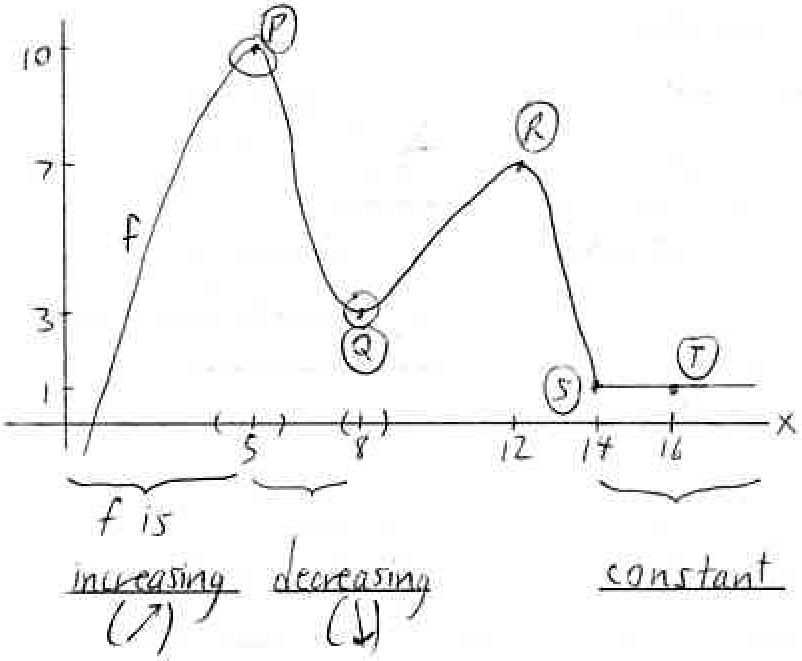
# CH.4: APPLICS OF f'

## 4.1: EXTREMA OF f

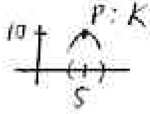
max, min

### Ⓐ Terminology



A = Absolute / Global
L = Local / Relative

Ⓟ is a L. Max Pt.



P: King of the Hill
(highest pt. in some neighborhood around it)

Book doesn't specify cont.! p.164

bec. 10 is the max value of f on some open interval containing x=5

like Jim Cameron

<u>and</u> an A. Max Pt.

P: King of the World
(highest pt. anywhere)

bec. 10 is the max value on the whole domain of f.

Ⓠ L. Min Pt.

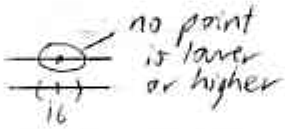
~~not~~ not A. Min Pt.
(Ⓢ lower)

Ⓡ L. Max Pt.
not A. Max Pt.

Simpsons: we don't take away hot sauce. Guys just as low as you are - in a bar

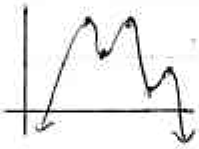
Ⓢ L. Min Pt.
"Ties are OK"

Ⓣ L. Min Pt.
and L. Max Pt.



We go to hell in both directions.

Ex



2 A. Max Pts.
0 A. Min Pts.

3 L. Max Pts.
2 L. Min Pts.

Ex

Don't need "open int" stuff for A. Min.
Larson agrees

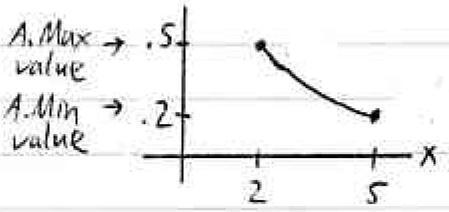
Ⓟ is an A. Min Pt.
but not a L. Min Pt. (there is no open interval containing 6 on which f is defined)
2 is an endpoint extremum.

## (B) Extreme Value Thm. (EVT)

Outside this interval, it can go nuts, like Pres. Bush in his hat days.

If $f$ is cont. on a closed interval $[a,b]$, then we are guaranteed an A. Max. value and an A. Min. value on $[a,b]$.

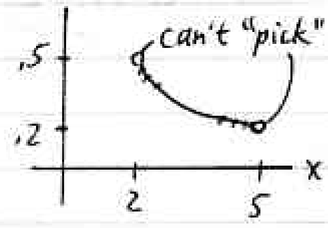
Ex $f(x)=\frac{1}{x}$ on (the interval) $\underbrace{[2,5]}_{\text{Closed}}$ ← Think: "Restricted domain"



$\frac{1}{x}$ cont. here
So, EVT applies

It doesn't mean we can't have A. Min/Max, but no guarantee. Anything goes
Adversary argument
If we're competing to get the highest y-coord., we'd never stop
Nelson - ha ha!
If you pick this pt...
We're not allowed to take (2, .5)

Ex $f(x)=\frac{1}{x}$ on $\underbrace{(2,5)}$
not closed
EVT does not apply (no guarantee of A. Max/Min values)

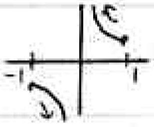


No A. Max. or A. Min. value.

.49
.499
.4999
⋮

Does the EVT apply?

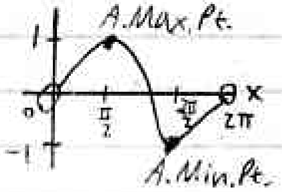
Ex $f(x)=\frac{1}{x}$ on $\underbrace{[-1,1]}$
$\frac{1}{x}$ is not cont. at 0
EVT does not apply



No A. Max. or A. Min. value.

Does this mean we can't have abs. extrema? No- We don't have that guarantee.

Ex $f(x)=\sin x$ on $\underbrace{(0,2\pi)}_{\text{EVT does not apply, yet}}$



(We get abs. extrema, anyway.)

Calculus more directly deals w/local behavior (lim as $x \to \infty$ notwithstanding)

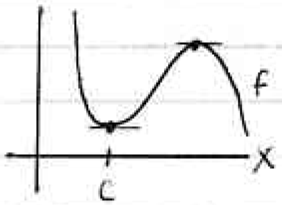
What kinds of tangent lines can we have there? What's the deal w/ f'? (slope)

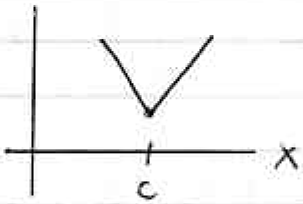
Think |x|

## Ⓒ Critical #s (CNs)

If f has a L.Max. or L.Min. at x=c, then...

Exs



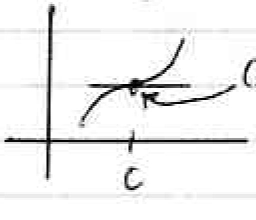


$$\boxed{f'(c)=0 \quad \underline{\text{or}} \quad f'(c)\text{ DNE}}$$

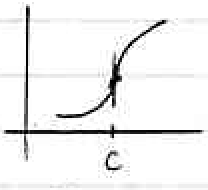
Can you think of a graph w/a horiz. tan line at a pt. that's neither? $(x^3)$

If f' is 0, do we have to have a local min/max? No- the graph could "snake in"

Converse is not generally true.



(Neither L.Max.Pt. nor L.Min.Pt.)



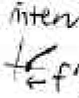
f'(c) DNE, but...

f'(c)=0, but not local ext.

The $\overset{(CNs)}{\underline{\text{critical \#s}}}$ of f are the #s in $\overbrace{\text{the domain of } f}^{Dom(f)}$ where f' is 0 or DNE.

L.Min., L.Max. can $\underline{\text{only}}$ appear at CNs.

A reason why we need "open interval"



← f' = 10, e.g. not even DNE

EVT: why find CNs in (a,b)

## Ⓓ Finding A.Max., A.Min. of f on [a,b]

cont. on

exist by EVT

"Critical pts." used in multivar calc.
M121 - Ch. 7,
Swok 862
Where can we
have abs. max/min?
At local ext and..

Candidates: $\underbrace{\text{Pts. at CNs}}_{\text{L.Max.? L.Min.?}}$, or endpoints

### Steps

Can't have CN
at a,b
if treat
[a,b] as
restricted
domain.

can't be
L.Max Pt.

Setting up the
lineup
One by one
we ask
what's your
func. value?
(y-coord.?)

① Find all CNs in (a,b).

② Evaluate f(c) for each critical "c"

③ Evaluate $\underbrace{f(a)}$ and $\underbrace{f(b)}$

endpoint values

④ Among the candidates from ② and ③,
the highest is the A.Max value on [a,b]
the lowest " " A. Min. "

I'll specify
on test
abs./local

Ex Find the [absolute] extrema of
$f(x) = x^{1/3} - x + 4$ on $[-1, 8]$.

① Find all critical #s (CNs) in $(-1, 8)$

Where is $f'(x) = 0$ or DNE?

Right now,
we can get
a critical #.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - 1$$
$$= \frac{1}{3x^{2/3}} - 1$$

in Dom(f), in $(-1, 8)$

$f'(0)$ DNE, so
$\boxed{0 \text{ is a CN}}$.

Set $$\frac{1}{3x^{2/3}} - 1 = 0$$

$\cdot 3x^{2/3}$

or $\frac{1}{3x^{4/3}} = 1$
$3x^{4/3} = 1$
$x^{2/3} = \frac{1}{3}$

Take recips. (if ≠0)

$$1 - 3x^{2/3} = 0$$
$$x^{2/3} = \frac{1}{3}$$
$$(x^{2/3})^3 = \left(\frac{1}{3}\right)^3$$
$$x^2 = \frac{1}{27}$$
$$x = \boxed{\pm\sqrt{\frac{1}{27}} \approx \pm 0.192}$$

CNs in $(-1, 8)$, in Dom(f)

If $x^2 = 9$...
$x = \pm 3$
If we get 9,
throw it out

Not simplified

②-④ Table

not $f'$

| $x$ | $f(x)$ | Candidates | |
|---|---|---|---|
| $a = -1$ | $f(-1) = (-1)^{1/3} - (-1) + 4$ | $= 4$ | |
| $-\sqrt{\frac{1}{27}}$ | $f(-\sqrt{\frac{1}{27}})$ | $\approx 3.615$ | |
| $0$ | $f(0)$ | $= 4$ | on $[-1, 8]$ |
| $\sqrt{\frac{1}{27}}$ | $f(\sqrt{\frac{1}{27}})$ | $\boxed{\approx 4.385}$ | ← A. Max. value |
| $b = 8$ | $f(8)$ | $= \boxed{-2}$ | ← A. Min. |

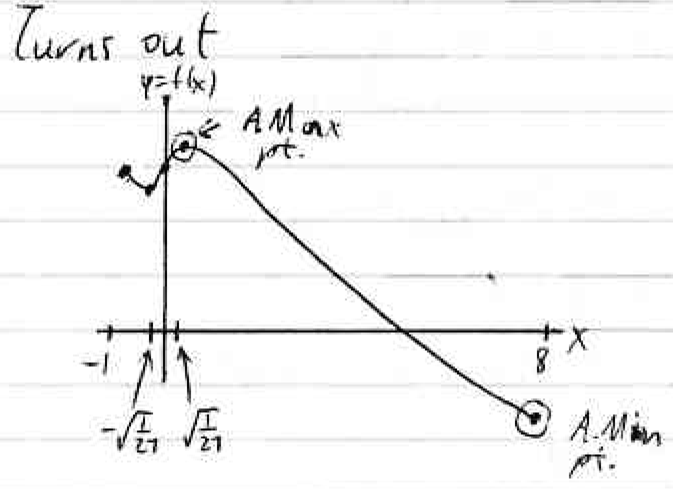
critical #s

Also look here
for abs. ext.
We may
find
L. Max/Min.
Pts. could be
A. Max/Min.
Pts.
Corr Condit

C4-4
4.1

Turns out

$f'$ at $\pm\sqrt{\frac{1}{27}}=0$
$f(0)$ undef



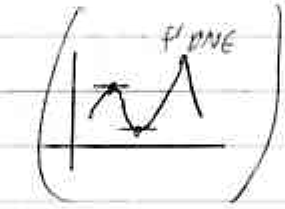
Up to 15

Can do 4.1 Hw, but we'll do some problems next time – algebraic tricks

Prereq

## Ⓔ Finding Critical #s of f (CNs)

in Dom(f) where $f'(x)=0$ or DNE



Me: $(3x+1)\sqrt{2x-1}$
$f'=\frac{9x-2}{\sqrt{2x-1}}$

Ex $f(x)=(3x+1)\sqrt{x^2-9}$

### ① Find Dom(f) - optional (but can help!)

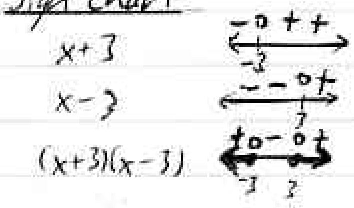
If I know the sign n? w/x+3...

Me: $a\geq b$ $(a,b\geq0)$
$\sqrt{a}\geq\sqrt{b}$
$|x|\geq3$

$$x^2-9\geq0$$
$$(x+3)(x-3)\geq0$$

"Parabola Method"

Ⓞⓡ Sign chart



$$\text{Dom}(f)=(-\infty,-3]\cup[3,\infty)$$

### ② Find f'(x)

$$f(x)=(3x+1)\sqrt{x^2-9}$$

Use Product Rule

$$f'(x)=(3)(\sqrt{x^2-9})+(3x+1)\cdot D_x(x^2-9)^{\frac{1}{2}}$$
$$=3\sqrt{x^2-9}+(3x+1)[\cancel{\tfrac{1}{2}}(x^2-9)^{-\frac{1}{2}}(\cancel{2}x)]$$

Exercise in algebra
Or factor: $(x^2-9)^{-\frac{1}{2}}[3(x^2-9)+(3x+1)(x)]$

$$= 3\sqrt{x^2-9} + \frac{x(3x+1)}{\sqrt{x^2-9}} \leftarrow \text{LCD}$$

→ One fraction

$$\left(= \frac{3\sqrt{x^2-9}}{1} \cdot \frac{\sqrt{x^2-9}}{\sqrt{x^2-9}} + \frac{3x^2+x}{\sqrt{x^2-9}}\right)$$

$$= \frac{3(x^2-9) + 3x^2+x}{\sqrt{x^2-9}}$$

$$= \frac{6x^2+x-27}{\sqrt{x^2-9}}$$

Yech, bad grammar
Mathematicians have bad grammar, anyway.

③ Where is $f'(x)$ DNE?

In Dom(f)

$\text{Dom}(f) = (-\infty,-3] \cup [3,\infty)$
Of these #s, -3 and 3 make $f'(x)$ DNE
→ CNs

④ Where is $f'(x)=0$?

$$f'(x) = \frac{6x^2+x-27}{\sqrt{x^2-9}}$$

Set numer.= 0 (hope denom. will be OK)
Solve $6x^2+x-27=0$ using QF : $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

649=11·59
(Don't be foolish!)

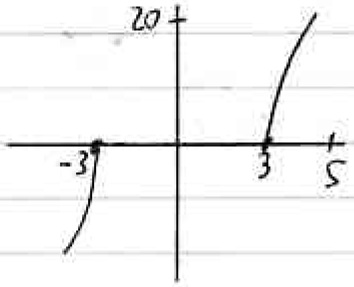
$$x = \frac{-1\pm\sqrt{649}}{12}$$

$x \approx -2.21$ , $2.04$ ← Not in Dom(f), so not CNs

Are there CN's
ou it f' DNE but
Literally: What's the pt.?

The only CNs are $\boxed{3,-3}$.

Turns out

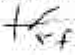


-2,2√6,2040
not in Dom(f)

Up to 2.3

This confused them!

If just  ← abs.min. pt.

Here, f' DNE at -3,3 but if  ← f'=0 one sided

That would drop the "f'=0 or DNE" need to have local ext.

Here, one-sided derivs happen to be DNE.

No A. Max, Min

No L. Max, Min

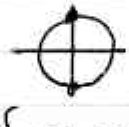
↳ Endpoints disqualified — need ⟷

$\underline{\text{Ex}}$ $f(x) = \sec x$

① $\underline{\text{Find Dom}(f)}$

$$f(x) = \frac{1}{\cos x}$$

$\text{Dom}(f) = \text{all reals } \underline{\text{except}} \ \underbrace{\text{where } \cos x = 0}_{x = \frac{\pi}{2} + \pi n,\ n \text{ int.}}$



$= \{x : x \neq \frac{\pi}{2} + \pi n,\ n \text{ int.}\}$

② $\underline{\text{Find } f'(x)}$

$$f'(x) = \sec x \tan x$$

③ $\underline{\text{Where is } f'(x) \text{ DNE?}}$

$f'(x) = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$ DNE when $\cos \textcircled{x} = 0$

$\underline{\text{not}}$ in Dom(f); $\underline{\text{can't}}$ be CNs!

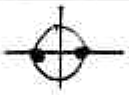
Are there CNs?

But what...?

④ Where is $f'(x)=0$?

$$f'(x)=\frac{\sin x}{\cos^2 x}$$

Where is $\sin x = 0$?



$\boxed{x = n\pi,\ n \text{ integer}}$ — CNs

in Dom(f) ✓

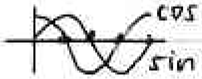
sin, cos shifted by $\frac{\pi}{2}$, so diff. zeros

Note

sin x, cos x are never simultaneously 0 (for the same x)

Reasons:

① Graphs shifted by $\frac{\pi}{2}$; different x-ints.

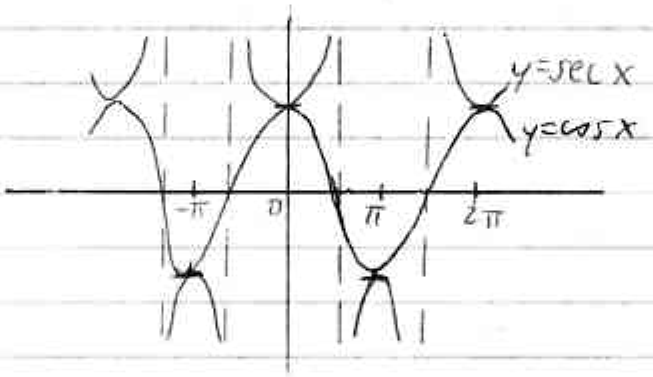


② $\sin^2 x + \cos^2 x = 1$ Not 0

③ 
(0,0) not on unit circle

Should Know

I might want to graph cos x 1st.



it makes sense that f'(x) is DNE only at the VAs. Other than the VAs, graph is very smooth! No corner pts, no pts. w/ vertical tangents.

Note: sec x is never 0
csc x

Ex $f(x)=\tan x \quad \left(=\frac{\sin x}{\cos x}\right)$

$f'(x)=\sec^2 x \quad \left(=\frac{1}{\cos^2 x}\right)$ $f'$ DNE ⇔ f DNE

(Anything that makes f' DNE is not in Dom(f), so can't be a CN.)

$\sec^2 x = 0$ NEVER!

$\boxed{\text{No CNs}}$

slope = $\sec^2(0)$ = 1

Not like $x^3$!!

Note: $\sec^2 x \geq 1 > 0$!! $\forall x \in$ Dom(f)

Between VAs, snakes always go up

We'll see in 4.4