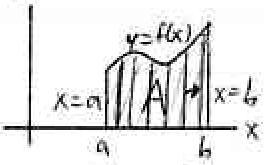
CH. 6: APPLICS. OF $\int$

6.1: AREA

(A) Integrating wrt x

$\int$, with respect to

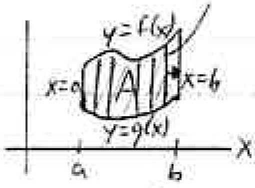


$A = \int_a^b f(x)\,dx$

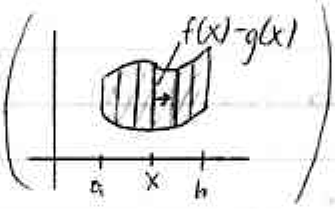
Scan in x-direction

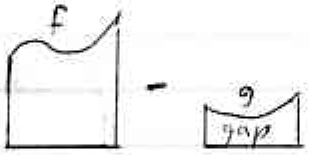
works if f cont.,
$f(x) \geq 0$ throughout (on) $[a,b]$

Don't be confused by vertical line segs. - they're scan lines.



Scan



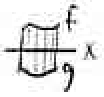


Imagine a Xerox machine

$$A = \int_a^b [f(x) - g(x)]\,dx$$
"top - bottom"

Comparing f to g, not f to 0

Don't need $f, g \geq 0$.



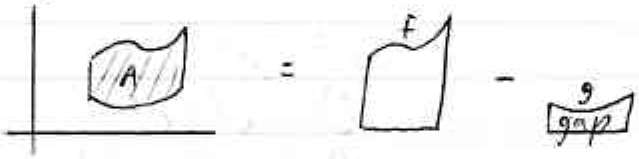
OK

Works if f, g cont.,
$f(x) \geq g(x)$ throughout (on) $[a,b]$

We can easily interpret in terms of areas.

Helps w/volume

(
Idea If $f, g \geq 0$



$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
or $\int_a^b [f(x) - g(x)]\,dx$
)

Like #10

Ex Find the area [of the region] bounded by [the graphs of] $y-x^2=0$ and $y-x^3=0$.
(May change order:)

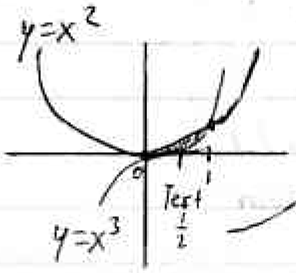
① Solve both eqs. for y

$$y = x^2$$
$$y = \underbrace{x^3}_{\text{funcs. of } x}$$

② Sketch the graphs (maybe do 1st)

If $x<6$, what's greater? Twilight Zone

I want left, right of region to be either vert. edge or pt.



$x^2 \geq x^3$ on $(0,1]$

(Twilight Zone: the unexpected happens)

y-coords. help w/a graph

③ Find intersection points (esp. x-coords.)

Here, they're pretty obvious.

When solving a system, you're finding pairs (x,y) that satisfy both eqs. Means point (x,y) lies on both graphs.

Solve the system $\begin{cases} y = x^2 \\ y = x^3 \end{cases}$

$$x^2 = x^3$$
$$0 = x^3 - x^2$$
$$0 = x^2(x-1)$$
$$x=0, \quad x=1$$
$$y=x^2 \Downarrow \qquad \Downarrow$$
$$(y=0) \quad (y=1)$$

Pt. $(0,0)$ Pt. $(1,1)$ ← I'd like it I tell you to graph the region.

④ Set up $\int$ (s)

$$\int_0^1 (x^2 - x^3)\,dx$$
(top - bottom)

⑤ Evaluate

Can do 1,5,9

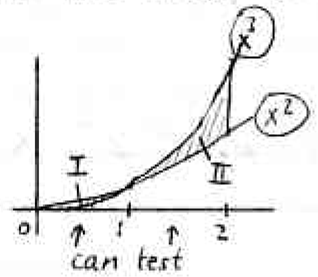
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$
$$= \left[\frac{1}{3} - \frac{1}{4}\right] - [0]$$
$$= \frac{1}{12} \text{ sq. units}$$

(Don't be surprised it's a nice fraction: nice poly. func., Power-Rule for $\int$, integer limits)

Doesn't work: $\int_0^2$ flip sign if -

Ex Find the area bounded by $y = x^2$, $y = x^3$, if x is restricted to $[0, 2]$.

(like pt., pt., vertical edge at 0,1,2.)

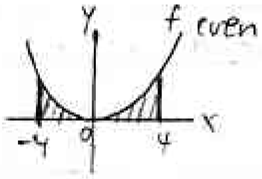


$$\int_0^1 (\underset{top}{x^2} - \underset{bot}{x^3})\,dx + \int_1^2 (\underset{top}{x^3} - \underset{bot}{x^2})\,dx$$

$\frac{1}{12}$ from before

$$= \frac{1}{12} + \frac{17}{12}$$
$$= \frac{3}{2} \text{ sq. units}$$

Ex

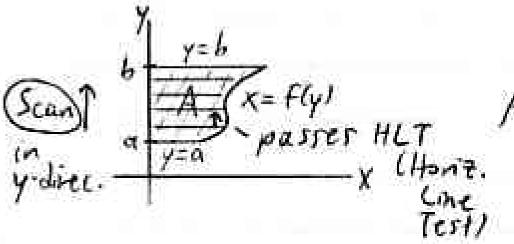


$$A = \int_{-4}^4 f(x)\,dx$$
$$\overset{sym.}{=} 2\int_0^4 f(x)\,dx \quad \text{(Easier?)}$$

Can do 1,5,7,9

Ⓑ Integrating wrt y

Does this curve rep. a func.? Doesn't rep. y as func. of x



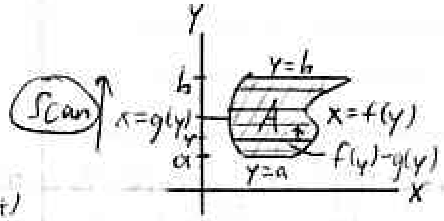
$A = \int_a^b f(y)\,dy$

Xerox ↑ scans

Before: ∫(top−bot)

Easy to mess up rt.−left on left

higher−lower



$A = \int_a^b [f(y) - g(y)]\,dy$

"right − left"

Works if f, g cont., $f(y) \geq g(y)$ throughout (on) $[a, b]$

x is rep. as a func. of y. This eq. does not rep. y as a func. of x

Ex (#14) Find the area bounded by $x = y^2$, $x - y = 2$.

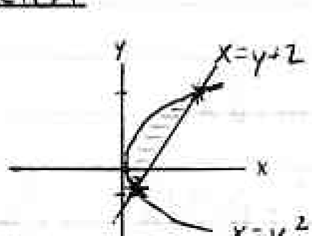
① Solve both eqs. for x

$x = y^2$

$x = \underbrace{y + 2}_{\text{funcs. of } y}$

② Sketch

So it may help to solve for y, anyway.



Graph $x = y + 2$

$y = x - 2$ (slope-int. form)

or find intercepts

Why ∫ wrt y? ① We can easily express x as funcs. of y.

② Easier to scan region ↑ (vs. →).

③ Intersection Points

Solve $\begin{cases} x = y^2 \\ x = y + 2 \end{cases}$

QF if desperate

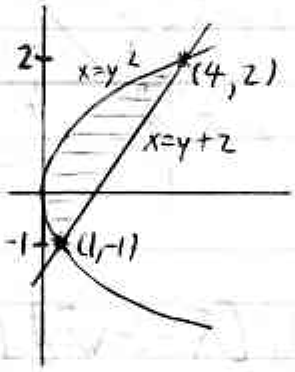
I want top, bot of region to be either horiz. edge or a pt.

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0$$
$$y = 2, \quad y = -1$$

$x = y^2 \Downarrow \qquad \Downarrow$

$$x = 4 \qquad x = 1$$

$$(4,2) \qquad (1,-1)$$



④, ⑤

Don't need, but need if had ≥2 terms

$$\int_{-1}^{2} [(y+2) - (y^2)]\, dy$$
right - left

You can do it!!

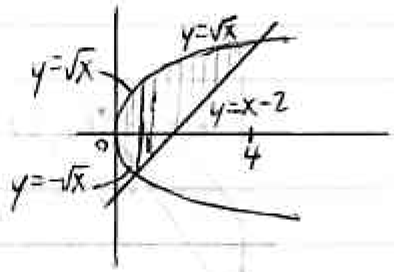
$$= \int_{-1}^{2} (y + 2 - y^2)\, dy$$
$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$
$$= \underbrace{\left[\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right]}_{\text{eval at } 2} - \underbrace{\left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right]}_{\text{eval at } -1}$$
$$= \frac{9}{2} \text{ sq. units}$$

Harder: $\int$ wrt x

if have heart set on $\int dx$, you're setting yourself up for heartbreak

$$x = y^2 \Rightarrow y = \pm\sqrt{x}$$
$$x - y = 2 \Rightarrow y = x - 2$$

change points at 1



What can you use? Sym:
$2\int_0^1 \sqrt{x}\,dx$

need

$$\underbrace{\int_0^1 [\underset{top}{\sqrt{x}} - (\underset{bot}{-\sqrt{x}})]}_{2\sqrt{x}}\,dx + \int_1^4 [\underset{top}{\sqrt{x}} - \underset{bot}{(x-2)}]\,dx$$

Correct, but...
HAVE FUN!! ☺

θ) sym

6.2: VOLUMES OF SOLIDS OF REVOLUTION

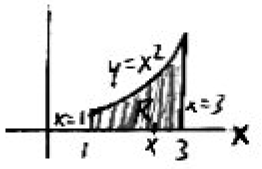
Bring Toys!!

Ⓐ Disk Method

Ex Sketch the region R bounded by [the graphs of] $y=x^2$, x-axis, $x=1$, and $x=3$.
Find the volume [of the solid generated] if R is revolved about the x-axis. $-x$

① Sketch R

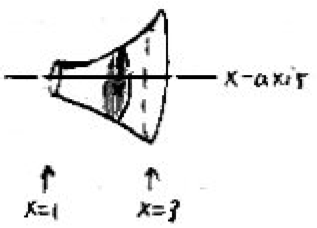
If we take a vertical line seg, rotate it

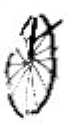


② Sketch the solid (optional)
③ Find the area of a cross-section

sideways volcano, bullhorn

If we slice the volcano at x, what do we get?





The cross-sec. at x $(1 \le x \le 3)$ is a <u>disk</u> of radius $f(x)$, or x^2.

Mine are infinitesimally thin; they have some thickness in the Riemann development (see next page)

$r(x)$
We'll need r later

Its area is $\pi(\text{radius})^2$
$= \pi[f(x)]^2$ ← More precisely, $\pi[|f(x)|]^2$
$= \pi[x^2]^2$
$= \pi x^4$

④ $\int$ cross-sec areas

(④ <u>Integrate the cross-sectional areas</u>)

(Volume of solid) $V = \int_1^3 \pi x^4 \, dx$

$= \pi \left[\frac{x^5}{5} \right]_1^3$

$\vdots$

$= \pi \left[\frac{(3)^5}{5} - \frac{(1)^5}{5} \right]$

$= \pi \left[\frac{242}{5} \right]$

$= \frac{242\pi}{5}$ (cubic units)

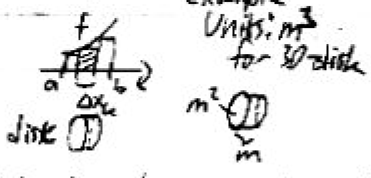
In Ch. 5, how did we approx. area

My digits are 2-11, like me.

<u>Theory (p. 314)</u>

Example

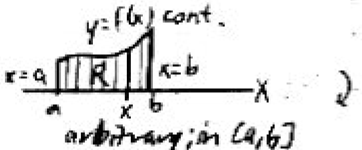
Units: m^3 for 3D disk



$V = \lim_{\|P\| \to 0}$ (sum of disk vols.)

$= \int_a^b \pi [f(x)]^2 dx$

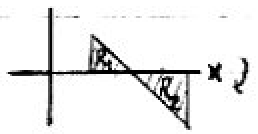
<u>In general</u>

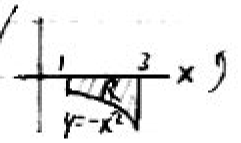


(If f is cont. on $[a,b]$,
R is bounded by $y=f(x)$, x-axis,
$x=a$, $x=b$,
then the volume of the solid gen. by revolving R about the x-axis is)

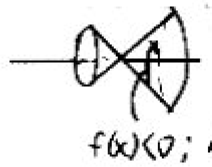
$$\boxed{V = \int_a^b \pi [f(x)]^2 dx}$$

<u>Note</u> Still works if $f(x) < 0$ sometimes.





(same as before)



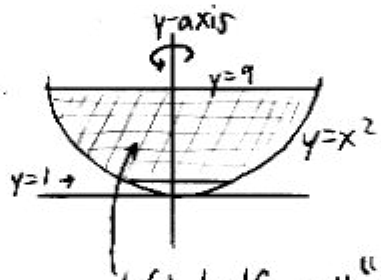
$f(x) < 0$; radius of disk at x is $|f(x)|$

Cross-sec. area $= \pi [\underbrace{f(x)}_{\text{OK if } <0}]^2$ because

Can do 1, 5, 7, 11

Ex R is bounded by $y=x^2$, $y=1$, and $y=9$.
(R is revolved about the y-axis.)
Find V.

① Sketch R



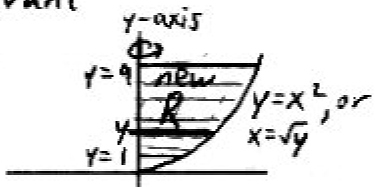
(Normally, we worry when the axis of revol. cuts through the region, but we can handle this.)

left half will "overlap" (what the right half "does")

Imagine revolving doors at a hotel. We're going to end up w/ twice what we want. It's implied that we're going around 1 full revol. 360° Double-counting. We must cut out ½ the figure. $\sqrt{y}-(-\sqrt{y})$ is the diameter.

We would "compensate" "$\cdot \frac{1}{2}$" if we had [I|II], rotate 360°. This is how we'd rotate. You may as well just rotate one of the regions.

Want



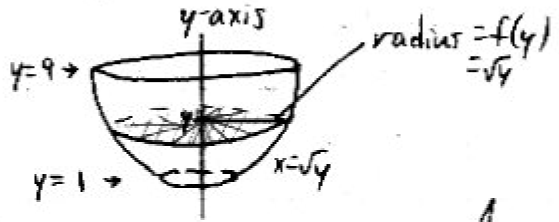
Solve $y=x^2$ for x

$x^2=y$

$x=\pm\sqrt{y}$

Scan ↑, so dy

②, ③

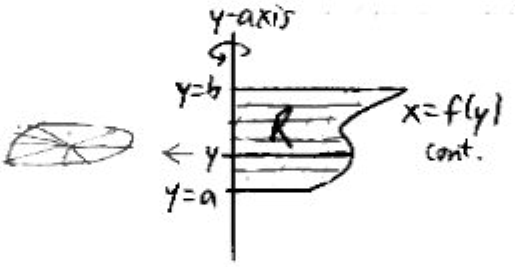


$$\underset{\text{at } y}{\text{Area}} = \pi(\sqrt{y})^2 = \pi y$$

④

$$\int_1^9 \pi y\,dy = \pi\left[\frac{y^2}{2}\right]_1^9 = \pi\left(\left[\frac{(9)^2}{2}\right]-\left[\frac{(1)^2}{2}\right]\right) = 40\pi \text{ cubic units}$$

In general



$$V=\int_a^b \pi\,[f(y)]^2\,dy$$

↑Some books use $\int_c^d$

Can do HI except 3

Donut

(B) Washer Method

You can factor out a π, but, You can't take out "2"

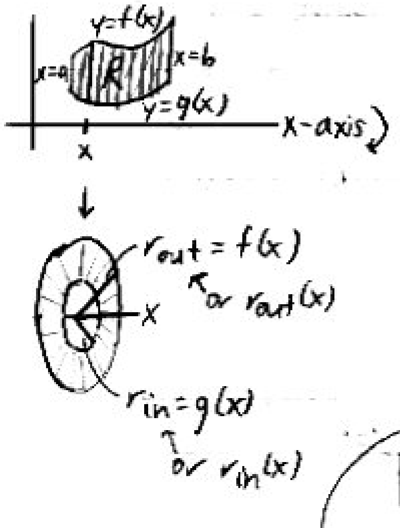
If we take this line seg, rotate it about x-axis

nature of overlap unclear

h-o-l-e not w-h-o-l-e

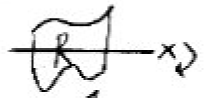
Remember →

many misplace ²



Note

In trouble:



weird self-overlapping

OK:

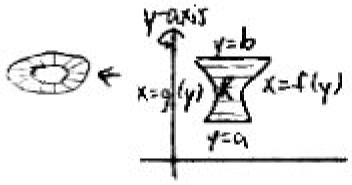
(←this takes care of whatever this does)

The cross-sec at x is a washer of area

= "big area" − "area of hole"

$= \pi r_{out}^2 - \pi r_{in}^2$

$$V=\int_a^b \left(\pi r_{out}^2-\pi r_{in}^2\right)dx = \pi\int_a^b \left([f(x)]^2-[g(x)]^2\right)dx$$



$$V=\int_a^b \left(\pi r_{out}^2-\pi r_{in}^2\right)dy = \pi\int_a^b \left([f(y)]^2-[g(y)]^2\right)dy$$

$\underline{Ex}$ (#14) R is bounded by $x=y^3$, $x^2+y=0$. $\longrightarrow x$
(R is revolved about the x-axis.)
Find V.

Scan "dx way" or "dy way"
x is changing

① Sketch R

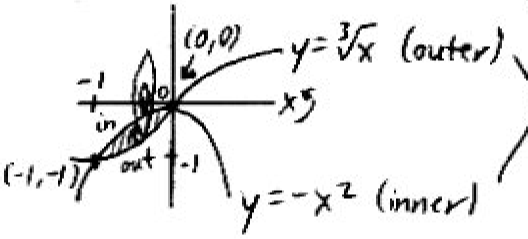
Scan →

dx

So, solve for y in terms of x.

$x=y^3 \Rightarrow y=\sqrt[3]{x}$
$x^2+y=0 \Rightarrow y=\underbrace{-x^2}_{\text{funcs. of } x}$

It's not a matter of top vs. bottom.



relative to x-axis (axis of revol.)

Intersection Pts. (x)
$\sqrt[3]{x}=-x^2$
$(\sqrt[3]{x})^3=(-x^2)^3$
$x=-x^6$
$x^6+x=0$
$x(x^5+1)=0$
$x=0 \quad x=-1$
$y=0 \quad y=-1$

How do you solve a poly. eq. What do you want on one side? 0.
No DeMoivre's Thm here.
What are the 5 5th roots of -1?
Only 1 real root.

Skip ②: Sketch solid

③, ④ $V=\int_{-1}^{0}(\pi r_{out}^2-\pi r_{in}^2)\,dx$

$=\pi\int_{-1}^{0}([\sqrt[3]{x}]^2-[x^2]^2)\,dx$
$=\pi\int_{-1}^{0}(x^{2/3}-x^4)\,dx$
$\vdots$
$=\frac{2\pi}{5}$

Up to 23

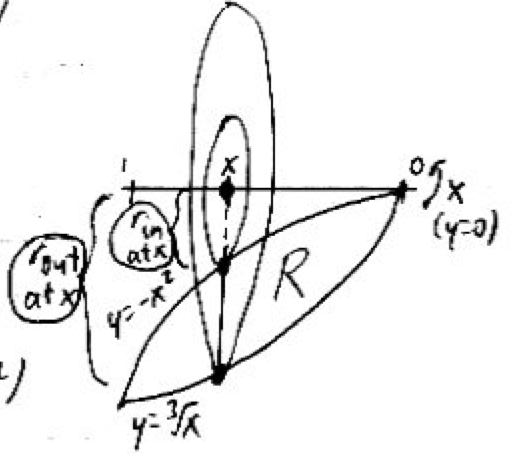
Radii, heights, lengths:

top − bottom, or right − left

They're never negative!

r_{out} (at x) $=y_{top}-y_{bot}$
$=0-\sqrt[3]{x}$
$=-\sqrt[3]{x}$

r_{in} (at x) $=y_{top}-y_{bot}$
$=0-(-x^2)$
$=x^2$

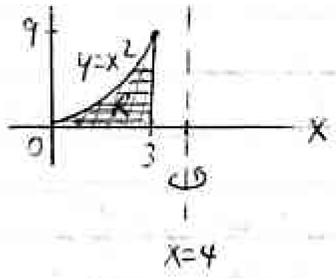


Ex (Weird axis)

R is bounded by $y=x^2$, $x=0$, and $x=3$.
R is revolved about $x=4$.
Find V.

x=4: hor or vert?

① Sketch R



②

Volcano/bullhorn

③, ④

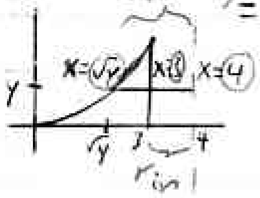
If you're not sure which way to do it, draw a cross-sec. x=4 ∥ y.

Is the outer radius $\sqrt{y}$? No.

For linear measures like height, radius right-left or top-bot

$\uparrow$ scan dy

$y=x^2$
$\Rightarrow x=\pm\sqrt{y}$



$r_{out} = x_{right} - x_{left}$
$= 4-\sqrt{y}$

r_{in}
$= 4-3$
$= 1$

$V = \int_0^9 (\pi r_{out}^2 - \pi r_{in}^2)\, dy$

$V = \pi \int_0^9 \left([4-\sqrt{y}]^2 - [1]^2\right) dy$

$= \frac{63\pi}{2}$

Could translate

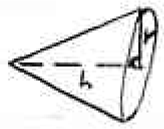
→ equiv. ∫

If you forget, can get them thru calc.

Ⓒ Famous Formulas

Ex (#37) HW Find the volume of a right circular cone of altitude h and base radius r.

Could △ of y



What would I need to rotate → x

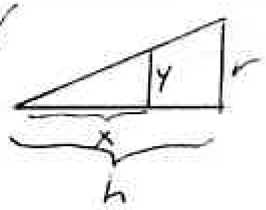
What's the slope?

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{r}{h}$$

$y = mx + b$
$y = \frac{r}{h}x + \cancel{0}$

commonly used in calculus

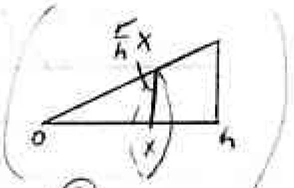
or similar triangles



$$\frac{y}{x} = \frac{r}{h}$$

$$y = \frac{r}{h}x$$

$$V = \int_0^h \pi \underbrace{\left(\frac{r}{h}x\right)}_{\text{radius of cross-sec at } x}^2 dx$$



Write all this out.

(You) r, h constants

⋮

$$= \frac{1}{3}\underbrace{\pi r^2}_{\text{base area}} h \qquad \left(\frac{1}{3}V_{cyl}\right)$$

$\underline{Ex}$ (#38) Sphere of radius r, Find V.
HW

what do we rotate $\longrightarrow x)$

(later)
$f(x) = \sqrt{r^2 - x^2}$

scan $\rightarrow$ dx

$\textcircled{D} \times 2$
(symmetry)

Circle of radius r
w/center $(0,0)$

$x^2 + y^2 = r^2$

Solve for y

$y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

$\textcircled{W}$ also works
$\textcircled{S} \times 2$

(You)

$V = \frac{4}{3}\pi r^3$

$\underline{Note}$ $D_r(V) = 4\pi r^2$ (surface area formula for a sphere!)

Like: $D_r(\underbrace{\pi r^2}_{\text{area of circle}}) = \underbrace{2\pi r}_{\text{Circumference of circle}}$

Not a coincidence!

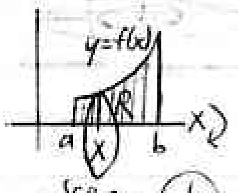
(D) →

(E) Summary

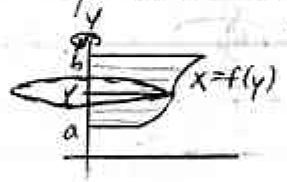
Axis of revolution

x-axis $\longrightarrow x$ | y-axis $\uparrow y$

Disk



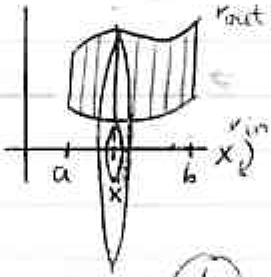
Scan → dx



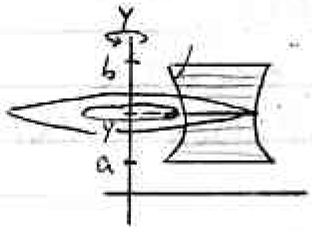
Scan ↑ dy

$$V=\int_a^b \pi [f(x)]^2 dx \qquad V=\int_a^b \pi [f(y)]^2 dy$$

Washer



→ dx



↑ dy

not a matter at top, bot.

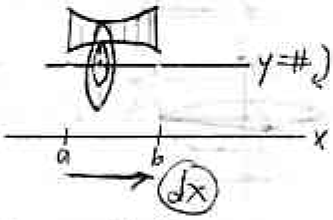
$$V=\int_a^b \Big(\overbrace{\pi r_{out}^2}^{\text{big area}} - \overbrace{\pi r_{in}^2}^{\text{area of hole}}\Big)\, dx \text{ or } dy$$

funcs. of x or y

CONT →

Weird Axis

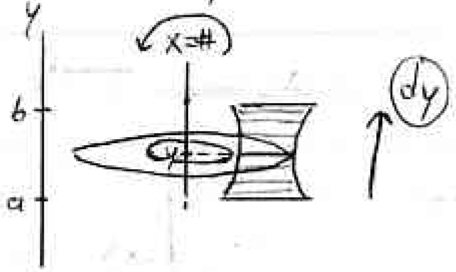
∥ to x-axis



$$V = \int_a^b \left(\pi (r_{out})^2 - \pi (r_{in})^2\right) dx$$

funcs. of x, "top-bot"

∥ to y-axis



$$V = \int_a^b \left(\pi (r_{out})^2 - \pi (r_{in})^2\right) dy$$

funcs. of y "right-left"

If R bounded by axis of revol. ⇒ Disk

(D) HW Tips

x as func of y - still freaky?

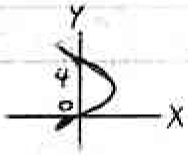
(#11) Graph $x = \underbrace{4y - y^2}_{}$ quadratic f(y)

leading coeff. = −1 < 0

⊃

find y-ints:

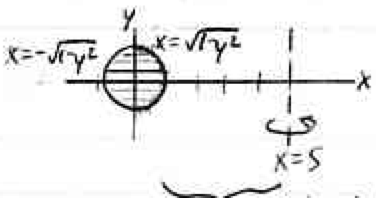
$$0 = 4y - y^2$$
$$0 = y(4-y)$$
$$y = 0, y = 4$$



(#33) R is bounded by $x^2+y^2=1$
R is revolved about $x=5$.

Think Homer

initially: surrounded by pawns



r_{in} = right-left $= 5 - \sqrt{1-y^2}$

r_{out} = right-left $= 5 - (-\sqrt{1-y^2})$

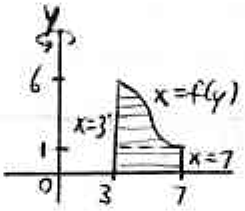
$$x^2+y^2=1$$
$$x^2 = 1-y^2$$
$$x = \pm\sqrt{1-y^2}$$

Get a donut/torus.
(Toroidal (wraparound) Chess:
Queen? Diff. initial setup
[N] Knight weird!!)

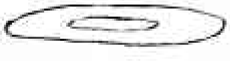
← (E)

6.3: MORE VOLUMES OF SOLIDS OF REVOL.: CYLINDRICAL SHELL METHOD

(A) Review Washer Method



$$V = \int_0^1 \left(\pi \underbrace{r_{out}}_{(7)}^2 - \pi \underbrace{r_{in}}_{(3)}^2\right) dy$$

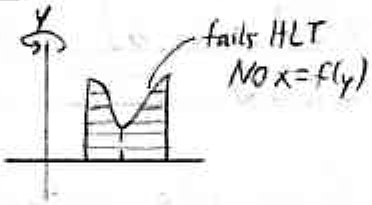


$$+ \int_1^6 \left(\pi \underbrace{r_{out}}_{[f(y)]}^2 - \pi \underbrace{r_{in}}_{[3]}^2\right) dy$$

(We're integrating areas of washers. )

Note

What do you get? Target



Washer method fails unless we split.

(B) Intro Cyl. Shells

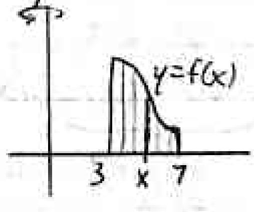
What if we take vertical line segments (scan lines) (∥ to axis of revol.)

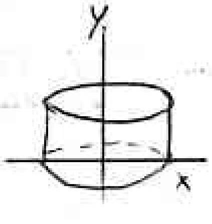
How would the cyls differ

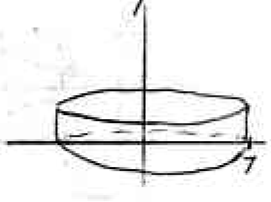
ripples of cylinder

manipulative

label of can







We'll integrate the [lateral] surface areas of cylinders. (A)

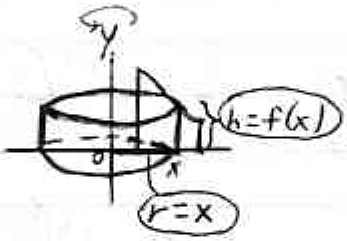
"Rippling out"

Can

☆

$\rightarrow$ $\boxed{A = 2\pi r h}$ h

$2\pi r$



$\boxed{A = 2\pi x f(x)}$

$$\boxed{V = \int_a^b 2\pi x f(x)\,dx}$$

no square
height ≥ 0
vs. Disk Method

if f cont. $\Rightarrow$ ensures integrability

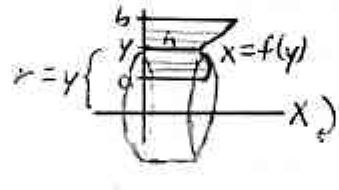
$f(x) \geq 0$ on $[a, b]$ $\Rightarrow$ height ≥ 0

$0 \leq a < b$

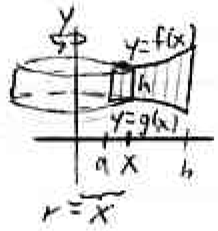
$\Rightarrow$ radius ≥ 0

C. Variations

(Key): Draw the picture! Memorization shaky...

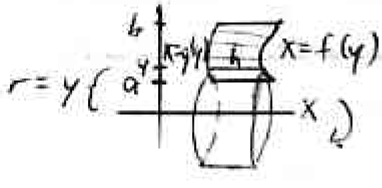


$$V = \int_a^b 2\pi y\, f(y)\,dy$$



$f - g \geq 0$

$$V = \int_a^b 2\pi \underset{\text{radius}}{x} \underbrace{\overset{\text{top - bot}}{[f(x) - g(x)]}}_{\text{height } (h, \text{ or } h(x)) \text{ at } x} dx$$



$$V = \int_a^b 2\pi \underset{\text{radius}}{y} \underbrace{\overset{\text{right - left}}{[f(y) - g(y)]}}_{\text{"height" } (h, \text{ or } h(y)) \text{ at } y} dy$$

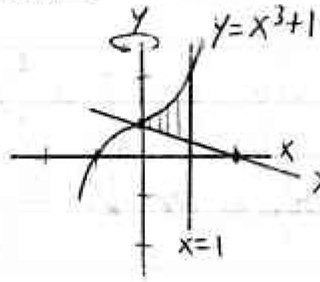
Don't need $f, g \geq 0$. Need $f \geq g$.

For the linear measures radius and height, it's "top-bottom or right-left."
(later: Weird axis) (here)

① Exs

Ex (#10) R is bounded by $y=x^3+1$, $x+2y=2$, $x=1$.
(R is revolved about y-axis.)
Find V.

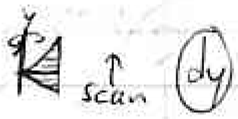
Sketch R



Intercept Method

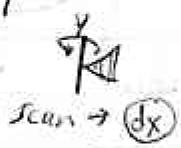
vertical, ∥ y-axis $\underbrace{}_{x=0}$

Washer Method

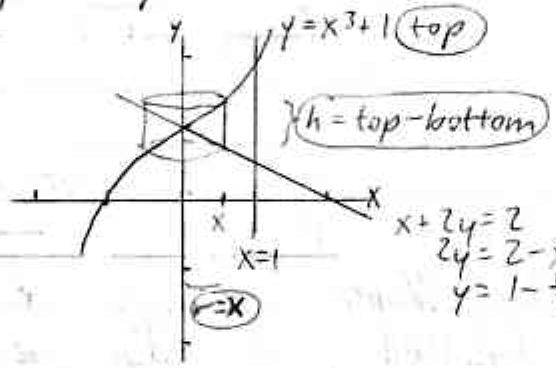
Split 

we scan R vertically.

Cyl Shells



Solve the eqs. for y in terms of x.



Scanner moves left-to-rt over R

$x+2y=2$
$2y=2-x$
$y=1-\frac{1}{2}x$ bottom

$$V=\int_0^1 \underbrace{2\pi x}\big[\overbrace{(x^3+1)}^{\text{top}}-\overbrace{(1-\tfrac{1}{2}x)}^{\text{bottom}}\big]\,dx$$

x-values

need

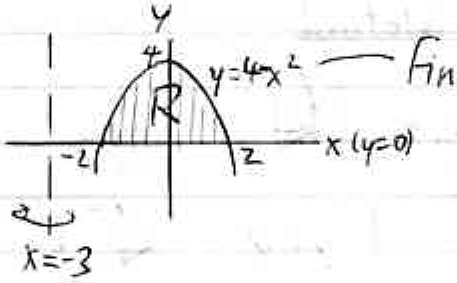
$$=\frac{11\pi}{15}$$

Ex (Weird axis)

(#20b) R is bounded by $y=4-x^2$, $y=0$.
R is revolved about $x=-3$.
Set up the integral for V. (So we get a ready-made Ch 5 problem, in principle.)

I'd rather scan R horizontally.

Sketch R



— find x-ints:

$0=4-x^2$

$x^2=4$

$x=\pm 2$

Scan → (dx)

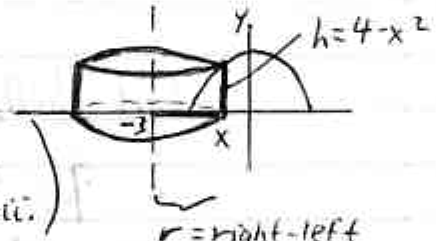
cyls of same height, diff radii

Disk

mean!

scanner moves left-to-rt over R

If $x=-3$, $x>0$ ... $3+x$



$r=$ right − left
$=x-(-3)$
$=x+3$

(Note: Symmetry about y-axis not helpful: cyls have diff. radii.)

$$\boxed{V=\int_{-2}^{2} 2\pi\,\underbrace{(x+3)}_{\text{radius}}\,\underbrace{(4-x^2)}_{\text{height}}\,dx}$$

x-values

Axis of revol.

	∥ to x-axis	∥ to y axis
Disk/Washer	dx	dy
Cyl. Shell	dy	dx

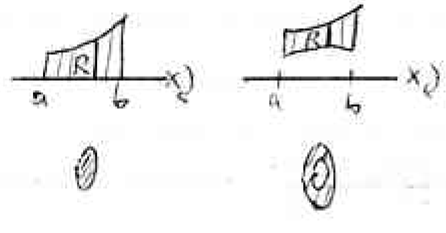
(dx): scan ||| (dy): scan ≡↑

(How does the scanner move over R? → ∫dx, ↑ ∫dy)

Challenge: Derive formula for the vol of a sphere using cyl. shells

6.4: VOLUMES BY CROSS-SECTIONS

6.2: Disks, Washers for Solids of Revol. 6.4: Other solids, too.



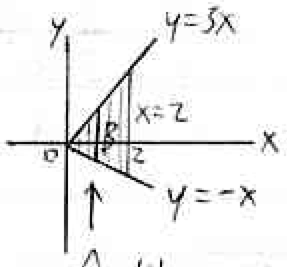
$V = \int_a^b \underbrace{A(x)}_{\text{area of cross-sec at } x} dx$ or $V = \int_a^b A(y)\,dy$

Ex B is bounded by (the graphs of) $y = 3x$, $y = -x$, and $x = 2$. Find the volume of the solid that has B as its base if every cross-sec. by a plane $\perp$ to the x-axis is an equilateral triangle.

Start x →

(Toys! Manips: Tri, Square Cross-secs)

Sketch B



Δ slide stands here in this slot. Slots ⊥ x-axis.

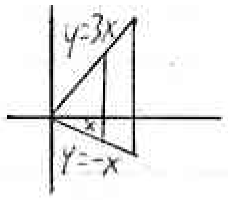
Top vertices of Δs do form nice straight edge.
edge lies above $y = \frac{3x-(-x)}{2}$
$y = x$

Solid

(rotated)

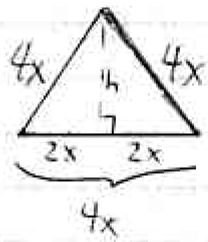
$\int$ can $\rightarrow$, so dx

<u>Find $A(x)$</u>



top-bot $= 3x-(-x)$
$= 4x$

$A(x)$ = area of an equil. triangle of side length $4x$.



find h:

$h = 2x\sqrt{3}$, or $2\sqrt{3}x$

Long Way:

$$\begin{aligned}
h^2 + (2x)^2 &= (4x)^2 \\
h^2 + 4x^2 &= 16x^2 \\
h^2 &= 12x^2 \\
h &= \pm\sqrt{12x^2} \\
h &= 2\sqrt{3}|x| \\
h &= 2\sqrt{3}x \qquad (x>0)
\end{aligned}$$

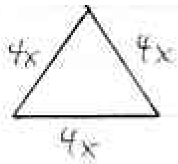
$$\begin{aligned}
A(x) &= \tfrac{1}{2}(\text{base})(\text{height}) \\
&= \tfrac{1}{2}(4x)(2\sqrt{3}x) \\
&= 4\sqrt{3}x^2
\end{aligned}$$

I know 3 sides of a triangle.

Method 2 (Heron's formula)

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where s = semi-perimeter $= \frac{a+b+c}{2}$



$s = \frac{12x}{2} = 6x$

$A = \sqrt{6x(6x-4x)(6x-4x)(6x-4x)}$

$= \sqrt{6x(2x)(2x)(2x)}$

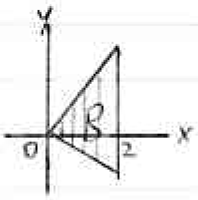
$= \sqrt{48x^4}$

$= 4\sqrt{3}\,x^2$

Don't need !!

Method 3 $A = \frac{1}{2}(4x)(4x)\sin(60°)$

Find V



$V = \int_0^2 A(x)\,dx$

$= \int_0^2 4\sqrt{3}\,x^2\,dx$

$\left(= 4\sqrt{3}\left[\frac{x^3}{3}\right]_0^2\right.$

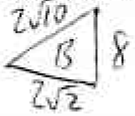
$\left.= 4\sqrt{3}\left[\frac{8}{3} - 0\right]\right)$ (You)

$= \frac{32\sqrt{3}}{3}$

Easy: 

Base area = 2

Harder: Heron



Height: $4\sqrt{3}$

Vol. of a pyramid $= \frac{1}{3}$ (Base area)(Height), like cone

(figure out! Think a lot.)

Proof uses similar triangles

 either could rep the base area.

---

or

Base $= \frac{1}{2}(8)(4\sqrt{3}) = \frac{32\sqrt{3}}{2}$

Height $= 2$

$\frac{1}{3}\left(\frac{32\sqrt{3}}{2}\right)(2)$

 $A(y) = \frac{1}{2}bh$, regardless of tri shape

Pyramids w/ polygonal bases can be decomposed into pyrs w/ tri bases

Write!

Works for curvy bases, too. Calculus idea

 general cone

Cavalieri

One pyr w/ tri base → whole family

6.5: ARC LENGTH and SURFACES OF REVOL.

(A) Arc Length

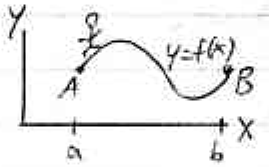
weird funcs where f' exist but not cont.

Edwards Gelbaum 36

$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Ch. 13 generalize to more general graphs: Cart., Polar

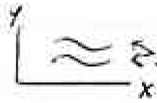


Assume f is <u>smooth</u> (f' is cont.) on $[a,b]$.

The <u>arc length</u> of the graph of f from A to B is

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

to unbelievy arc length

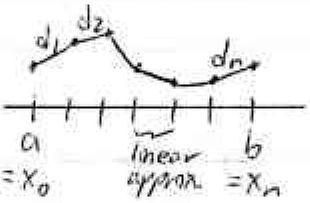


$\approx$ same L $\quad$ f' is the key.

Idea of Proof (pp. 333-4)

Sample (n+1) points based on a partition P of $[a,b]$.

These are pieces of secant lines going through the graph.



piecewise-linear approx. to f

Total length $= "L_P" = \sum_{k=1}^{n} d_k$

$L = \lim_{\|P\| \to 0} L_P = \int_a^b \underbrace{\sqrt{1 + [f'(x)]^2}\,dx}_{\text{See (B)}}$

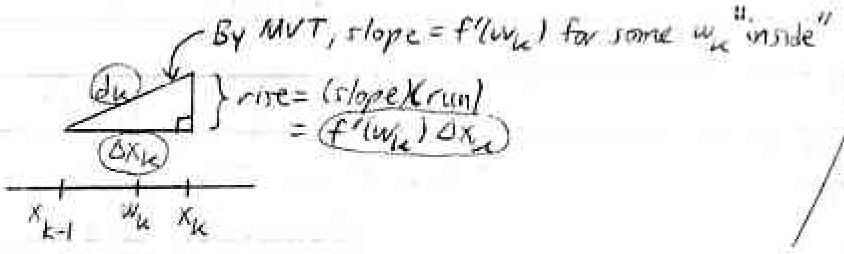
limit of sums

Note on Rigorous Proof

Approx. L by L_P, the sum of lengths of secant line segments.

Find d_k

Idea from differentials:



Note (cont.)

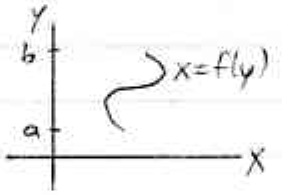
By Pyth. Thm.,

$$d_k = \sqrt{(\Delta x_k)^2 + [f'(w_k)\Delta x_k]^2}$$
$$= \sqrt{(1 + [f'(w_k)]^2)(\Delta x_k)^2}$$
$$= \sqrt{1 + [\underset{\uparrow \text{cont.}}{f'}(w_k)]^2}\ \Delta x_k$$

$$L_P = \sum_{k=1}^{n} d_k$$

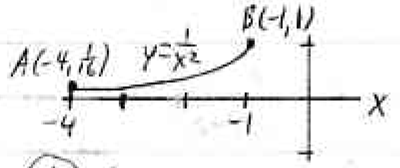
$$L = \lim_{\|P\| \to 0} L_P = \underset{\uparrow \text{limit of sums}}{\int_a^b} \sqrt{1 + [f'(x)]^2}\, dx$$

Also



$$\boxed{L = \int_a^b \sqrt{1 + [f'(y)]^2}\, dy}$$

Ex (#4)



Set up an integral for L

dx ∫ wrt x

If fails, VLT, can't

$\int_1^4$ same # by sym/conv

$$L = \int_{-4}^{-1} \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$
$$f'(x) = -2x^{-3}$$

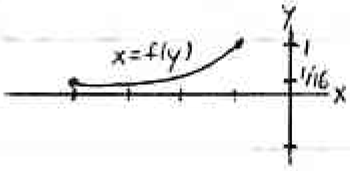
Trap Rule gives different result from straight piecewise linear approx

has nothing to do w/ area under original curve

$$= \int_{-4}^{-1} \sqrt{1 + [-2x^{-3}]^2}\, dx$$
$$= \int_{-4}^{-1} \sqrt{1 + 4x^{-6}}\, dx$$

Numerical approx. (Trapezoidal, Simpson)

(dy)
or $\int$ wrt y



$y = \frac{1}{x^2}$

$x^2 = \frac{1}{y}$

$x = \pm\sqrt{\frac{1}{y}}$

$= \underbrace{-y^{-\frac{1}{2}}}_{f(y)}$

$f'(y) = \frac{1}{2}y^{-\frac{3}{2}}$

Sign doesn't matter: pops out of deriv.; we square deriv. Also, sym $\Rightarrow$ same L

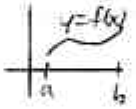
$$L = \int_{1/16}^{1} \sqrt{1 + [f'(y)]^2}\, dy$$

$$= \int_{1/16}^{1} \sqrt{1 + [\tfrac{1}{2}y^{-\frac{3}{2}}]^2}\, dy$$

Do 1, 3, 7

(B) ds Notation

Good review really appreciated this!



$$L = \int_a^b \underbrace{\sqrt{1 + [f'(x)]^2}\, dx}_{= ds,\ \text{the differential of arc length}}$$

$$(ds)^2 = (dx)^2 + (dy)^2$$

Memory $L = \int_{x=a}^{x=b} ds$, where

ds > 0

$$ds \underset{\text{def}}{=} +\sqrt{(dx)^2 + (dy)^2}$$

"Factor out" $(dx)^2$.

$6x+8 = 2(3x+4)$ we divided by 2.

$$= \sqrt{\left[\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}\right](dx)^2}$$

dx > 0

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (dx > 0)$$

$$\approx \sqrt{1 + [f'(x)]^2}\, dx$$

Idea: $dx \to 0$

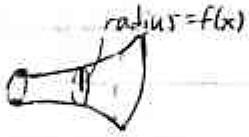
dy = dx approx. f'(x)
Do 1, 3, 7

ⓒ Surfaces of Revol.

$f(x) \geq 0$ Need for radius idea

Assume f is smooth, $f(x) \geq 0$ on $[a,b]$.

lateral! um...we have a curve



(For V, we $\int$ cross-sec. areas $\underbrace{\text{wrt}}_{\text{with respect to}}$ x or y.)

S = surface area

bong

Instead of the area of a ⊖, we deal w/...

We $\int$ integrate circumferences ($2\pi \cdot$ radius) wrt arc length.

Shorthand

Bad: Book omits!

$$\boxed{S = \int_{x=a}^{x=b} \underbrace{2\pi \underbrace{f(x)}_{\text{radius}}}_{\text{circum.}} \, ds} \qquad ds = \sqrt{1+[f'(x)]^2}\,dx$$

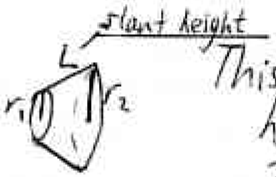
Why?

Sample points

min ⌊ bet. a, b

cylinder: could do dx

$ds = \sqrt{1+(0)^2}\,dx$



This frustum of a cone has [lateral] surface area $2\pi \underset{\substack{\uparrow \\ \text{average of } r_1, r_2}}{r} L$. (Cylinder: $2\pi r h$)

not including these disks

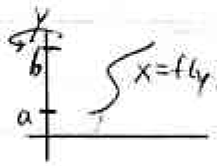
do cyl 1st

What do you think r is?

we have to deal w/ slant height issue even if we let $\|P\| \to 0$. This wasn't true with volume.

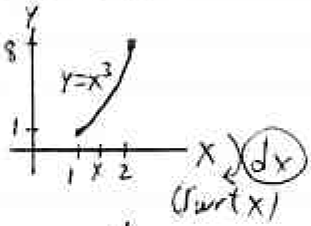
Because of "L", we $\int$ integrate wrt arc length.

Also



$$\boxed{S = \int_{y=a}^{y=b} 2\pi f(y) \underbrace{ds}_{\sqrt{1+[f'(y)]^2}\,dy}}$$

Ex (#30)



$$y = \underbrace{x^3}_{f(x)}$$

$$f'(x) = 3x^2$$

$\begin{pmatrix} C = 2\pi r \\ = 2\pi f(x) \\ \text{or } 2\pi x^3 \end{pmatrix}$

$$S = \int_{x=1}^{x=2} 2\pi f(x)\, ds$$
$$= 2\pi \int_1^2 f(x)\sqrt{1+[f'(x)]^2}\, dx$$
$$= 2\pi \int_1^2 x^3 \sqrt{1+[3x^2]^2}\, dx$$
$$= 2\pi \int_1^2 x^3 \underbrace{\sqrt{1+9x^4}}_{u = 1+9x^4}\, dx$$

$$du = 36x^3\, dx$$
$$\Rightarrow \tfrac{1}{36} du = x^3\, dx$$
$$x = 1 \Rightarrow u (= 1+9(1)^4) = 10$$
$$x = 2 \Rightarrow u (= 1+9(2)^4) = 145$$

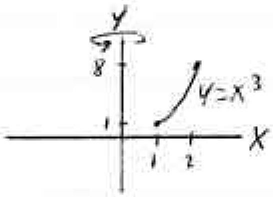
$$= 2\pi \int_{10}^{145} \sqrt{u} \cdot \tfrac{1}{36}\, du$$

$$\left(= \frac{\pi}{18} \int_{u=10}^{u=145} \sqrt{u}\, du \right)$$

$\vdots$

$$\approx 199.48$$

Ex



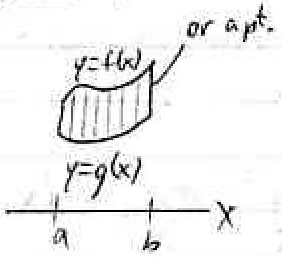
$$y = x^3$$
$$(\sqrt[3]{y} = x)$$
$$x = \underbrace{\sqrt[3]{y} \text{ or } y^{1/3}}_{f(y)}$$

$$f'(y) = \tfrac{1}{3} y^{-\frac{2}{3}}$$

$$S = \int_{y=1}^{y=8} 2\pi f(y)\, ds$$
$$= 2\pi \int_1^8 f(y) \sqrt{1+[f'(y)]^2}\, dy$$
$$= 2\pi \int_1^8 y^{\frac{1}{3}} \sqrt{1 + \underbrace{[\tfrac{1}{3} y^{-\frac{2}{3}}]^2}_{\frac{1}{9} y^{-\frac{4}{3}}}}\, dy$$
$$\approx 71.41$$

Numerical methods

(6.1) Area

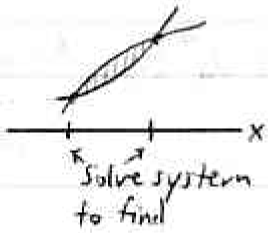


$A = \int_a^b [f(x) - g(x)]\,dx$ ($\int$ lengths of line segments)

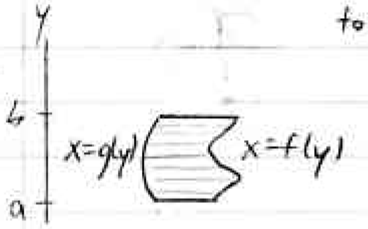
top - bot

If g=0

flush against axis
no gap



If left-right... flips sign
Do it right!

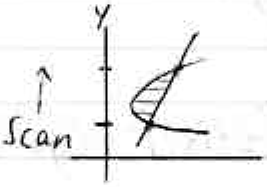


$A = \int_a^b [f(y) - g(y)]\,dy$

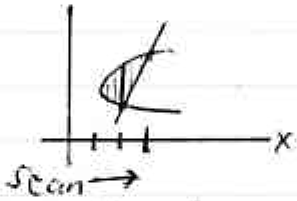
right - left

If g=0

If you sort x, what would you have to do?
Also depends what f, g look like
what can you solve for?



vs.



$\int \quad dy$

$\int \quad dx$
$+\int \quad dx$

↑
Better?
Maybe not if hard to take eqr. and write as $x = f(y)$.

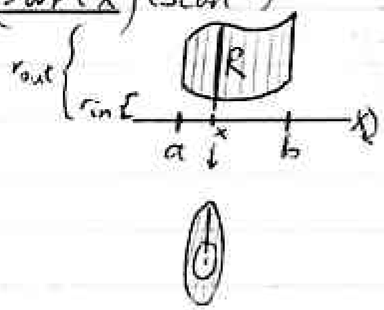
(6.2/6.3) Volumes of Solids of Revol.

Given: Axis of rev.
Know how scan $\Leftrightarrow$ Know method

(6.2) Disk / Washer

regardless of method

dx (Swrt x) (Scan →)

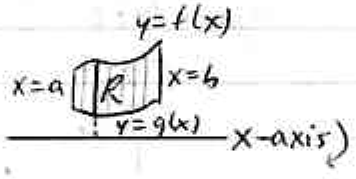


$$V = \int_a^b (\pi r_{out}^2 - \pi r_{in}^2)\, dx$$

funcs. of x

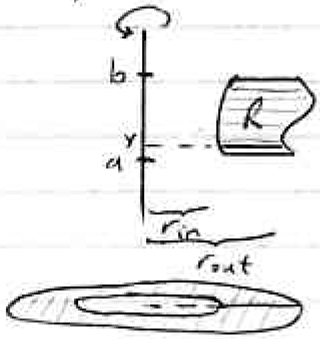
If = 0 ⇒ Disk

If



$$V = \int_a^b (\pi[f(x)]^2 - \pi[g(x)]^2)\, dx$$

dy (Swrt y) (Scan ↑)



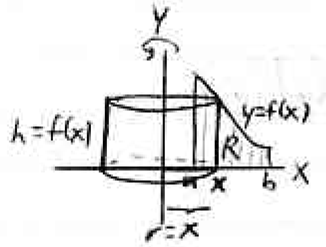
$$V = \int_a^b (\pi r_{out}^2 - \pi r_{in}^2)\, dy$$

funcs. of y

(6.3) Cyl. Shells

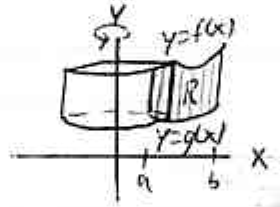
Lateral Surface Area $= 2\pi r h$

funcs. of x (or y)



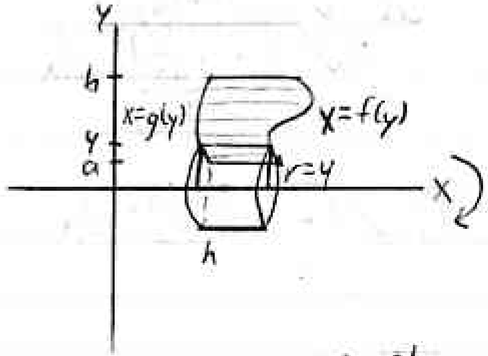
$$V = \int_a^b 2\pi x f(x)\,dx$$

what's diff about this cyl?



$$V = \int_a^b 2\pi x \overbrace{[f(x) - g(x)]}^{h\text{, or } h(x)}\,dx$$

top − bot



$$V = \int_a^b 2\pi y \overbrace{[f(y) - g(y)]}^{h}\,dy$$

right − left

Weird axis

Draw cylinder

Find radius, height.

doesn't affect

shifted axis of revol, affects radius

6.2 also

r_{in}, r_{out}

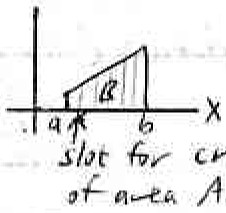
Use right−left or top−bottom (← for nonnegative linear measures)

Axis of revol. won't cross R unless you can exploit symmetry.

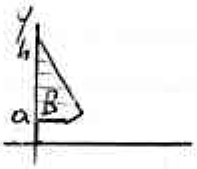
(using symmetry optional)

(← does what top does)

(6.4) Volumes by General Cross-Sections

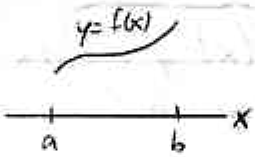


$$V=\int_a^b A(x)\,dx$$

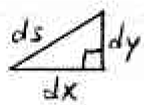


$$V=\int_a^b A(y)\,dy$$

(6.5) Arc Length

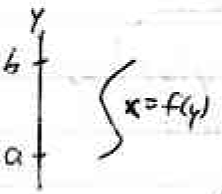


$$L=\int_{x=a}^{x=b} ds$$



$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\underbrace{\left(\frac{dy}{dx}\right)^2}_{\approx f'(x)}}\,dx$$

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$



x→y in
dx→dy

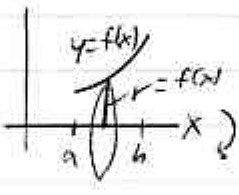
("→" means "replaced by", not "limit")

$$L=\int_{y=a}^{y=b} ds$$

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{\underbrace{\left(\frac{dx}{dy}\right)^2}_{\approx f'(y)}+1}\;dy$$

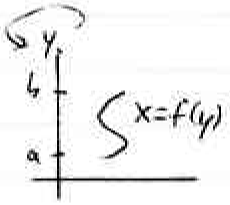
$$L=\int_a^b \sqrt{1+[f'(y)]^2}\,dy$$

Surfaces of Revol.



$$S = \int_{x=a}^{x=b} \underbrace{C(x)}_{\substack{=\text{circum.} \\ =2\pi r}}\, ds$$

$$= \int_{x=a}^{x=b} 2\pi f(x)\, \underset{\uparrow \atop \sqrt{\cdots}\,dx}{ds}$$



$x \to y$ $\sqrt{\cdots}\,dy$

SKIP

(6.6) Work

Motor Ex

(6.7) Centers of Mass