5.4/5.3: AREA and DEFINITE $\int$s

Def. $\int$s: Area
D_x: slope

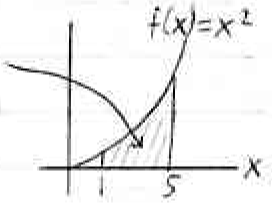
Ⓐ Intro

Ex $\int_1^5 x^2\,dx$ = definite integral of x^2 from 1 to 5
($x=1$) ($x=5$)

upper limit

lower limit of integration

= Area



because $x^2 \geq 0$ on $[1,5]$

turns out $= 41\frac{1}{3}$

we'll see in 5.6

$\frac{124}{3}$

Ex $\int_1^5 t^2\,dt = 41\frac{1}{3}$, also

t is the dummy variable
can replace → same result

If I replace "x"s w/ t, will that change the result?

what do you do w/a dummy employee? You replace him w/ another dummy. And nothing really changes. Like airport security.

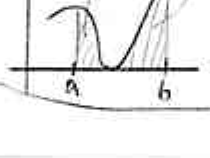
In general

(From now on, assume f is integrable on $[a,b]$.)
we'll define later

If $f(x) \geq 0$ on $[a,b]$, then

$\int_a^b f(x)\,dx$ = area (between graph of f and x-axis on $[a,b]$)

meaning the graph of f does not dip below x-axis

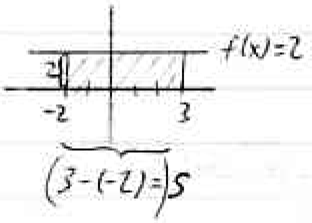


Efficient way of indicating an area

Ex $\int_{-2}^{3} 2\,dx$

$= (5)(2)$

$= 10$



They don't recognize!

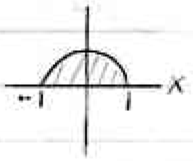
Like Concentration reveal— In 150, can only do geometrically. Mech in 151

Ex $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$y = \sqrt{1-x^2}$

$y^2 = 1-x^2 \quad (y \geq 0)$

$x^2+y^2 = 1 \quad (y \geq 0)$



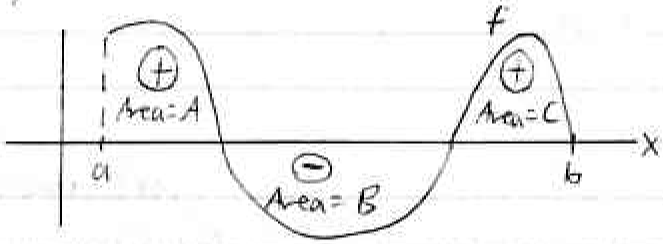
$= \frac{1}{2}\left[\pi(1)^2\right]$

$= \frac{\pi}{2}$

What happens if the graph falls below the x-axis?

(B) Signed Areas

Areas that contribute a neg. # to the def. integral



$\int_a^b f(x)\,dx = A - B + C$

A=2

Ex $\int_0^{2\pi} \sin x\,dx = 0$

(By the way, A=2 !!)

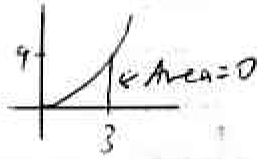
Ex $\int_{-1}^{1} -\sqrt{1-x^2}\,dx$ (linearity / splitting rules for $\int$ hold)

$= -\int_{-1}^{1}\sqrt{1-x^2}\,dx$

$= -\frac{\pi}{2}$

Ⓒ Some Props.

Ex $\int_3^3 x^2\,dx = 0$



If you have an asym. there, what's the int. literally? We assumed integrability, but cont. not relevant here.

$\int_c^c f(x)\,dx = 0$ if $f(c)$ exists

Ex $\int_1^5 x^2\,dx = 41\frac{1}{3}$

$\Rightarrow \int_5^1 x^2\,dx = -41\frac{1}{3}$

if f integrable on [a,b] - already assumed

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

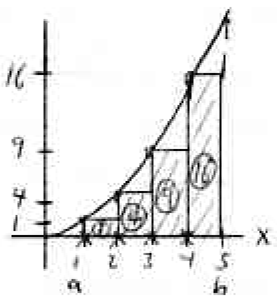
5.4: Can do 17-35 odd

Ⓓ Approximating Definite ∫s

Ex 1 $\int_1^5 x^2\,dx \quad (= 41\frac{1}{3})$

[1,5] used in #5 P = {1,3,4,5}

①



"LRA" = Left-hand Riemann approx. using the partition "P" of [1, 5] determined by {1, 2, 3, 4, 5}

Area of 1st rect.

$$\int_1^5 x^2\,dx \approx f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$$
$$= (1)^2(1) + (2)^2(1) + (3)^2(1) + (4)^2(1)$$
$$= 1 + 4 + 9 + 16$$
$$= 30$$

Underest.

Height of rect is $f(2)$, or 4

overest.

trap. rule

we've had an underest., overest. How can we get a more balanced est.?

Balance bet. extra areas, missing areas

More missing than extra, so underest.

Here, 1 trap works as an overest.

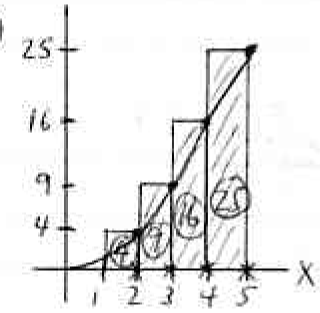
worse

Also works: $\frac{LRA+RRA}{2}$

$MRA = 42$

This is not $\approx$ trapezoid nonlinearity betrays this.

(2)

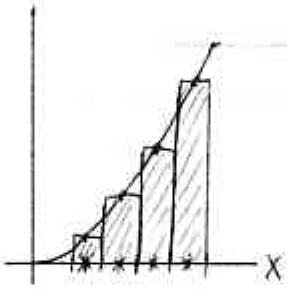


"RRA" = Right-hand

$$\int_1^5 x^2\,dx \approx f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1$$
$$= 4 + 9 + 16 + 25$$
$$= 54$$

Avg. of LRA, RRA $= \frac{30+54}{2} = 42$ (=Result from Trapezoidal Rule - 5.1)

(3)



"MRA" = Midpoint

$$\int_1^5 x^2\,dx \approx f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 + f(4.5)\cdot 1$$
$$= 41$$ (Winner for this Ex)

Exact: $41\frac{1}{3}$

named after R. (1826-66)

Riemann Sums

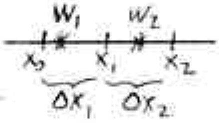
$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{\overset{\#\text{rects.}}{n}} f(w_k)\,\Delta x_k \;,\; w_k \text{ is in } k^{th} \text{ subinterval.}$$

We pick something like a test value.

Δ_0 # lines before write $f(w_k)\Delta x_k$

Remember precalc! (sum over all k, the index of summation (dummy))

$\underbrace{\underbrace{f(w_k)}_{\pm\text{height of } k\text{th rect.}}\ \underbrace{\Delta x_k}_{\text{width}}}_{\pm\text{ Area of } k\text{th rect.}}$



Where do we sample f values w/in each subint.?

regular partition Δx_k are equal

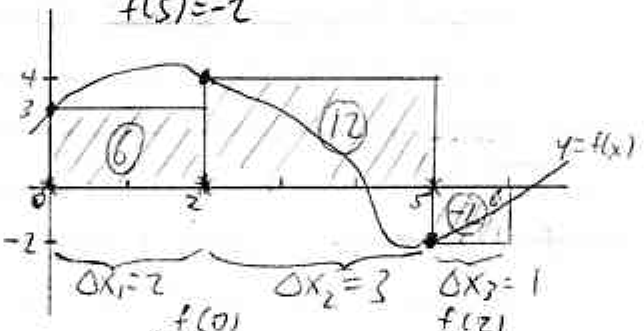
In Ex 1, $\Delta x_k = 1$ for all k ← Regular partition: all Δx_k are =.

Ex 2 $\int_0^6 f(x)dx$, P determined by $\{0,2,5,6\}$

Find LRA.

Given: $f(0)=3$
$f(2)=4$
$f(5)=-2$

x time
f(x) = how high Tweety is
Limited info, experiment
or x time
f(x): velocity

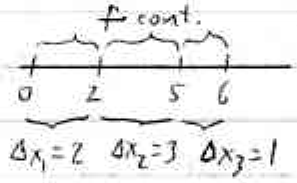


$$\begin{aligned} LRA &= \overbrace{f(w_1)}^{f(0)}\Delta x_1 + \overbrace{f(w_2)}^{f(2)}\Delta x_2 + \overbrace{f(w_3)}^{f(5)}\Delta x_3 \\ &= (3)(2) + (4)(3) + (-2)(1) \\ &= 6 + 12 - 2 \\ &= 16 \end{aligned}$$

(It's like we have limited info from an experiment ⇒ Approx. $\int$)
(Life is not always $\int x^2 dx$ ....)

© Using R. Sums to Define Definite $\int$s

Ex 2 P:



$\|P\|$ = norm of P = $\max \Delta x_i$
= length of longest subint.
= 3, here

measures how fine the partition is

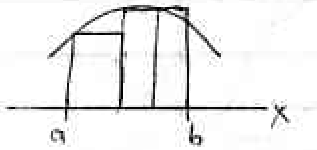
What Ch. 2 concept?

≈ useful in practice
Tweety Ex
Just 1 more step to get official def'n of $\int_a^b$ (Def. 1)

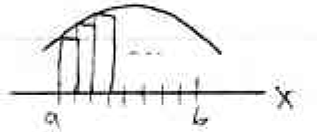
$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_k \underbrace{f(w_k)\,\Delta x_k}_{\pm \text{Area of } k^{th} \text{ rect}}$$

$\Rightarrow$ (let n = # rects. $\to \infty$)

if this limit exists
(i.e., if f is integrable on $[a, b]$)
guaranteed if f is cont. on $[a, b]$

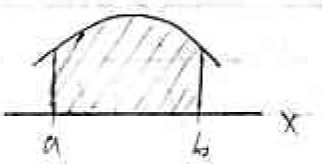


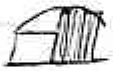
We're probably getting a much better approx. of true area



$\downarrow$ $\|P\| \to 0$
So, $n \to \infty$

It's not like we have one big rect. and the other rects ~~does~~ get thinner. We have to have this thinning all across the interval

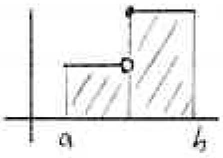


NO!  Not sufficient: $\lim_{n \to \infty}$

Note

jump discont.
removable OK
(more of a problem: —o—)
if f defined

f still integrable:

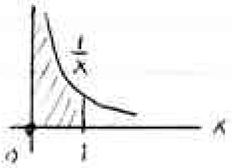


f is at least defined on $[0,1]$.

f not int. on $[0,1]$:

$$f(x) = \begin{cases} 0, & x=0 \\ \frac{1}{x}, & 0<x\le 1 \end{cases}$$

p. 273
∞ discont.
$\to f$ not int.



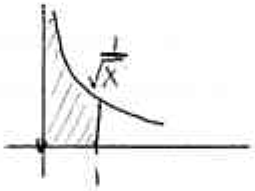
Ch. 10: $\int_0^1 f(x)\,dx = \infty$

The curve is approaching the x-axis in a way (fast enough) that traps a finite area.

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\infty$

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=2$

<u>but</u>



$\int_0^1 f(x)\,dx = 2$

(still not int.)

Calc II

(F) Σ Notes

$$\sum_{k=1}^{n} k = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

\# terms; avg. of 1st, last terms: $\frac{n(n+1)}{2}$ } Arithmetic series ($d=1$)

Ex $\sum_{k=1}^{100} k = 1+2+\ldots+100 = \frac{100(101)}{2} = 5050$

Show for $n=100$

p. 259 $\sum_{k=1}^{n} k^2$, $\sum_{k=1}^{n} k^3$

$= (1)^2+(2)^2+\ldots+(n)^2$

Gauss:

$$\begin{aligned} S &= 1+2+\ldots+100 \\ S &= 100+99+\ldots+1 \\ \hline 2S &= 101+101+\ldots+101 \end{aligned}$$

$2S = 100(101)$

$S = \frac{100(101)}{2}$

Beat him up? Went on to great things

Cool!