FUND. THM. OF ARITHMETIC: Every integer ≥ 2 is prime or can be factored/decomposed as a product of primes uniquely, up to a reordering of factors.

FUND. THM. OF ALGEBRA: Every n^{th}-degree poly. func. $(n \geq 1)$ w/complex (incl. real) coeffs. has a complex root/zero. ($\leq n$ distinct sols.)

5.6: FUNDAMENTAL THM. OF CALCULUS (FTC)

Assume f is cont. on $[a,b]$.

Ⓐ Old Ex (5.1)

I inherit \$1000.
After t years, I spend at the rate of $10t \frac{\$}{yr.}$
How much do I spend from $t=5$ to $t=6$?

$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$
Proof without Words
Great!
AMATYC Fall 2002 p. 65

\$ in a box.

Let $f(t) =$ \$ left after t years

$f'(t) = -10t, \quad f(0) = 1000$

see back ① ⇒ $f(t) = -5t^2 + 1000$

Change in \$ $= f(6) - f(5)$ ②
$= [-5(6)^2 + 1000] - [-5(5)^2 + 1000]$
$= -180 + \cancel{1000} + 125 - \cancel{1000}$
③ Cancel!

$= -55$

I spend \$55

Notation Shift

Big daddy func. and his deriv.

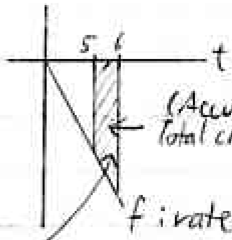
$f \rightarrow F$

$f' \rightarrow f$

Before: f — f'; Now: F — f

When you integrate the rate of change from here to here, you get ~~what represents~~ the total change... Area of trap.

Let $f(t) = -10t$ (rate)



$$\int_5^6 f(t)\,dt = \int_5^6 -10t\,dt$$

$$= \left[-5t^2\right]_5^6$$

$-5t^2 = F(t)$

Don't need +C !!

(Turns out $\int_0^x -10t\,dt = -5x^2$)

You'll get +C, −C just like +1000, −1000

$$= F(6) - F(5)$$
$$= [-5(6)^2] - [-5(5)^2]$$
$$= -55$$

Connections w/ 5.1 approach

1. Like working out indefinite $\int$
2. Like $f(6) - f(5)$, Old Notation
3. Don't need +C

(B) FTC, Part II

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$

$F(x)$: any AD of f on $[a, b]$

$$= F(b) - F(a)$$

even if $b < a$!!

Calc II: maybe diff. looking expressions

Evaluate at b, a, subtract the results

No Quotient Rule for ∫s.

Ex Evaluate $\int_1^2 \underbrace{\frac{x^5+7}{x^3}}_{\text{cont. on } [1,2]} dx$

$$= \int_1^2 \left(\frac{x^5}{x^3} + \frac{7}{x^3}\right) dx$$

$$= \int_1^2 (x^2 + 7x^{-3}) dx$$

$$= \left[\frac{x^3}{3} + 7\left(\frac{x^{-2}}{-2}\right)\right]_1^2$$

$$= \left[\frac{x^3}{3} - \frac{7}{2x^2}\right]_1^2$$

top result − bottom even if top < bottom

$$= \underbrace{\left[\frac{(2)^3}{3} - \frac{7}{2(2)^2}\right]}_{\text{eval at 2}} - \underbrace{\left[\frac{(1)^3}{3} - \frac{7}{2(1)^2}\right]}_{\text{eval at 1}}$$

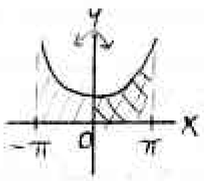
1-21 odd wait on 11

$$= \boxed{\frac{119}{24}}$$

Ex $\int_{-\pi}^{\pi} \underbrace{(x^2 + \cos x)}_{\substack{f \text{ even} \\ f(-x)=f(x)}} dx$

If I replace x w/ −x, nothing really happens. Get s-thing equiv. Graph sym about y axis. Instead of plugging in −π, we'd probably rather plug in 0.

Turns out



$$= 2\int_0^{\pi} (x^2 + \cos x)\, dx \quad \text{by symmetry}$$ (Doesn't work for: $\int_{-1}^{\pi} \dots$)

Whose deriv is cos x?

$$= 2\left[\frac{x^3}{3} + \sin x\right]_0^{\pi}$$

$$= 2\left(\left[\frac{\pi^3}{3} + \underbrace{\sin \pi}_{=0}\right] - \underbrace{\left[\frac{0^3}{3} + \sin 0\right]}_{=0}\right)$$

See if your HS geometry teacher can do this!

$$= \boxed{\frac{2\pi^3}{3}}$$

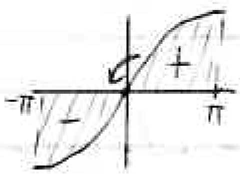
$$\text{If } f \text{ is even, cont. on } [-a,a]$$
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

Ex $\int_{-\pi}^{\pi} \underbrace{(x+\sin x)}_{f \text{ odd}}\,dx = \textcircled{0}$

$f(-x) = -f(x)$

when you replace x w/ -x, you get the opp.
graph sym about origin (not x-axis)

Turns out



1-21 odd

$$\text{If } f \text{ is odd, cont. on } [-a,a]$$
$$\int_{-a}^{a} f(x)\,dx = 0$$

Ex (u subs)

(#28) $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

Why?

Let $u = x^2+9$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,du$ (or "compensation")

(Solve for what you want to kill.)

New limits (for u)

$x = 0 \Rightarrow u = (0)^2+9 = 9$

$x = 4 \Rightarrow u = (4)^2+9 = 25$

$\left(= \frac{1}{2} \int_{x=0}^{x=4} \frac{2x}{\sqrt{x^2+9}}\,dx \right)$ if use compensation

$$= \frac{1}{2} \int_{u=9}^{u=25} \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2} \int_9^{25} u^{-1/2}\,du$$

$$= \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right]_9^{25}$$
$$= \left[\sqrt{u} \right]_9^{25}$$
$$= \sqrt{25} - \sqrt{9}$$
$$= 5 - 3$$
$$= \boxed{2} \quad \text{(wow!!)}$$

Use for (#37).

<u>Another way</u> (if you don't want to change limits)

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$$

Work out indefinite $\int$, 1st, to get an AD in x.

$$\int \frac{x}{\sqrt{x^2+9}}\,dx \qquad \begin{aligned} u &= x^2+9 \\ du &= 2x\,dx \Rightarrow x\,dx = \tfrac{1}{2}du \end{aligned}$$

SO WRONG

$$= \frac{1}{2} \int \frac{du}{\sqrt{u}}$$
$$= \frac{1}{2} \int u^{-1/2}\,du$$
$$= \frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right] + C$$
$$= \sqrt{u} + C$$
$$= \underbrace{\sqrt{x^2+9}}_{\text{an AD}} + C$$

NOW, $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

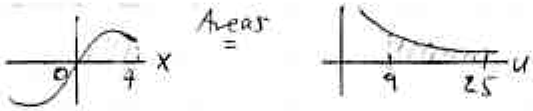
$\overset{FTC}{=} \underbrace{\left[\sqrt{x^2+9}\right]_0^4}_{\text{Our AD!!}}$

In a way, we have to work out u-limits, anyway. 1-37 odd

$= \sqrt{4^2+9} - \sqrt{0^2+9}$

$= \boxed{2}$

$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx = \int_9^{25} \frac{1}{2\sqrt{u}}\,du$



Change of variable

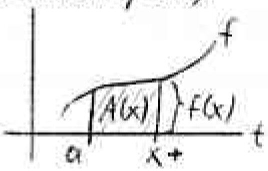
like Star Trek: I not as good...

(C) FTC, Part I

Assume f cont., "a" constant

$D_x\Big(\underbrace{\int_a^x f(t)\,dt}_{=A(x)}\Big) = f(x)$

(Funcs. can be defined by $\int$s!!)



$D_x A(x) = f(x)$

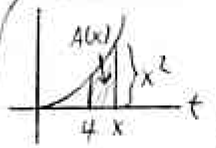
$\begin{pmatrix}\text{rate of change of } A(x) \text{ wrt } x\end{pmatrix} = \begin{pmatrix}\text{func. value at } x\end{pmatrix}$

we could've started at -10M

If x is out here, A(x) growing at a faster rate

A(x) solves $A'(x) = x^2$, $A(4) = 0$, $A(x) = \frac{1}{3}x^3 - \frac{64}{3}$

Ex $D_x\Big(\int_4^x \underbrace{t^2}_{f(t)}\,dt\Big) = \underbrace{\boxed{x^2}}_{f(x)}$



Skip? Irritating... Kill 53, 55

In general

$D_x\Big(\int_{k(x)}^{g(x)} f(t)\,dt\Big) = \underbrace{f(g(x))}_{\text{replace } t \text{ w/ } g(x) \text{ in integrand}}\underbrace{g'(x)}_{\text{tail}} - \underbrace{f(k(x))}_{k(x)}\underbrace{k'(x)}_{\text{tail}}$

(both diff'e)

$\int_{k(x)}^{g(x)} = \int_0^{g(x)} - \int_0^{k(x)}$, $u = g(x)$, $G(u)$, $D_x = \frac{dG}{du}\frac{du}{dx}$

Ex $D_x \int_{2x}^{\sin x} t^2\,dt = \underbrace{(\sin x)^2}_{\text{Replace } t \text{ w/ } (\sin x)}\underbrace{(\cos x)}_{\text{tail}} - \underbrace{(2x)^2}_{\text{Replace } t \text{ w/ } (2x)}\underbrace{(2)}_{\text{tail}}$

$= \boxed{\sin^2 x \cos x - 8x^2}$

limits