Reverse D_x

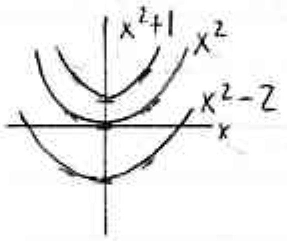
CH. 5: INTEGRALS ($\int$s)

Later, taking derivs.

5.1: ANTIDERIVATIVES (ADs) and INDEFINITE $\int$s

What's the deriv. of x^2? If I tell you $2x$ is the deriv of some func, does it have to be x^2?

(A) ADs



$\xrightarrow{D_x}$ $\quad f'(x) = 2x$

$\xleftarrow[\text{Find ADs}]{\text{NOW}}$

Wouldn't you agree the slope situation is the same consistent w/ $D_x(\text{const.}) = 0$

Given the deriv func, how do we reconstruct the "original func."? Actually, we have a whole family of ADs.

Given: $f(x) = 2x$

Then, $F(x) = x^2 + \underset{\text{arbitrary constant}}{C}$ gives <u>all</u> the ADs of f. (a family)

$$\boxed{F'(x) = f(x)}$$

Why dx? What if $\int$?, Calc III.

(B) Indefinite $\int$s

Swok. Book.

It's because of this C that we call these Indefinite $\int$s / How to Acc: family of funcs, Definite $\int$ = #

$$\underbrace{\int}_{\text{integral sign}} \underbrace{2x}_{f(x),\ \text{the integrand}}\ \underbrace{dx}_{\text{says } x \text{ is the variable of integration}} = \underbrace{x^2}_{F(x),\ \text{an AD of } f} + \underset{\text{constant of integration}}{C}$$

read as CAND vs. INT as grand

Evaluating the $\int$
Integrating $f(x)$

$$\boxed{\text{Form: } \int f(x)\,dx = \underset{\text{an AD of } f}{F(x)} + C}$$

© Basic Rules

Deriv of x is 1 ⇒ … ∫ of 1 is

$$D_x(x) = 1 \qquad \int 1\,dx = \int dx = x + C$$
$$D_x(3x) = 3 \qquad \int 3\,dx = 3x + C$$

D_x to ✓

Power Rule for ∫

What if we want ∫ plain x^3 dx? How do we adjust? Not "sing" Don't need $\frac{c}{4}$

$$D_x(x^4) = 4x^3 \Rightarrow \int 4x^3\,dx = x^4 + C$$

(Beresford) "Sing at sight"

$$D_x\left(\frac{x^4}{4}\right) = \frac{1}{4}\cdot 4x^3 = x^3 \Rightarrow \int x^3\,dx = \frac{x^4}{4} + C$$

Beresford: $\frac{1}{n+1}x^{n+1} + C$ There's one value of n for which the formula fails.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1$$

$$\frac{x^{\text{add } 1}}{\text{copy}} + C$$

Shocking…

Note: If $n = -1$:
$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$
Ch. 7

Trig Rules

Helps to write

$$D_x(\sin x) = \cos x \qquad D_x(\cos x) = -\sin x$$
$$D_x(\tan x) = \sec^2 x \qquad D_x(\cot x) = -\csc^2 x$$
$$D_x(\sec x) = \sec x \tan x \qquad D_x(\csc x) = -\csc x \cot x$$

Recognize Deriv. of what? Don't memorize all 12 → confuse signs. Mind dump the derivs.

When do I flip sign?

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$
$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

have sin, csc ⇒ −

What was $e^{\ln x}$?

Oh, man!

don't go:
$\int \sec^2 x \sec^2 x \, dx$
No!!
$\tan \tan x + C$

Inverse Props

Exs $D_x\left[\int \sec^4 x \, dx\right] = \sec^4 x$.

$\int [D_x(\sec^4 x)] \, dx = \sec^4 x + C$

(In general
$D_x\left[\underbrace{\int f(x)\,dx}_{\text{if exists}}\right] = f(x)$
$\int [\underbrace{D_x f(x)}_{\text{if exists}}] \, dx = f(x) + C$)

Splitting Rules

Can $\int$ (operate) term-by-term
warning:
$\int(\text{product}) \neq$ product of $\int$s, usually.

$\int$ is a linear operator, like D_x, lim

Can $\int$ term-by-term

$\int c f(x) \, dx$

($\int$ of Sum = Sum of $\int$s
Diff. Diff.
$\int (f \pm g)\,dx = \int f\,dx \pm \int g\,dx$) Can $\int$ term-by-term

① Exs

Need!!

Ex Evaluate $\int (x^6 + 3x^5 - 2) \, dx$

$= \int x^6 dx + 3\int x^5 dx - \int 2 \, dx$ ← can skip

$= \frac{x^7}{7} + 3\left(\frac{x^6}{6}\right) - 2x + C$ ← absorbs all consts. of $\int$

$= \frac{1}{7}x^7 + \frac{1}{2}x^6 - 2x + C$

Ex $\int \left(\sqrt[3]{x^2} + \frac{4}{\sqrt{x}} \right) dx$

① Rewrite

$= \int \left(x^{2/3} + 4x^{-1/2}\right) dx$

② Integrate (+C)

$= \frac{x^{5/3}}{5/3} + 4\left(\frac{x^{1/2}}{1/2}\right) + C$

③ Simplify

How do you $\div$ by a frac.?
What's $4 \div \frac{1}{2}$?
Many: "2".

$= \frac{3}{5}x^{5/3} + 4(2)\sqrt{x} + C$

$= \frac{3}{5}x^{5/3} + 8\sqrt{x} + C$

1-9 odd

This isn't technically a binomial (just a poly.), but it's like squaring a binomial

Ex (#20) $\displaystyle\int \frac{(\sqrt{t}+2)^2}{t^3}\,dt$

Now what - still alg.

$$= \int \frac{t + 4\sqrt{t} + 4}{t^3}\,dt$$

$$= \int \left(\frac{t^1}{t^3} + \frac{4t^{1/2}}{t^3} + \frac{4}{t^3} \right) dt$$

Quotient rule for exps.

$$= \int \left(t^{-2} + 4t^{-5/2} + 4t^{-3} \right) dt$$

$$= \frac{t^{-1}}{-1} + 4\left(\frac{t^{-3/2}}{-3/2}\right) + 4\left(\frac{t^{-2}}{-2}\right) + C$$

$$= -t^{-1} + 4\left(-\tfrac{2}{3}\right)t^{-3/2} - 2t^{-2} + C$$

$$= \underbrace{-t^{-1}}_{-\frac{1}{t}} - \tfrac{8}{3}t^{-3/2} \underbrace{-2t^{-2}}_{-\frac{2}{t^2}} + C$$

1-19 odd

peel out sin x

oh yeah, he's the deriv of.

Ex $\displaystyle\int \frac{\cos x}{7\sin^2 x}\,dx = \frac{1}{7}\int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) dx$

$$= \frac{1}{7}\int \csc x \cot x\,dx \qquad D_x(\csc x) = -\csc x \cot x$$

$$\left(= \frac{1}{7}(-\csc x) + C \right)$$

$$= -\frac{1}{7}\csc x + C$$

Don't need "$-\frac{1}{7}C$"

1-47 odd

(E) Differential Eqs.

Uncle Ernie/Lester dies

Ex I inherit \$1000.
After t years, I spend it at the rate of $10t \frac{\$}{yr.}$
In how many years will it be gone?

After 2 yrs. 20 $/yr.
2.5 25
3 30

My spending always accelerates
I'm kinda mean-today's not even budget and I'm already spending the \$. Slowly at 1st and then more later

① Setup
Let $f(t)$ = \$ left after t years
(or y)

Solve the diff. eq.

$$f'(t) = -10t \quad \text{or} \quad \frac{dy}{dt} = -10t$$

What is the rate of change of f? Careful!

subject to the initial condition (IC)

$$f(0) = 1000 \quad \text{or} \quad y = 1000 \text{ when } t = 0$$

What else do we know about f?

② Find the general sol'n of $f'(t) = -10t$

$$f'(t) = -10t \qquad \text{or} \qquad \frac{dy}{dt} = -10t$$

Integrate(S) both sides wrt t.

$$dy = -10t\,dt$$

"dt" only
Differential form

Separation of variables

$$\int f'(t)\,dt = \int -10t\,dt$$

$$f(t) = -10\left(\frac{t^2}{2}\right) + C \qquad \int dy = \int -10t\,dt$$

$$f(t) = -5t^2 + C \qquad y = -5t^2 + C$$

f(t)+C, but we'll write +C at end

y+C
$D_t = -10t$

Use the IC to solve for C, and
③ Find the particular sol'n

$$f(0) = -5(0)^2 + C$$

$$1000 = C$$

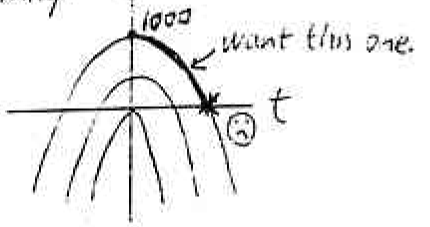
$$C = 1000$$

If your financial advisor says you have \$10K + C
What's C?
"any real #"
You're fired!

$$f(t) = -5t^2 + 1000$$

start off by spending slowly + spendthrift

Family: $-5t^2+C$



what do I want to know: when does the inheritance run out!

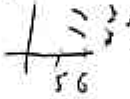
④ Solve $f(t)=0$.

$$-5t^2+1000=0$$
$$t^2=200$$
$$t=\pm\sqrt{200}$$
$$t=10\sqrt{2}\text{ years}$$
$$\approx 14.1\text{ years}$$

1-51 odd

Stumbling onto FTC:
How much \$ spent in 1 yr.?
e.g. t=5 to t=6
$\int_5^6 f'(t)\,dt = f(6)-f(5)$

indep. of $f(0)=C$
since parallel graphs



} same Δy

2nd-order diff. eq.

<u>Ex</u> Solve $\underbrace{f''(x)=4}$ (reconstruct $f(x)$)
subject to

$$\left.\begin{array}{l} f'(2)=1 \\ f(2)=3 \end{array}\right\} \text{ICs}$$

(OK if we have info at 2, not 0)
(<u>Note</u> In "boundary-value" problems, you're given 2 f-values or 2 f'-values. Harder!)

<u>Find $f'(x)$</u>

Before you find f, you have to find... what do we do to both sides? $f'(x)+C$

$$\int f''(x)\,dx = \int 4\,dx$$
$$f'(x) = 4x + C$$

<u>Find C</u>: Use $f'(2)=1$

$$f'(2) = 4(2)+C$$
$$1 = 8+C$$
$$C = -7$$

$$f'(x) = 4x-7$$

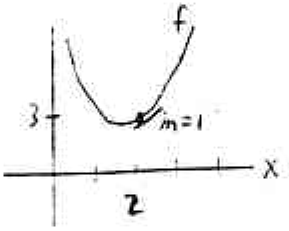
<u>Find $f(x)$</u>

$$\int f'(x)\,dx = \int (4x-7)\,dx$$
$$f(x) = 4\left(\tfrac{x^2}{2}\right) - 7x + D$$
$$f(x) = 2x^2 - 7x + D$$

<u>Find D</u>: Use $f(2)=3$

$$f(2) = 2(2)^2 - 7(2) + D$$
$$3 = 8 - 14 + D$$
$$D = 9$$

$$f(x) = 2x^2 - 7x + 9$$



Given: f'' } ⇒ Reconstruct f,
Pt. Get graph.
Slope (of tan line)

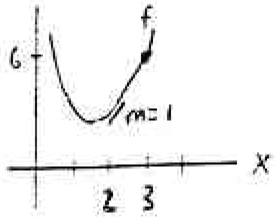
<u>Variation</u>

$f''(x) = 4$

s.t. $f'(2) = 1$

$f(3) = 6$

⇒ same f



<u>Ex (Projectiles)</u>

BEFORE:

s
$\downarrow$
$s' = v$
$\downarrow$
$s'' = v' = a$

↑ NOW

As you $\int$, you pick up constants of integration that you determine using ICs.

(HATE) At time 0, throw down @ $10 \frac{ft.}{sec.}$

100 ft.

$s = 0$

$s(t)$ = height of brick t secs after thrown

Solve $a(t) = -g$ ← local grav. constant (for Earth; slightly higher at poles

$a(t) \approx -32 \ (\frac{ft.}{sec.^2})$

(or $-9.8 \frac{m}{sec^2}$ if have m) ("wide")

- "bec. brick going faster in the neg direction

What does $\downarrow 10 \frac{ft}{sec.}$ tell us, using mathem. notation?

s.t. $\begin{cases} v(0) = -10 & (\frac{ft.}{sec.}) \\ s(0) = 100 & (ft.) \end{cases}$ ICs

$a(t) = -32$

$\int a(t)\,dt = \int -32\,dt$

$v(t) = -32t + C$

$t=0$: $v(0) = 0 + C$

$-10 = C$

Dangerous: $C = v(0)$?? What if IC somewhere else? Students blow it... (if not poly...

$v(t) = -32t - 10$

$\int v(t)\,dt = \int (-32t - 10)\,dt$

$s(t) = -16t^2 - 10t + D$

$t=0$: $s(0) = 0 - 0 + D$

$100 = D$

63: find general formula

$s(t) = -16t^2 - 10t + 100$

Then, you can answer interesting ?s.
When will it hit my head/foot.