# 5.7: NUMERICAL APPROX. OF DEFINITE $\int$s

## (A) Intro

Useful for $\int_a^b f(x)\,dx$ when

Hard to find $F$
$u$ subs don't work
No nice geometry like
Just have a table of $f$ values (from experimental sampling?)

You're forced to approximate, anyway.

| $x$ | $f(x)$ |
|---|---|
| 7 | 40 |
| 8 | 50 |
| ⋮ | |

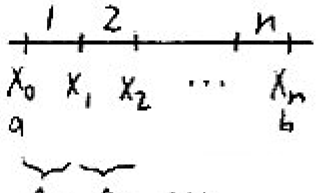
**Ex** $\int_0^4 \sqrt{x^3+1}\,dx$

No $x^2$!!

$$\left( \begin{aligned} u &= x^3-1 \\ du &= 3x^2\,dx \end{aligned} \right) \text{ NO!}$$

What did we use before?

Let $x_0, x_1, \ldots, x_n$ determine a <u>regular partition</u> of $[a,b]$.



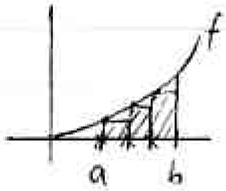

How can I express $\Delta x$ in terms of $a, b, n$?

Each $\Delta x_k = \Delta x = \frac{b-a}{n}$

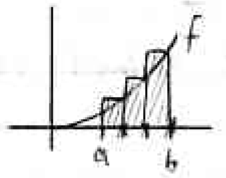
Assume $f$ is cont. on $[a,b]$.

## Ⓑ Review Riemann Sums

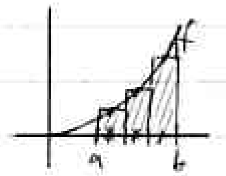
Approx. $\int_a^b f(x)\,dx$




LRA, $n=3$




RRA




MRA

Flat top

Larson

We're approx $f$ by using a piecewise constant func.

For each subinterval, we approx. $f$ by using a constant function. Rectangles!

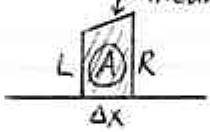
(We approx. $f$ by a piecewise constant func.)

Instead of constant funcs, what if we use linear funcs? Instead of rects, we get... You may get some rects, also. We're approx f by using a piecewise linear func. An option in Mathematica. More adv. methods sometimes uses irregular partitions: f is changing rapidly on a subint, it will get broken down.

## © Trapezoidal Rule

For now, assume $f(x) \geq 0$ on $[a,b]$.

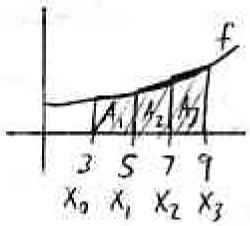
linear (or constant)




$$A = \underbrace{\Delta x}_{\text{"height"}} \underbrace{\left(\frac{L+R}{2}\right)}_{\text{avg. of bases}} \quad (\leftarrow \text{Tilt your head!})$$
$$= \tfrac{1}{2}\Delta x (L+R)$$

<u>Ex</u> Approx. $\int_3^9 f(x)\,dx$

I'm motivating the general formula




$n=3$

$$\int_3^9 f(x)\,dx \approx A_1 + A_2 + A_3$$
$$= \tfrac{1}{2}(2)[f(3)+f(5)] + \tfrac{1}{2}(2)[f(5)+f(7)] + \tfrac{1}{2}(2)[f(7)+f(9)]$$

$$= \tfrac{1}{2}\underbrace{(2)}_{=\Delta x = \frac{b-a}{n}}[f(3)+2f(5)+2f(7)+f(9)]$$

On a test, I'll give you this

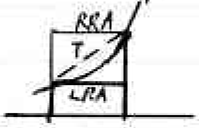
$$\int_a^b f(x)\,dx \approx \underbrace{\tfrac{1}{2}\Delta x}_{\text{or } \frac{b-a}{2n}} \underbrace{\left[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{n-1})+f(x_n)\right]}_{\text{Coeffs: } 1\ 2\ 2 \dots 2\ 1}$$

$= \text{“}T\text{”}$

The trap. approx based on this partition

Notes ① $T$ = average of LRA, RRA

Splits the difference




LRA takes $f(x_0)$ but not $f(x_n)$
RRA

All others taken by both

Explains $1\ 2\ 2 \dots 2\ 1$ (LRA takes $f(a)$, not $f(b)$. RRA $f(b)$ $f(a)$.)

② Don't need $f(x) \geq 0$.

Trap. picture → may fall apart

Thinking of LRA, RRA

③ As $n \to \infty$, $T \to$ exact

④ Mathematica may use irregular partitions:


finer where f changing rapidly

We have regular partitions. If $n \to \infty$, $\|P\| \to 0$.

Ex Approx. $\int_0^{12} \underbrace{(\sqrt{x}-1)}_{f(x)}\,dx$, $n=6$

We can find this exactly

$n=6$ subints. Don't say trapezoids.

$\Delta x = \frac{b-a}{n} = \frac{12-0}{6} = 2$

$T = \frac{1}{2}(2)\,[\ f(0)$ ← $(\sqrt{0}-1) = -1$

$+2f(2)$ ← $2(\sqrt{2}-1) \approx 0.8284$

$+2f(4)$ ← $2(\sqrt{4}-1) = 2$

$+2f(6)$ ⋮ 2.8990

$+2f(8)$ 3.6569

$+2f(10)$ 4.3246

$+ f(12)\,]$ ← $(\sqrt{12}-1) \approx 2.4641$

$\approx \frac{1}{2}(2)[15.1730]$

$T \approx 15.1730$ (Book: $\approx 15.17$)

Exact: $\approx 15.7128$

In HW, final ans. to 2 dec. places

$\frac{2}{3}(\sqrt{12})^3 - 12$

named after Thomas Simpson (1710-61)
Lesson 301
formalized Newton's method

We've approx f piecewise using constant funcs., linear funcs. What's the next step?

Draw parabolas 1st (for me) before plotting pts.

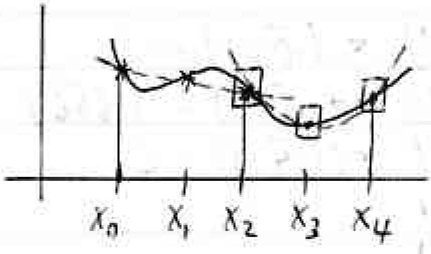
(He also formalized Newton's Method!)

## ① Simpson's Rule

often more accurate

(Riemann: piecewise constant approx.
Trap: linear
Now... quad.)

Use pieces of parabolas or lines.



1 parabola $*$ 1 parabola $\square$

Let $n$ be even.

$$\int_a^b f(x)\,dx \approx \underbrace{\tfrac{1}{3}\Delta x}_{\text{or } \frac{b-a}{3n}} \left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\dots + 2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

Coeffs: 1 4 2 4 2 ... 4 2 4 1

$=$ "$S$"

As $n\to\infty$, $S\to$ exact

$\underline{Ex}$ Approx $\int_0^{12} \underbrace{(\sqrt{x}-1)}_{f(x)} dx$, $n=6$

$\Delta x = \frac{b-a}{n} = 2$

$S = \frac{1}{3}(2) \cdot [\ f(0) \leftarrow (\sqrt{0}-1) = -1$

$+4f(2) \leftarrow 4(\sqrt{2}-1) \approx 1.6569$

$+2f(4) \leftarrow 2(\sqrt{4}-1) = 2$

5.7980 $+4f(6)$ $\vdots$

3.6569 $+2f(8)$

8.6491 $+4f(10)$

2.4641 $+\ f(12)]$

$\approx \frac{2}{3}[23.2250]$

$\approx 15.4833$

$T \approx 15.17$

$S \approx 15.48$

Exact $\approx 15.71$