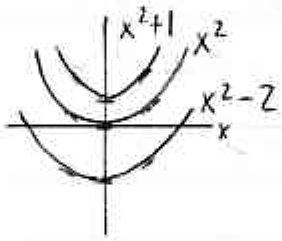
Later, taking derivs.

5.1: ANTIDERIVATIVES (ADs) and INDEFINITE $\int$s

What's the deriv. of x^2? If I tell you $2x$ is the deriv of some func, does it have to be x^2?

Ⓐ ADs



$\xrightarrow{D_x}$ $\quad f'(x) = 2x$

$\xleftarrow[\text{Find ADs}]{\text{NOW}}$

Wouldn't you agree the slope situation is the same consistent w/ $D_x(\text{const.}) = 0$. Given the deriv func, how do we reconstruct the "original" func.? Actually, we have a whole family of ADs.

Given: $f(x) = 2x$

Then, $F(x) = x^2 + C$ gives <u>all</u> the ADs of f. (a family)

C: arbitrary constant

$$\boxed{F'(x) = f(x)}$$

Why dx? What if $\int f$, Calc III.

Ⓑ Indefinite $\int$s

Swok. Book.

It's because of this C that we call these Indefinite (def/How to Acc; family of funcs, Definite $\int$ = #)

$$\int \underbrace{2x}_{\substack{f(x),\\ \text{the}\\ \text{integrand}}} \underbrace{dx}_{\substack{\text{says } x\\ \text{is the}\\ \text{variable}\\ \text{of integration}}} = \underbrace{x^2}_{\substack{F(x),\\ \text{an AD}\\ \text{of } f}} + C$$

$\int$: <u>integral sign</u>

C: <u>constant of integration</u>

read as CAND vs. W/T as grand

Evaluating the $\int$

Integrating $f(x)$

$$\boxed{\text{Form: } \int f(x)\,dx = F(x) + C}$$

$F(x)$: an AD of f

© Basic Rules

Deriv of x is 1 ⇒ … $\int$ of 1 is

$$D_x(x) = 1 \qquad \int 1\,dx = \int dx = x + C$$
$$D_x(3x) = 3 \qquad \int 3\,dx = 3x + C$$

D_x to ✓

Power Rule for $\int$

What if we want $\int$ plain $x^3\,dx$? How do we adjust? Not "sing" Don't need $\frac{c}{4}$

(Beresford) "Sing at sight"

$$D_x(x^4) = 4x^3 \Rightarrow \int 4x^3\,dx = x^4 + C$$
$$D_x\left(\frac{x^4}{4}\right) = \frac{1}{4}\cdot 4x^3 = x^3 \Rightarrow \int x^3\,dx = \frac{x^4}{4} + C$$

Beresford: $\frac{1}{n+1}x^{n+1} + C$ There's one value of n for which the formula fails.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1$$

$$\frac{x^{\text{add } 1}}{\text{copy}} + C$$

Shocking…

Note: If $n = -1$:
$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$ Ch. 7

Trig Rules

Helps to write

$$D_x(\sin x) = \cos x \qquad D_x(\cos x) = -\sin x$$
$$D_x(\tan x) = \sec^2 x \qquad D_x(\cot x) = -\csc^2 x$$
$$D_x(\sec x) = \sec x \tan x \qquad D_x(\csc x) = -\csc x \cot x$$

Recognize Deriv. of what? Don't memorize all 12 & confuse signs. Mind dump the derivs.

When do I flip sign?

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$
$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

(have sin, csc ⇒ −)

What was $e^{\ln x}$?

Oh, man!

Don't go:
$\int \sec^2 x \sec^2 x\,dx$
No!!
$\tan \tan x + C$

Inverse Props

Exs $D_x\left[\int \sec^4 x\,dx\right] = \sec^4 x$

$\int [D_x(\sec^4 x)]\,dx = \sec^4 x + C$

In general

$$D_x\Big[\underbrace{\int f(x)\,dx}_{\text{if exists}}\Big] = f(x)$$

$$\int [\underbrace{D_x f(x)}_{\text{if exists}}]\,dx = f(x) + C$$

Splitting Rules

$\int$ is a linear operator, like D_x, lim

Can $\int$ term-by-term

$\int c f(x)\,dx$

Can $\int$ (operate) term-by-term
warning: $\int(\text{product}) \neq$ product of $\int$s, usually.

$\int$ of Sum = Sum of $\int$s
Diff. Diff.
$\int (f \pm g)\,dx = \int f\,dx \pm \int g\,dx$
Can $\int$ term-by-term

(1) Exs

Need!!

Ex Evaluate $\int (x^6 + 3x^5 - 2)\,dx$

$= \int x^6\,dx + 3\int x^5\,dx - \int 2\,dx$ ← can skip

$= \frac{x^7}{7} + 3\left(\frac{x^6}{6}\right) - 2x + C$ ← absorbs all consts. of $\int$

$= \frac{1}{7}x^7 + \frac{1}{2}x^6 - 2x + C$

Ex $\int \left(\sqrt[3]{x^2} + \frac{4}{\sqrt{x}}\right) dx$

(1) Rewrite

$= \int \left(x^{2/3} + 4x^{-1/2}\right) dx$

(2) Integrate (+C)

$= \frac{x^{5/3}}{5/3} + 4\left(\frac{x^{1/2}}{1/2}\right) + C$

(3) Simplify

How do you $\div$ by a frac?
What's $4 \div \frac{1}{2}$?
Many: "2".

$= \frac{3}{5}x^{5/3} + 4(2)\sqrt{x} + C$

$= \frac{3}{5}x^{5/3} + 8\sqrt{x} + C$

1-9 odd

This isn't technically a binomial (just a poly.), but it's like squaring a binomial

Ex (#20) $\displaystyle\int \frac{(\sqrt{t}+2)^2}{t^3}\,dt$

$\displaystyle = \int \frac{t + 4\sqrt{t} + 4}{t^3}\,dt$

Now what - still alg.

$\displaystyle = \int \left(\frac{t^1}{t^3} + \frac{4t^{1/2}}{t^3} + \frac{4}{t^3} \right) dt$

Quotient Rule for Exps.

$\displaystyle = \int (t^{-2} + 4t^{-5/2} + 4t^{-3})\,dt$

$\displaystyle = \frac{t^{-1}}{-1} + 4\left(\frac{t^{-3/2}}{-3/2}\right) + 4\left(\frac{t^{-2}}{-2}\right) + C$

$= -t^{-1} + 4(-\tfrac{2}{3})t^{-3/2} - 2t^{-2} + C$

$= \underbrace{-t^{-1}}_{-\frac{1}{t}} - \tfrac{8}{3}t^{-3/2} \underbrace{-2t^{-2}}_{-\frac{2}{t^2}} + C$

1-19 odd

peel out sin x

Ex $\displaystyle\int \frac{\cos x}{7\sin^2 x}\,dx = \frac{1}{7}\int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) dx$

oh yeah, he's the deriv of.

$\displaystyle = \frac{1}{7}\int \csc x \cot x\,dx$ $\qquad D_x(\csc x) = -\csc x \cot x$

$\left(= \frac{1}{7}(-\csc x) + C\right)$

1-47 odd

$= -\frac{1}{7}\csc x + C$

Don't need "$-\frac{1}{7}C$"

(E) Differential Eqs.

Uncle Ernie/Lester dies

Ex I inherit \$1000.
After t years, I spend it at the rate of $10t \frac{\$}{yr.}$
In how many years will it be gone?

After 2 yrs. 20 $/yr.
2.5 25
3 30

My spending always accelerates
I'm kinda mean — body's not even cold yet and I'm already spending the \$. Slowly at 1st and then more later

① Setup
Let $\underset{\text{or } y}{f(t)}$ = \$ left after t years

Solve the diff. eq.

$$f'(t) = -10t \quad \text{or} \quad \frac{dy}{dt} = -10t$$

What is the rate of change of f? Careful!

subject to the initial condition (IC)

$$f(0) = 1000 \quad \text{or} \quad y = 1000 \text{ when } t = 0$$

What else do we know about f?

② Find the general sol'n of $f'(t) = -10t$

$f'(t) = -10t$ or	$\frac{dy}{dt} = -10t$
Integrate (S) both sides wrt t.	$dy = -10t\,dt$
$\int f'(t)\,dt = \int -10t\,dt$	Separation of variables
$f(t) = -10\left(\frac{t^2}{2}\right) + C$	$\int dy = \int -10t\,dt$
$f(t) = -5t^2 + C$	$y = -5t^2 + C$

".dt" notation
Differential form

f(t)+C, but we'll write +C at end

y+C
$D_t = -10t$

Use the IC to solve for C, and

③ Find the particular sol'n

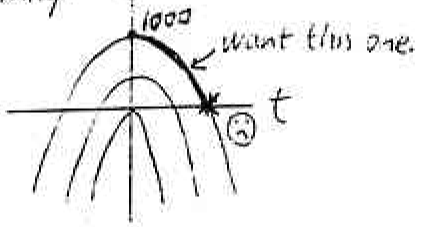
$$f(0) = -5(0)^2 + C$$
$$1000 = C$$
$$C = 1000$$

$$f(t) = -5t^2 + 1000$$

If your financial advisor says you have \$10K + C
What's C?
"any real #"
You're fired!

start off by spending slowly + spendthrift

Family. $-5t^2+C$



what do I want to know? when does the inheritance run out?

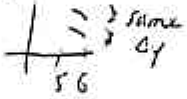
④ Solve $f(t)=0$.

$$-5t^2+1000=0$$
$$t^2=200$$
$$t=\pm\sqrt{200}$$
$$t=10\sqrt{2}\text{ years}$$
$$\approx 14.1\text{ years}$$

1-51 odd

Stumbling onto FTC:
How much \$ spent in 1 yr.?
e.g. $t=5$ to $t=6$
$\int_5^6 f'(t)\,dt = f(6)-f(5)$

indep. of $f(0)=C$
since parallel graphs



Ex Solve $\underbrace{f''(x) = 4}_{\text{2nd-order diff. eq.}}$ (reconstruct $f(x)$)

subject to

$$\left.\begin{aligned} f'(2) &= 1 \\ f(2) &= 3 \end{aligned}\right\} \text{ICs}$$

(OK if we have info at 2, not 0.)

(Note In "boundary-value" problems, you're given 2 f-values or 2 f'-values. Harder!)

Find $f'(x)$

Before you find f, you have to find... what do we do to both sides? $f'(x)+C$

$$\int f''(x)\,dx = \int 4\,dx$$
$$f'(x) = 4x + C$$

Find C: Use $f'(2)=1$

$$f'(2) = 4(2) + C$$
$$1 = 8 + C$$
$$C = -7$$

$$f'(x) = 4x - 7$$

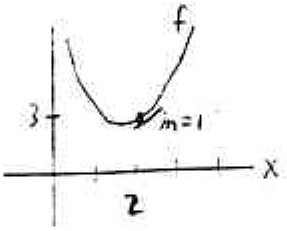
Find $f(x)$

$$\int f'(x)\,dx = \int (4x-7)\,dx$$
$$f(x) = 4\left(\tfrac{x^2}{2}\right) - 7x + D$$
$$f(x) = 2x^2 - 7x + D$$

Find D: Use $f(2)=3$

$$f(2) = 2(2)^2 - 7(2) + D$$
$$3 = 8 - 14 + D$$
$$D = 9$$

$$f(x) = 2x^2 - 7x + 9$$



Given: f''
Pt.
Slope (of tan line) $\Bigg\} \Rightarrow$ Reconstruct f, Get graph.

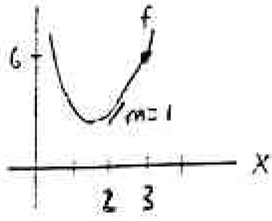
Variation

$f''(x) = 4$

s.t. $f'(2) = 1$

$f(3) = 6$

$\Rightarrow$ same f



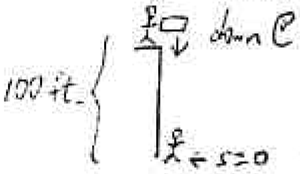
Ex (Projectiles)

BEFORE:

s
$\downarrow$
$s' = v$
$\downarrow$
$s'' = v' = a$

$\uparrow$ NOW

As you $\int$, you pick up constants of integration that you determine using ICs.



At time 0, thrown down @ $10 \frac{ft.}{sec.}$

$s(t)$ = height of brick t secs after thrown

Solve $a(t) = -g$ ← local grav. constant (for Earth; slightly higher at poles)

$a(t) \approx -32 \left(\frac{ft.}{sec.^2}\right)$

(or $-9.8 \frac{m}{sec^2}$ if have m) ("wide")

s.t. $\begin{cases} v(0) = -10 \left(\frac{ft.}{sec.}\right) \\ s(0) = 100 \ (ft.) \end{cases}$ ICs

– "bec. brick going faster in the neg direction"
What does ↓$10 \frac{ft}{sec.}$ tell us, using mathem. notation?

$a(t) = -32$

$\int a(t)\,dt = \int -32\,dt$

$v(t) = -32t + C$

$t=0$: $v(0) = 0 + C$
$-10 = C$

$v(t) = -32t - 10$

$\int v(t)\,dt = \int (-32t - 10)\,dt$

$s(t) = -16t^2 - 10t + D$

$t=0$: $s(0) = 0 - 0 + D$
$100 = D$

$s(t) = -16t^2 - 10t + 100$

Dangerous: $C = v(0)$?? What if IC somewhere else? Students blow it ... (if not poly...)

63: find general formula

Then, you can answer interesting ?s. When will it hit my head/foot.

so powerful
like Chain Rule for $\int$s

5.2: "u" SUBS (are to $\int$s what the Chain Rule is to D_x)

#8
$u=\sec x$ works $\frac{\sec^2 x}{2}+C$
#63: $\int \sin x \cos x\, dx$
shows trig equiv.

Ex $\int \tan x \sec^2 x\, dx$

Put in form $\int \underbrace{f(u)}_{\text{nice}}\, du$

$\tan x, \sec^2 x$
How are these related?

Let $u = \tan x$

Idea: Quotient of differentials $dx \to 0$ (like $h \to 0$ in limit defn of deriv.)

$\frac{du}{dx} = \sec^2 x$

technically inappropriate to say we're mult. by dx
"Differential form"

$\Rightarrow du = \sec^2 x\, dx$

$u = \sec x$ works & equivalent.

$$\int \underbrace{\tan x}_{u} \underbrace{\sec^2 x\, dx}_{du}$$

$$= \int \underbrace{u}_{\text{Nice!}}\, du$$

What? +C
Am I done?

$$= \frac{u^2}{2} + C$$

$$= \frac{\tan^2 x}{2} + C$$

Go back to x!
Can ✓ by D_x

You could've done in 5.1 - take you a day. Maybe Bin Thm, Pascal's Δ.

Ex $\int x^2 (4x^3 - 5)^{10}\, dx$

Let $u = 4x^3 - 5$ ← Why?

D_x in $\int$, except constant factor
u^{10} nicer
Common: u = "inside" of a composite func., exp., denom.
(Not always, though!)
(Do HW!!)

$\Rightarrow du = 12x^2\, dx$

I can't just throw in a 12 arbitrarily Const. factor pop out anyway. Like CTS

I guess mixing u, x OK, but awkward.

Method 1 (Compensation)

Balance

$$\frac{1}{12}\int \underbrace{12x^2\overbrace{(4x^3-5)^{10}}^{u^{10}}dx}_{du}$$ Only pop out constants. NO! $\frac{1}{x}\int x \cdots$

$$= \frac{1}{12}\int u^{10}\,du$$
$$= \frac{1}{12}\left(\frac{u^{11}}{11}\right) + C$$ Don't need $\frac{C}{12}$
$$= \frac{u^{11}}{132} + C$$
$$= \frac{(4x^3-5)^{11}}{132} + C$$

Method 2 (Solve for whomever you have to kill)

$$\int \underbrace{x^2\overbrace{(4x^3-5)^{10}}^{u^{10}}dx}_{\text{KILL!}}$$

$u = 4x^3-5$

$du = 12x^2\,dx$

$\Rightarrow \frac{1}{12}du = \underbrace{x^2\,dx}_{\text{KILL!}}$

Is $\frac{du}{12}$ OK?

$$= \int u^{10}\cdot\frac{1}{12}\,du$$
$$= \frac{1}{12}\int u^{10}\,du$$

etc. (See Method 1)

Your answer better be equiv. to prev. answer. C can absorb consts.

Ex $\int \sin(2x+1)\,dx$

$u = 2x+1$

$du = 2\,dx$

$$= \frac{1}{2}\int \sin(2x+1)\cdot 2\,dx$$
$$= \frac{1}{2}\int \sin u\,du$$
$$= \frac{1}{2}(-\cos u) + C$$
$$= -\frac{1}{2}\cos(2x+1) + C$$

Better? 2

$u = 2x+1$

$du = 2\,dx \Rightarrow dx = \frac{1}{2}du$

$$= \int \sin u\cdot\frac{1}{2}du$$
$$= \frac{1}{2}\int \sin u\,du$$
$$= \frac{1}{2}(-\cos u) + C$$
$$= -\frac{1}{2}\cos(2x+1) + C$$

sin and sec?
1^{st} step - not calc.

Ex (#38) $\int \sin(2x)\sec^5(2x)\,dx$

$$= \int \frac{\sin(2x)}{\cos^5(2x)}\,dx$$

Could do $u = 2x$
→ would need 2 subs.
Try to let u = biggest guy whose deriv. is in there.

$$u = \cos(2x)$$
$$du = -\sin(2x)\cdot 2\,dx$$
$$= -2\sin(2x)\,dx$$

$$= \left(-\frac{1}{2}\right)\int \frac{(-2)\sin(2x)}{\cos^5(2x)}\,dx$$

$$= -\frac{1}{2}\int \frac{du}{u^5}$$
$$= -\frac{1}{2}\int u^{-5}\,du$$
$$= -\frac{1}{2}\left[\frac{u^{-4}}{-4}\right] + C$$
$$= \frac{1}{8}\left(\frac{1}{u^4}\right) + C$$
$$= \frac{1}{8\cos^4(2x)} + C$$

or $\frac{1}{8}\sec^4(2x) + C$

Book doesn't yet have this, but standard trick. Ch. 9
$du = 1\,dx$

Ex $\int x\sqrt{x-1}\,dx$

$u = x - 1 \Rightarrow x = u + 1$ (Take our sub statement, solve for x.)
$du = dx$

What do we do w/ x?
Does $\int$prod = prod $\int$s? NO
Does $\int$sum = Sum $\int$s? YES if ∃

$$= \int (u+1)\sqrt{u}\,du$$
$$= \int (u+1)u^{1/2}\,du \qquad \left(\neq \int (u+1)\,du \cdot \int u^{1/2}\,du\right)$$
$$= \int \left(u^{3/2} + u^{1/2}\right) + C$$

& stick w/ algebra

Don't skip. Many blow coeffs. When you ÷ by a frac ...

$$= \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} + C$$
$$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

Get practice on Hw!!

5.4/5.3: AREA and DEFINITE ∫s

Def. ∫s: Area
Dx: slope

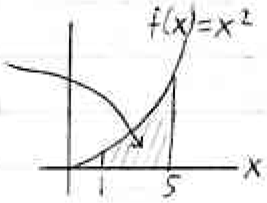
Ⓐ Intro

Ex $\int_1^5 x^2\,dx$ = definite integral of x^2 from 1 to 5
(x=1) (x=5)

upper limit

lower limit of integration

= Area



because $x^2 \geq 0$ on [1, 5]

we'll see in 5.6 $\frac{124}{3}$

$\overset{\text{turns out}}{=} 41\frac{1}{3}$

If I replace "x"s w/ t, will that change the result?

Ex $\int_1^5 t^2\,dt = 41\frac{1}{3}$, also

t is the dummy variable
can replace → same result

what do you do w/a dummy employee? You replace him w/ another dummy. And nothing really changes. Like airport security.

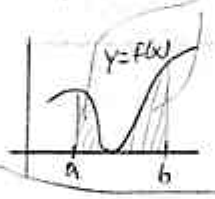
In general

(From now on, assume f is integrable on [a, b].) we'll define later

meaning the graph of f does not dip below x-axis

If $f(x) \geq 0$ on [a, b], then

$\int_a^b f(x)\,dx$ = area (between graph of f and x-axis on [a, b])

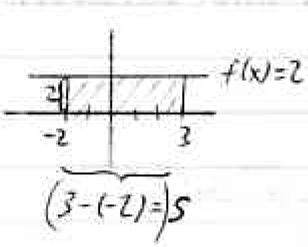


Efficient way of indicating an area

Ex $\int_{-2}^{3} 2\,dx$

$= (5)(2)$

$= 10$



They don't recognize!

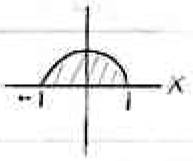
Like Concentration reveal—
In 150, can only do geometrically.
Mech in 151

Ex $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$y = \sqrt{1-x^2}$

$y^2 = 1-x^2 \quad (y \geq 0)$

$x^2+y^2 = 1 \quad (y \geq 0)$



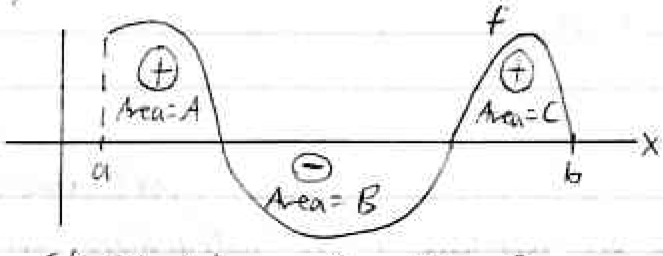
$= \frac{1}{2}\left[\pi(1)^2\right]$

$= \frac{\pi}{2}$

What happens if the graph falls below the x-axis?

(B) Signed Areas

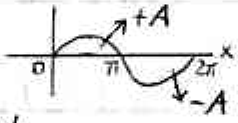
Areas that contribute a neg. # to the def. integral



$\int_a^b f(x)\,dx = A - B + C$

A=2

Ex $\int_0^{2\pi} \sin x\,dx = 0$

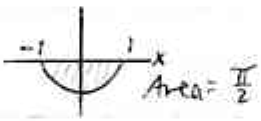


(By the way, A=2 !!)

Ex $\int_{-1}^{1} -\sqrt{1-x^2}\,dx$ (linearity/splitting rules for $\int$ hold)

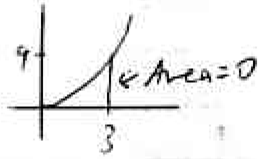
$\left(= -\int_{-1}^{1} \sqrt{1-x^2}\,dx\right)$

$= -\frac{\pi}{2}$



Ⓒ Some Props.

Ex $\int_3^3 x^2\,dx = 0$



$\int_c^c f(x)\,dx = 0$ if $f(c)$ exists

If you have an asym. there, what's the pt. literally? We assumed integrability, but cont. not relevant here.

Ex $\int_1^5 x^2\,dx = 41\frac{1}{3}$
$\Rightarrow \int_5^1 x^2\,dx = -41\frac{1}{3}$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

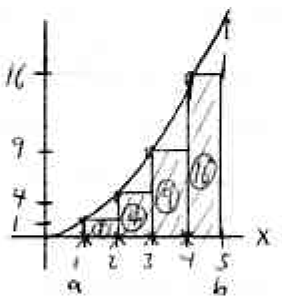
if f integrable on [a,b] - already assumed

5.4: Can do 11-35 odd

Ⓓ Approximating Definite $\int$s

Ex 1 $\int_1^5 x^2\,dx \quad (=41\frac{1}{3})$

[1,5] used in #5
$P = \{1,3,4,5\}$

① 

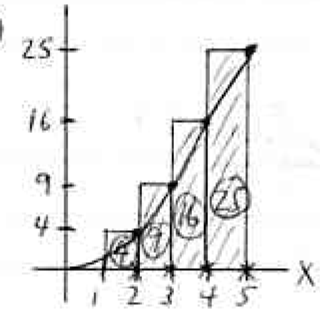
"LRA" = left-hand Riemann approx. using the partition "P" of [1, 5] determined by $\{1, 2, 3, 4, 5\}$

Area of 1st rect.

$$\int_1^5 x^2\,dx \approx f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$$
$$= (1)^2(1) + (2)^2(1) + (3)^2(1) + (4)^2(1)$$
$$= 1 + 4 + 9 + 16$$
$$= 30$$

Underest.

②



"RRA" = Right-hand

Height of rect is f(2), or 4

$$\int_1^5 x^2\,dx \approx f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1$$
$$= 4 + 9 + 16 + 25$$
$$= 54$$

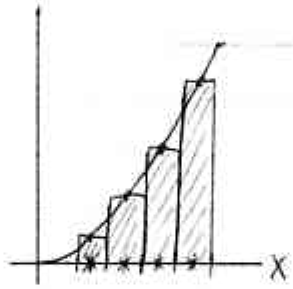
overest.

trap. rule

Avg. of LRA, RRA $= \frac{30+54}{2} = 42$ (=Result from Trapezoidal Rule - 5.7)

we've had an underest., overest. How can we get a more balanced est.?

③



"MRA" = Midpoint

Balance bet. extra areas, missing areas

More missing than extra, so underest.

Here, 1 trap works as an overest.

worse

$$\int_1^5 x^2\,dx \approx f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 + f(4.5)\cdot 1$$
$$= 41 \quad \text{(Winner for this Ex)}$$

Exact: $41\frac{1}{3}$

Also works: $\frac{LRA+RRA}{2} = 42$

MRA

This is not ≈ trapezoid nonlinearity betrays this.

named after R. (1826-66)

Riemann Sums

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(w_k)\,\Delta x_k \quad , w_k \text{ is in } k^{th} \text{ subinterval.}$$

#rects.

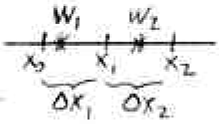
We pick something like a test value.

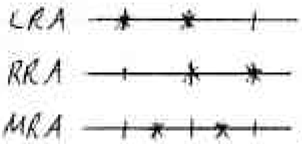
Do # lines before write $f(w_k)\Delta x_k$

Remember precalc! (sum over all k, the index of summation (dummy))

± height of kth rect. width

± Area of kth rect.





Where do we sample f values w/in each subint.?

regular partition Δx_k are equal

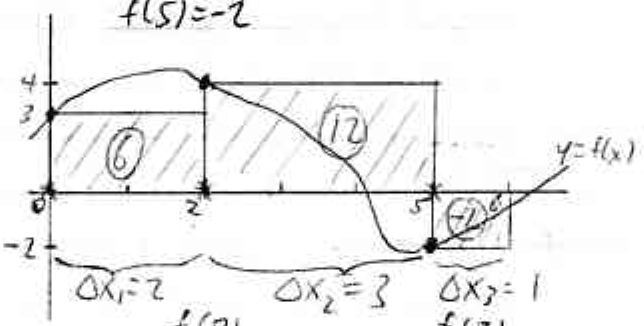
In Ex 1, $\Delta x_k = 1$ for all k ← Regular partition: all Δx_k are =.

Ex 2 $\int_0^6 f(x)dx$, P determined by {0, 2, 5, 6}

Find LRA.

Given: $f(0)=3$
$f(2)=4$
$f(5)=-2$

x time
f(x) = how high Tweety is
Limited info, experiment
or x time
f(x): velocity

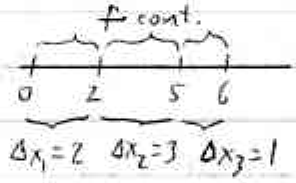


$$LRA = \overbrace{f(w_1)}^{f(0)}\Delta x_1 + \overbrace{f(w_2)}^{f(2)}\Delta x_2 + \overbrace{f(w_3)}^{f(5)}\Delta x_3$$
$$= (3)(2) + (4)(3) + (-2)(1)$$
$$= 6 + 12 - 2$$
$$= 16$$

(It's like we have limited info from an experiment ⇒ Approx. ∫)
(Life is not always $\int x^2 dx$ ....)

Ⓒ Using R. Sums to Define Definite $\int$s

Ex 2 P:



$\|P\|$ = norm of P = max Δx_i
= length of longest subint.
= 3, here

} measures how fine the partition is

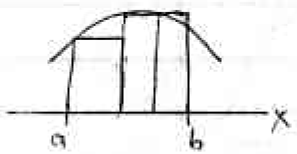
What Ch. 2 concept.

$\approx$ useful in practice
Tweety Ex
Just 1 more step to get official def'n of $\int_a^b$ (Def. 1)

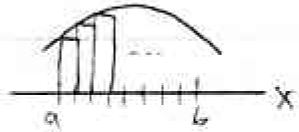
$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_k \underbrace{f(w_k)\,\Delta x_k}_{\pm \text{Area of } k^{th} \text{ rect}}$$

$\Rightarrow$ (let n = # rects. $\to \infty$)

if this limit exists
(i.e., if f is integrable on $[a,b]$)
guaranteed if f is cont. on $[a,b]$

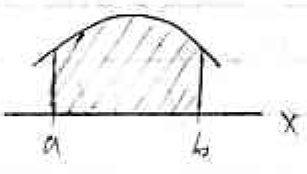


We're probably getting a much better approx. of true area



$\downarrow$ $\|P\| \to 0$
So, $n \to \infty$

It's not like we have one big rect. and the other rects ~~does~~ get thinner
we have to have this thinning all across the interval



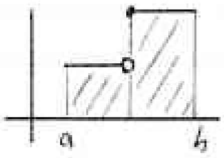
NO!  Not sufficient: $\lim_{n \to \infty}$

LS-10
5.4/5.3

Note

jump discont. removable OK (more of a problem:) if f defined

f still integrable:

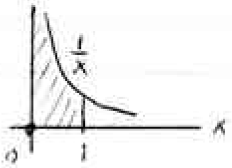


F not int. on $[0,1]$:

f is at least defined on $[0,1]$.

p. 273 ∞ discont. → f not int.

$$f(x) = \begin{cases} 0 & , x = 0 \\ \frac{1}{x} & , 0 < x \le 1 \end{cases}$$

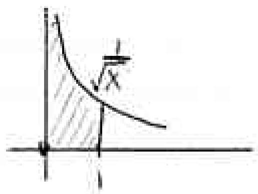


Ch. 10: $\int_0^1 f(x)\,dx = \infty$

The curve is approaching the x-axis in a way (fast enough) that traps a finite area.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$

but



$\int_0^1 f(x)\,dx = 2$

(still not int.)

Calc II

(F) Σ Notes

$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ — # terms, avg. of 1st, last terms: $\frac{n(n+1)}{2}$ } Arithmetic series ($d=1$)

Ex $\sum_{k=1}^{100} k = 1 + 2 + \ldots + 100 = \frac{100(101)}{2} = 5050$

Show for $n=100$

p. 259 $\sum_{k=1}^{n} k^2$, $\sum_{k=1}^{n} k^3$
$= (1)^2 + (2)^2 + \ldots + (n)^2$

Gauss: $S = 1 + 2 + \ldots + 100$
$S = 100 + 99 + \ldots + 1$
$2S = 101 + 101 + \ldots + 101$

$2S = 100(101)$
$S = \frac{100(101)}{2}$

Beat him up? Went on to great things. Cool!

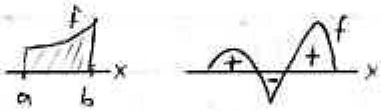
5.5: PROPERTIES OF DEFINITE $\int$s

(A) Review

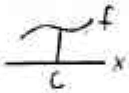
cont. on [a,b] ⇒ int. there

Assume integrability.

$\int_a^b f(x)\,dx$



① $\int_c^c (\quad) = 0$



$\int_a^b + \int_b^a = \int_a^a = 0$

② $\int_b^a (\quad) = -\int_a^b (\quad)$

SKIP; see back

If I'm $\int$ something complicated, I may be able to break it down.

(B) Integrand-Splitting Rules

like for $\int$, D_x

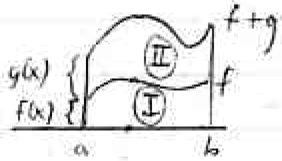
③ $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

$f \geq 0$

Shorthand

④ $\int_a^b (f+g)\,dx = \underbrace{\int_a^b f\,dx}_{\text{I}} + \underbrace{\int_a^b g\,dx}_{\text{II}}$

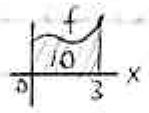
$f, g \geq 0$



⑤ $-$ $-$

(B) Integrand-Splitting (Linearity)

Ex Given: $\int_0^3 f(x)\,dx = 10$



Then, $\int_0^3 [4f(x)+1]\,dx$

*linear operator, like $\int, D_x, \lim$

$$\left(= \int_0^3 4f(x)\,dx + \int_0^3 1\,dx\right)$$

When I graph "1", I'm finding the area of a...

$$= 4\underbrace{\int_0^3 f(x)\,dx}_{=10} + \underbrace{\int_0^3 1\,dx}$$

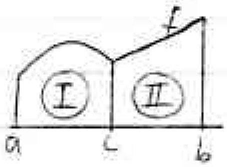
Up to 9

$$= 4(10) + 3$$
$$= 43$$

We can split the integrand, like for $\int s$, what can we also split for definite $\int s$?

(C) Interval-Splitting

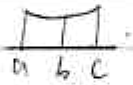
(maybe f is piecewise-defined)



$$\int_a^b f(x)\,dx = \underbrace{\int_a^{c} f(x)\,dx}_{\text{I}} + \underbrace{\int_{c}^b f(x)\,dx}_{\text{II}}$$

Even if c is outside the interval.

Even if



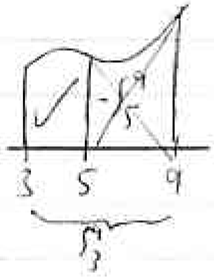
Can "cancel" $\int^x \int_x$ or $\int_x \int^x$ — more intuitive

How can rewrite $\int_9^5$?

Ex Simplify $\int_9^5 f(x)\,dx + \int_3^9 f(x)\,dx = \int_3^{9} \quad + \int_{9}^5$

$$= \int_3^5 f(x)\,dx$$

You don't need to know what happens from 5 to 9, as long as you have integrability.



Name from Larson 278

(D) Inequality-Preserving Rules

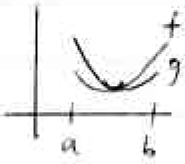
Throughout $[a,b]$,

(i) If $f(x) \geq 0$,
then $\int_a^b f(x)\,dx \geq 0$.

nonnegativity preserved

Up to 21
Gain (7):
$\int [f-g] \geq 0$

(ii) If $f(x) \geq g(x)$,
then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.



dominance preserved

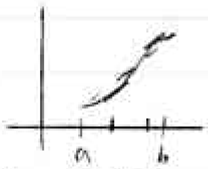
True even if f and/or g falls below 0
If one or both graphs fall below x-axis.
(i) Comparing f to 0
(ii) Comparing f to g
See m151!
Comparison Idea

(E) MVT for Definite $\int$s, f_{av}

In words do you know 'em?

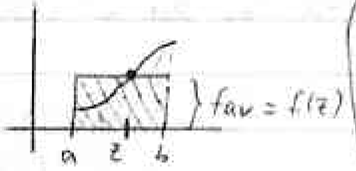
How do you avg. so many func. values?
Range game from Pre-calc is right.
Fixed width
Raising the bar until area of rect = area of piano
balance
cut out hole

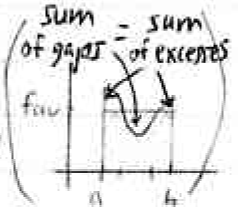
MVT for D_x



(somewhere in here, we get the avg. rate of change)

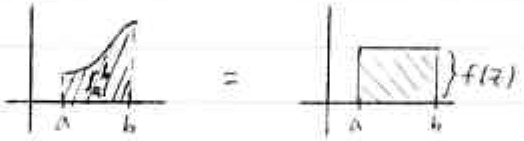
Now,





(somewhere in here, we get the avg. func. value)

If f is cont. on $[a,b]$, then $\exists z$ in (a,b) such that



equiv.; if $a=b$, (a,a) is silly.

$$\int_a^b f(x)\,dx = f(z)(b-a) \quad \leftarrow \text{MVT concl. in book}$$

i.e.,

$$\underbrace{f(z) = \frac{\int_a^b f(x)\,dx}{b-a}}_{= f_{av}\text{, the average value of } f \text{ on } [a,b]} \quad \begin{matrix} \leftarrow \text{"Sum"} \\ \leftarrow \text{"Input Size"} \end{matrix}$$

(like in "discrete averaging")

$= f_{av}$, the <u>average value of f on $[a,b]$</u>

Ex (#26)

Given: $\int_{-1}^{3} \underbrace{(3x^2 - 2x + 3)}_{f(x)}\, dx = 32$

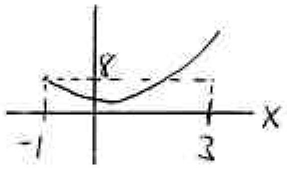
(b) Find f_{av} on $[-1, 3]$.

Do 1st

$$f_{av} = \frac{\int_{-1}^{3} f(x)\,dx}{3-(-1)}$$

$$= \frac{32}{4}$$

$$= \boxed{8}$$



What's Part (a)? where is f_{av} taken on?

How do you solve? There's no "8-factor property."

QF if desperate. Either way, $\ldots = 0$

(a) (Find all z such that $f(z) = f_{av}$.)

$$f(z) = 8$$
$$3z^2 - 2z + 3 = 8$$
$$3z^2 - 2z - 5 = 0$$
$$\underbrace{(3z-5)}_{\Downarrow}\underbrace{(z+1)}_{\Downarrow} = 0$$

$\boxed{z = \frac{5}{3}}$ $\quad$ $\cancel{z = -1}$ not in $(-1, 3)$

FUND. THM. OF ARITHMETIC: Every integer ≥ 2 is prime or can be factored/decomposed as a product of primes uniquely, up to a reordering of factors.

FUND. THM. OF ALGEBRA: Every n^{th}-degree poly. func. $(n \geq 1)$ w/complex (incl. real) coeffs. has a complex root/zero. ($\leq n$ distinct sols.)

5.6: FUNDAMENTAL THM. OF CALCULUS (FTC)

Assume f is cont. on $[a,b]$.

(A) Old Ex (5.1)

$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$
Proof w/out Words
Great!
AMATYC Fall 2002
p. 65

\$ in a box.

I inherit \$1000.
After t years, I spend at the rate of $10t \ \frac{\$}{yr.}$
How much do I spend from $t=5$ to $t=6$?

Let $f(t) =$ \$ left after t years

$f'(t) = -10t, \quad f(0) = 1000$

see back (1) $\Rightarrow f(t) = -5t^2 + 1000$

Change in \$ $= f(6) - f(5)$ (2)
$= [-5(6)^2 + 1000] - [-5(5)^2 + 1000]$
$= -180 + \cancel{1000} + 125 - \cancel{1000}$
(3) Cancel!
$= -55$

I spend \$55

Notation Shift

Big daddy func. and his deriv.

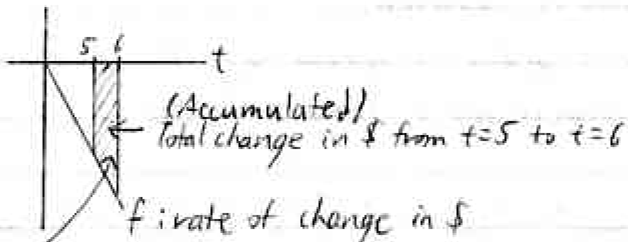
$$\begin{matrix} f \\ f' \end{matrix} \rightarrow \begin{matrix} F \\ f \end{matrix}$$

Now: F, f — Before: f, f'

Let $f(t) = -10t$ (rate)

When you integrate the rate of change from here to here, ~~what represents~~ you get the total change...

Area of trap.



$$\int_5^6 f(t)\,dt = \int_5^6 -10t\,dt$$

$$= \Big[\underbrace{-5t^2}_{=F(t)}\Big]_5^6$$

Don't need +C !!

(Turns out $\nabla - \nabla$ $\int_0^x -10t\,dt = -5x^2$)

You'll get +C, -C just like +1000, -1000

$$= F(6) - F(5)$$
$$= [-5(6)^2] - [-5(5)^2]$$
$$= -55$$

Connections w/ 5.1 approach

① Like working out indefinite $\int$

② Like $f(6) - f(5)$, Old Notation

③ Don't need +C

Ⓑ FTC, Part II

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$

any AD of f on $[a,b]$

$$= F(b) - F(a)$$

even if $b < a$!!

Calc II: maybe diff. looking expressions

Evaluate at b, a, subtract the results

No Quotient Rule for ∫s.

Ex Evaluate $\int_1^2 \underbrace{\frac{x^5+7}{x^3}}_{\text{cont. on } [1,2]} dx$

$$= \int_1^2 \left(\frac{x^5}{x^3} + \frac{7}{x^3}\right) dx$$

$$= \int_1^2 (x^2 + 7x^{-3})\,dx$$

$$= \left[\frac{x^3}{3} + 7\left(\frac{x^{-2}}{-2}\right)\right]_1^2$$

$$= \left[\frac{x^3}{3} - \frac{7}{2x^2}\right]_1^2$$

top result - bottom
even if top < bottom

$$= \underbrace{\left[\frac{(2)^3}{3} - \frac{7}{2(2)^2}\right]}_{\text{eval at 2}} - \underbrace{\left[\frac{(1)^3}{3} - \frac{7}{2(1)^2}\right]}_{\text{eval at 1}}$$

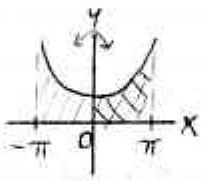
1-21 odd
wait on 11

$$= \frac{119}{24}$$

Ex $\int_{-\pi}^{\pi} \underbrace{(x^2 + \cos x)}_{\substack{f \text{ even} \\ f(-x) = f(x)}} dx$

If I replace x w/ -x, nothing really happens. Get same-thing equiv. Graph sym about y axis. Instead of plugging in $-\pi$, we'd probably rather plug in 0.

Turns out



$$= 2\int_0^{\pi} (x^2 + \cos x)\,dx \quad \text{by symmetry}$$

(Doesn't work for: $\int_{-1}^{\pi} \dots$)

Whose deriv is cos x?

$$= 2\left[\frac{x^3}{3} + \sin x\right]_0^{\pi}$$

$$= 2\left(\left[\frac{\pi^3}{3} + \underbrace{\sin \pi}_{=0}\right] - \underbrace{\left[\frac{0^3}{3} + \sin 0\right]}_{=0}\right)$$

See if your HS geometry teacher can do this!

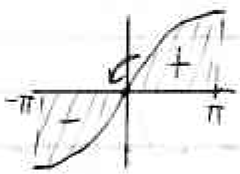
$$= \frac{2\pi^3}{3}$$

$$\boxed{\text{If } f \text{ is even, cont. on } [-a,a] \quad \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

Ex $\displaystyle\int_{-\pi}^{\pi} \underbrace{(x+\sin x)}_{\substack{f \text{ odd} \\ f(-x)=-f(x)}}\,dx = 0$

when you replace x w/ -x, you get the opp. graph sym about origin (not x-axis)

Turns out



1-21 odd

$$\boxed{\text{If } f \text{ is odd, cont. on } [-a,a] \quad \int_{-a}^{a} f(x)\,dx = 0}$$

Ex (u subs)

(#28) $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{x^2+9}}\,dx$

Why?

Let $u = x^2+9$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,du$ (or "compensation")

(Solve for what you want to kill.)

New limits (for u)

$x = 0 \Rightarrow u = (0)^2+9 = 9$

$x = 4 \Rightarrow u = (4)^2+9 = 25$

$$\left(= \frac{1}{2}\int_{x=0}^{x=4} \frac{2x}{\sqrt{x^2+9}}\,dx\right) \text{ if use compensation}$$

$$= \frac{1}{2}\int_{u=9}^{u=25} \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\int_9^{25} u^{-1/2}\,du$$

$$= \cancel{\tfrac{1}{2}}\left[\frac{u^{1/2}}{\cancel{1/2}}\right]_9^{25}$$
$$= \left[\sqrt{u}\,\right]_9^{25}$$
$$= \sqrt{25} - \sqrt{9}$$
$$= 5 - 3$$
$$= 2 \quad \text{(wow!!)}$$

Use for #37.

<u>Another way</u> (if you don't want to change limits)

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$$

Work out indefinite $\int$, 1st, to get an AD in x.

$$\int \frac{x}{\sqrt{x^2+9}}\,dx \qquad \begin{aligned} u &= x^2+9 \\ du &= 2x\,dx \Rightarrow x\,dx = \tfrac{1}{2}du \end{aligned}$$

SO WRONG

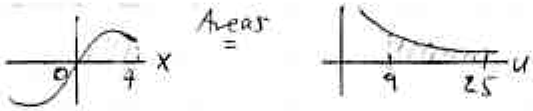
$$= \frac{1}{2}\int \frac{du}{\sqrt{u}}$$
$$= \frac{1}{2}\int u^{-1/2}\,du$$
$$= \cancel{\tfrac{1}{2}}\left[\frac{u^{1/2}}{\cancel{1/2}}\right] + C$$
$$= \sqrt{u} + C$$
$$= \underbrace{\sqrt{x^2+9}}_{\text{an AD}} + C$$

NOW,
$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$$
$$\overset{FTC}{=} \underbrace{\left[\sqrt{x^2+9}\right]}_{\text{Our AD!!}}{}_0^4$$

In a way, we have to work out u-limits, anyway. 1–37 odd

$$= \sqrt{4^2+9} - \sqrt{0^2+9}$$
$$= 2$$

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx = \int_9^{25} \frac{1}{2\sqrt{u}}\,du$$



Change of variable

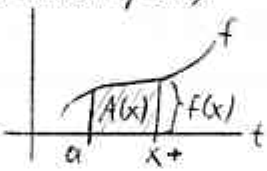
like Star Trek: I not as good...

(C) FTC, Part I

Assume f cont., "a" constant

$$D_x\Big(\underbrace{\int_a^x f(t)\,dt}_{=A(x)}\Big) = f(x)$$

(Funcs. can be defined by $\int$s!!)



$$D_x A(x) = f(x)$$

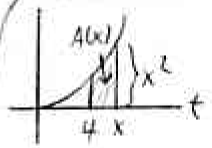
$$\begin{pmatrix}\text{rate of}\\ \text{change of}\\ A(x) \text{ wrt } x\end{pmatrix} = \begin{pmatrix}\text{func.}\\ \text{value}\\ \text{at } x\end{pmatrix}$$

we could've started at −10M

If x is out here, A(x) growing at a faster rate

A(x) solves $A'(x) = x^2$, $A(4) = 0$, $A(x) = \frac{1}{3}x^3 - \frac{64}{3}$

Ex $D_x\Big(\int_4^x \underbrace{t^2}_{f(t)}\,dt\Big) = \underbrace{x^2}_{f(x)}$



Skip? Irritating... Kill 53, 55

$\int_{k(x)}^{g(x)} = \int_0^{g(x)} - \int_0^{k(x)}$, $u = g(x)$, $G(u)$

$D_x = \frac{dG}{du}\frac{du}{dx}$

limits

<u>In general</u>

$$D_x\Big(\int_{k(x)}^{g(x)} f(t)\,dt\Big) = \underbrace{f(g(x))}_{\substack{\text{replace } t\\ \text{w/ } g(x) \text{ in}\\ \text{integrand}}}\underbrace{g'(x)}_{\text{tail}} - \underbrace{f(k(x))}_{\text{'}k(x)\text{'}}\underbrace{k'(x)}_{\text{tail}}$$

both diff'e

Ex $D_x \int_{2x}^{\sin x} t^2\,dt = \underbrace{(\sin x)^2}_{\substack{\text{Replace}\\ t \text{ w/}(\sin x)}}\underbrace{(\cos x)}_{\text{tail}} - \underbrace{(2x)^2}_{\substack{\text{Replace}\\ t \text{ w/}(2x)}}\underbrace{(2)}_{\text{tail}}$

$$= \sin^2 x \cos x - 8x^2$$

5.7: NUMERICAL APPROX. OF DEFINITE $\int$s

Ⓐ Intro

Useful for $\int_a^b f(x)\,dx$ when

Hard to find F
u subs don't work
No nice geometry like
Just have a table of f values (from experimental sampling?)

You're forced to approximate, anyway.

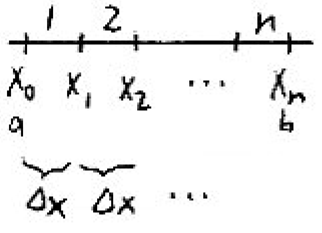
x	$f(x)$
7	40
8	50
$\vdots$	

Ex $\int_0^4 \sqrt{x^3+1}\,dx$

$\left(\begin{array}{l} u = x^3 - 1 \\ du = 3x^2\,dx \end{array} \right\} \text{NO!} \quad$ ← No x^2!!

What did we use before?

Let $x_0, x_1, \ldots, x_n$ determine a <u>regular partition</u> of $[a, b]$.



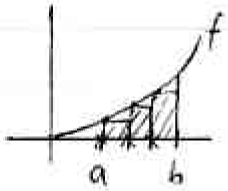
How can I express Δx in terms of a, b, n?

Each $\Delta x_k = \Delta x = \frac{b-a}{n}$

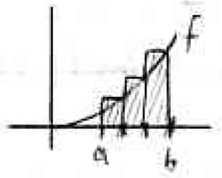
Assume f is cont. on $[a, b]$.

(B) Review Riemann Sums

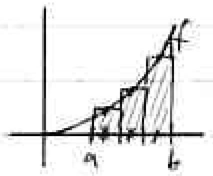
Approx. $\int_a^b f(x)\,dx$



LRA, $n=3$



RRA



MRA

Flat top

Larson

We're approx f by using a piecewise constant func.

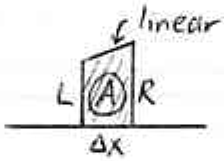
For <u>each</u> subinterval, we approx. f by using a <u>constant</u> function. Rectangles!

(We approx. f by a piecewise constant func.)

Instead of constant funcs, what if we use linear funcs?
Instead of rects, we get...
You may get some rects, also.
We're approx f by using a piecewise linear func.
An option in Mathematica
More adv. methods
Simpson uses irregular partitions:
f is changing rapidly on a subint, it will get broken down.

© Trapezoidal Rule

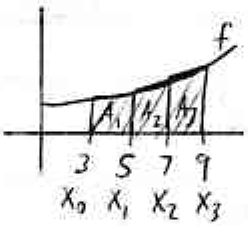
For now, assume $f(x) \geq 0$ on $[a,b]$.



$$A = \underbrace{\Delta x}_{\text{"height"}} \underbrace{\left(\frac{L+R}{2}\right)}_{\text{avg. of bases}} \quad (\leftarrow \text{Tilt your head!})$$
$$= \tfrac{1}{2}\Delta x\,(L+R)$$

Ex Approx. $\int_3^9 f(x)\,dx$

I'm motivating the general formula



$n=3$

$$\int_3^9 f(x)\,dx \approx A_1 + A_2 + A_3$$
$$= \tfrac{1}{2}(2)[f(3)+f(5)]$$
$$+ \tfrac{1}{2}(2)[f(5)+f(7)]$$
$$+ \tfrac{1}{2}(2)[f(7)+f(9)]$$

$$= \tfrac{1}{2}\underbrace{(2)}_{\substack{=\Delta x \\ =\frac{b-a}{n}}}[f(3)+2f(5)+2f(7)+f(9)]$$

On a test, I'll give you this

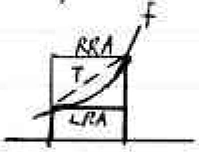
$$\int_a^b f(x)\,dx \approx \underbrace{\tfrac{1}{2}\Delta x}_{\text{or } \frac{b-a}{2n}} \underbrace{\left[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{n-1})+f(x_n)\right]}_{\text{Coeffs: } 1\ 2\ 2 \dots 2\ 1}$$

$= \text{"}T\text{"}$

The trap. approx based on this partition

Notes ① T = average of LRA, RRA

Split the difference



LRA takes $f(x_0)$ but not $f(x_n)$; RRA

Explains 1 2 2 ... 2 1 (LRA takes $f(a)$, not $f(b)$. RRA $f(b)$ $f(a)$.)

All others taken by both

② Don't need $f(x) \geq 0$.

Trap. picture → may fall apart. Thinking of LRA, RRA

③ As $n \to \infty$, $T \to$ exact

④ Mathematica may use irregular partitions:

finer where f changing rapidly

We have regular partitions. If $n \to \infty$, $\|P\| \to 0$.

Ex Approx. $\int_0^{12} \underbrace{(\sqrt{x}-1)}_{f(x)}\,dx$, $n=6$

We can find this exactly

n=6 subints. Don't say trapezoids.

$\Delta x = \frac{b-a}{n} = \frac{12-0}{6} = 2$

$T = \frac{1}{2}(2)\cdot[\ f(0) \leftarrow (\sqrt{0}-1) = -1$

$+2f(2) \leftarrow 2(\sqrt{2}-1) \approx 0.8284$

$+2f(4) \leftarrow 2(\sqrt{4}-1) = 2$

2.8990 $\quad +2f(6) \quad \vdots$

3.6569 $\quad +2f(8)$

4.3246 $\quad +2f(10)$

$+ f(12)\,] \leftarrow (\sqrt{12}-1) \approx 2.4641$

$\approx \frac{1}{2}(2)[15.1730]$

In HW, final ans. to 2 dec. places

$T \approx 15.1730$ (Book: ≈ 15.17)

$\frac{2}{3}(\sqrt{12})^3 - 12$ $\quad$ Exact: ≈ 15.7128

named after Thomas Simpson (1710-61)
Lesson 301
formalized Newton's method

We've approx f piecewise using constant funcs., linear funcs. What's the next step?

Draw parabolas 1st (for me) before plotting pts.

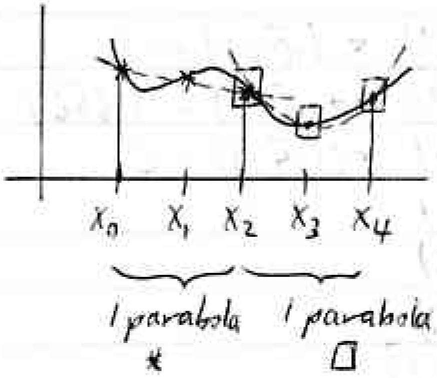
(He also formalized Newton's Method!)

① Simpson's Rule

often more accurate

(Riemann: piecewise constant approx.
Trap: linear
Now... quad.)

Use pieces of parabolas or lines.



Let n be even.

$$\int_a^b f(x)\,dx \approx \overbrace{\tfrac{1}{3}\Delta x}^{\text{or } \frac{b-a}{3n}}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots + 2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

Coeffs: 1 4 2 4 2 ... 4 2 4 1

= "S"

As $n \to \infty$, $S \to$ exact

$\underline{Ex}$ Approx $\int_0^{12} \underbrace{(\sqrt{x}-1)}_{f(x)} dx$, $n=6$

$\Delta x = \frac{b-a}{n} = 2$

$S = \frac{1}{3}(2) \cdot [\ f(0) \leftarrow (\sqrt{0}-1) = -1$

$+4f(2) \leftarrow 4(\sqrt{2}-1) \approx 1.6569$

$+2f(4) \leftarrow 2(\sqrt{4}-1) = 2$

5.7980 $+4f(6)$ $\vdots$

3.6569 $+2f(8)$

8.6491 $+4f(10)$

2.4641 $+\ f(12)]$

$\approx \frac{2}{3}[23.2250]$

≈ 15.4833

$T \approx 15.17$

$S \approx 15.48$

Exact ≈ 15.71

CH.5 REVIEW

(5.1) Indefinite $\int$s
$\int f(x)dx = F(x) + C$
Power, Trig, $\int$ linearity
Diff. Eqs.

(5.2) u subs

(5.4/5.3) Definite $\int$s
$\int_a^b f(x)dx = \#$, Def$^{\underline{n}}$: $\lim_{\|P\|\to 0} \sum_k f(w_k)\Delta x_k$ in k^{th} subinterval
signed areas, Geometry
Approx. by LRA, MRA, RRA

(5.5) Props. of $\int$
MVT, fav

(5.6) FTC
Part II Evaluate $\int_a^b f(x)dx$
u subs $\Rightarrow \int_c^d g(u)du$

Part I $D_x\left(\int_a^x f(t)dt\right) = f(x)$

(5.7) Numerical Approx of $\int_a^b f(x)dx$
If can't find F, or have f table
Before: LRA, MRA, RRA 
Trapezoidal Rule  (Mathem: break down intervals where f changing rapidly)
Simpson's Rule