4.1) Critical #s (CNs : #s in Dom(f) where $f' = 0$ or DNE)
$\xrightarrow{?}$ L. Max/Min

EVT

If f cont. on closed interval, we're guaranteed what?



✓ for $\underbrace{\text{A. Max/Min}}_{\exists \text{ (there exist)}}$

(4.2) Rolle
MVT

f cont. on [a,b], diff'e on (a,b), f(a)=f(b) ⇒ $\exists c$ in (a,b): $f'(c)=0$
⇒ $\exists c$ in (a,b): $f'(c) = \frac{f(b)-f(a)}{b-a}$

(4.3) f' → Find CNs.
Where f ↗, ↘
1st DT to classify pts. at CNs. L. Min., L. Max., Neither
(Plug into f to get y-coords.)

2nd DT: Only f'–f'' bridge: You can use f'' to tell you about pts. at critical #s *assoc. w/ f'. Plug in CNs into f'' for 2nd DT. o.w., f' issue ⇒ f' f'' f''

(4.4) f'' ⇒ Find PINs.
Where f graph CU, CD. IPs : f cont., concavity changes if
(Plug into f to get y-coord.)
2nd DT to classify pts. at CNs (faster than 1st DT?)
Need $f'=0$. f'' ⊕ CU L. Min., f'' ⊖ CD L. Max. else no info

(4.5) Graphing

(4.6) Optimization (handout)

(4.7) Rectilinear Motion $\underrightarrow{s}_{s}$

(4.8) Newton's Method
Approx. sol'n of $f(x) = 0$.

x
tan line

x_1 = seed
$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ← ≠0