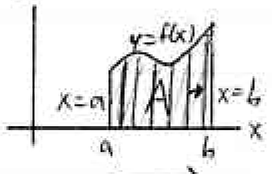
CH. 6: APPLICS. OF $\int$

6.1: AREA

(A) Integrating wrt x ($\int$, with respect to)

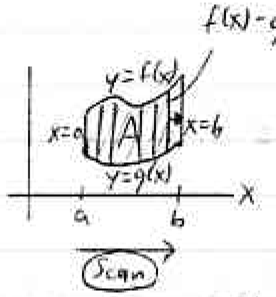


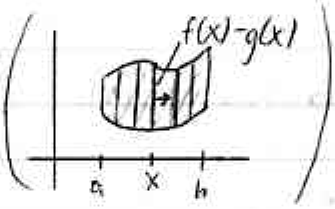
$A = \int_a^b f(x)\,dx$

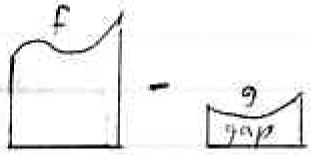
Scan in x-direction

works if f cont., $f(x) \geq 0$ throughout (on) $[a,b]$

Don't be confused by vertical line segs. - they're scan lines.







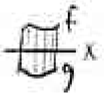
Imagine a Xerox machine

$$A = \int_a^b [f(x) - g(x)]\,dx$$

"top - bottom"

Comparing f to g, not f to 0

Don't need $f, g \geq 0$.

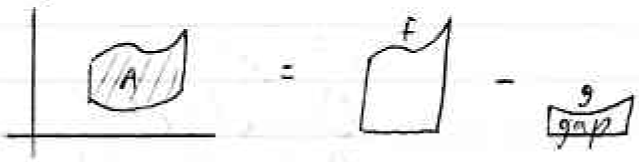


Works if f, g cont., $f(x) \geq g(x)$ throughout (on) $[a,b]$

We can easily interpret in terms of areas.

Helps w/volume

(Idea If $f, g \geq 0$



$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

or $\int_a^b [f(x) - g(x)]\,dx$)

Like #10

<u>Ex</u> Find the area [of the region] bounded by [the graphs of] $y-x^2=0$ and $y-x^3=0$.

(May change order:)

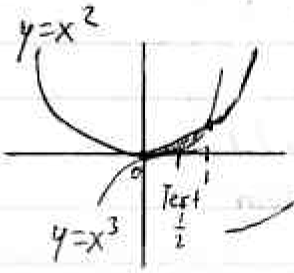
<u>① Solve both eqs. for y</u>

$$y=x^2$$
$$y=\underbrace{x^3}_{\text{funcs. of } x}$$

<u>② Sketch the graphs</u> (maybe do 1st)

If $x<1$, what's greater? Twilight Zone

I want left, right of region to be either vert. edge or pt.



$x^2 \geq x^3$ on $[0,1]$

(Twilight Zone: the unexpected happens)

y-coords. help w/a graph

<u>③ Find intersection points (esp. x-coords.)</u>

Here, they're pretty obvious. When solving a system, you're finding pairs (x,y) that satisfy both eqs. Means point (x,y) lies on both graphs.

Solve the system $\begin{cases} y=x^2 \\ y=x^3 \end{cases}$

$$x^2=x^3$$
$$0=x^3-x^2$$
$$0=x^2(x-1)$$
$$x=0,\ x=1$$
$$y=x^2 \Downarrow \qquad \Downarrow$$
$$(y=0) \quad (y=1)$$

Pt. $(0,0)$ Pt. $(1,1)$ ← I'd like it I tell you to graph the region.

④ Set up $\int$ (s)

$$\int_0^1 (x^2 - x^3)\,dx$$
(top − bottom)

⑤ Evaluate

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$
$$= \left[\frac{1}{3} - \frac{1}{4}\right] - [0]$$

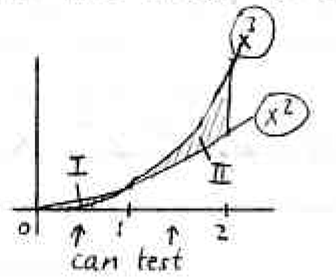
Can do 1,5,9

$= \frac{1}{12}$ sq. units (Don't be surprised it's a nice fraction: nice poly. funcs., Power-Rule for $\int$, integer limits)

Doesn't work: $\int_0^2$ flip sign if −

Ex Find the area bounded by $y = x^2$, $y = x^3$, if x is restricted to $[0, 2]$.

(I like pt.-ph, vertical edge at 0,1,2.)

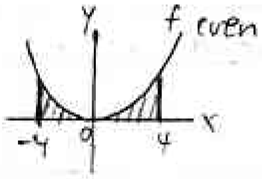


$$\int_0^1 (\underset{top}{x^2} - \underset{bot}{x^3})\,dx + \int_1^2 (\underset{top}{x^3} - \underset{bot}{x^2})\,dx$$

$\frac{1}{12}$ from before

$$= \frac{1}{12} + \frac{17}{12}$$
$$= \frac{3}{2} \text{ sq. units}$$

Ex

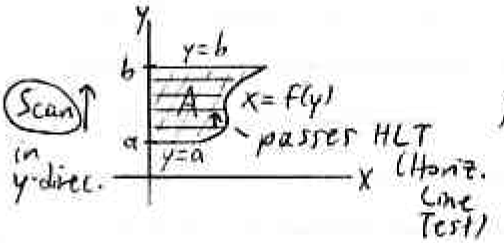


$$A = \int_{-4}^{4} f(x)\,dx$$
$$\overset{sym.}{=} 2\int_0^4 f(x)\,dx \quad \text{(Easier?)}$$

Can do 1,5,7,9

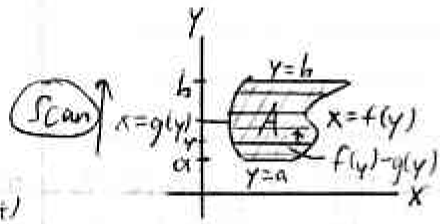
ⓑ Integrating wrt y

Does this curve rep. a func.? Doesn't rep. y as func. of x



$$A = \int_a^b f(y)\,dy$$

Xerox ↑ scans
Before: ∫ (top−bot)
Easy to mess up
rt.−left
on left
higher−lower



$$A = \int_a^b [f(y) - g(y)]\,dy$$
"right − left"

Works if f, g cont., $f(y) \geq g(y)$ throughout (on) $[a, b]$

x is rep. as a func. of y. This eq. does not rep. y as a func. of x

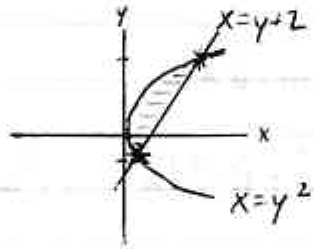
Ex (#14) Find the area bounded by $x = y^2$, $x - y = 2$.

① Solve both eqs. for x

$$x = y^2$$
$$x = \underbrace{y + 2}_{\text{funcs. of } y}$$

② Sketch

So it may help to solve for y, anyway.



Graph $x = y + 2$
$y = x - 2$ (slope-int. form)
or find intercepts

Why ∫ wrt y? ① We can easily express x as funcs. of y.
② Easier to scan region ↑ (vs. →).

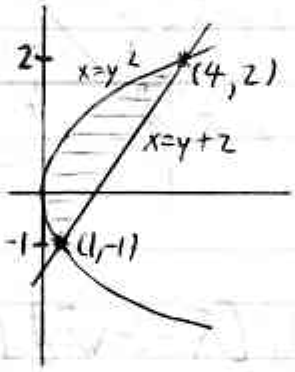
③ Intersection Points

Solve $\begin{cases} x = y^2 \\ x = y + 2 \end{cases}$

QF if desperate

I want top, bot of region to be either horiz. edge or a pt.

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0$$
$$y = 2, \quad y = -1$$
$$x=y^2 \Downarrow \qquad \Downarrow$$
$$x = 4 \qquad x = 1$$
$$(4,2) \qquad (1,-1)$$



④, ⑤

Don't need, but need if had ≥2 terms

$$\int_{-1}^{2} [(y+2) - (y^2)]\, dy$$
right − left

You can do it!!

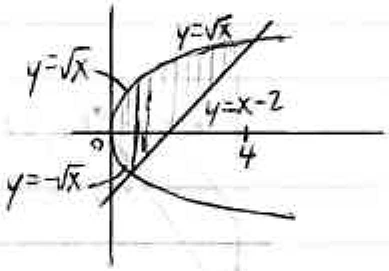
$$= \int_{-1}^{2} (y + 2 - y^2)\, dy$$
$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$
$$= \underbrace{\left[\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right]}_{\text{eval at } 2} - \underbrace{\left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right]}_{\text{eval at } -1}$$
$$= \frac{9}{2} \text{ sq. units}$$

Harder: $\int$ wrt x

if have heart set on $\int dx$, you're setting yourself up for heartbreak

$$x = y^2 \Rightarrow y = \pm\sqrt{x}$$
$$x - y = 2 \Rightarrow y = x - 2$$

change parts at 1



What can you use? sym: $2\int_0^1 \sqrt{x}\,dx$

$$\int_0^1 \underbrace{[\underset{top}{\sqrt{x}} - (\underset{bot}{-\sqrt{x}})]}_{2\sqrt{x}}\,dx + \int_1^4 [\underset{top}{\sqrt{x}} - \overset{need}{(}\underset{bot}{x-2}\overset{need}{)}]\,dx$$

Correct, but... HAVE FUN!! ☺

sym