6.2: VOLUMES OF SOLIDS OF REVOLUTION

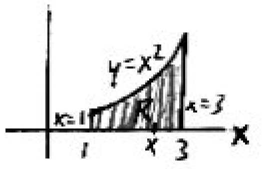
Bring Toys!!

Ⓐ Disk Method

<u>Ex</u> Sketch the region R bounded by [the graphs of] $y=x^2$, x-axis, $x=1$, and $x=3$.
Find the volume [of the solid generated] if R is revolved about the x-axis. — x

① Sketch R

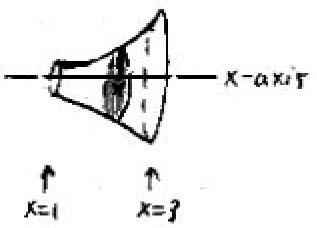
If we take a vertical line seg, rotate it

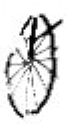


② Sketch the solid (optional)
③ Find the area of a cross-section

sideways volcano, bullhorn

If we slice the volcano at x, what do we get?





The cross-sec. at x $(1 \le x \le 3)$ is a <u>disk</u> of radius $f(x)$, or x^2.

Mine are infinitesimally thin; they have some thickness in the Riemann development (see next page)

Its area is $\pi(\text{radius})^2$
$= \pi[f(x)]^2$ ← More precisely, $\pi[|f(x)|]^2$
$= \pi[x^2]^2$
$= \pi x^4$

r(x)
We'll need r later

④ $\int$ cross-sec areas

(④ Integrate the cross-sectional areas)

(Volume of solid) $V = \int_1^3 \pi x^4 \, dx$

$= \pi \left[\frac{x^5}{5} \right]_1^3$

$\vdots$

$= \pi \left[\frac{(3)^5}{5} - \frac{(1)^5}{5} \right]$

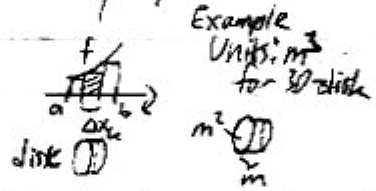
$= \pi \left[\frac{242}{5} \right]$

$= \frac{242\pi}{5}$ (cubic units)

In Ch. 5, how did we approx. area

My disks are 2-D, like me.

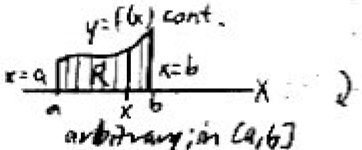
Theory (p. 314)



$V = \lim_{\|P\| \to 0}$ (sum of disk vols.)

$= \int_a^b \pi [f(x)]^2 \, dx$

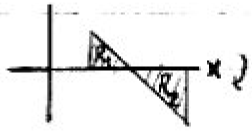
In general

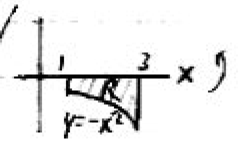


(If f is cont. on $[a,b]$,
R is bounded by $y=f(x)$, x-axis, $x=a$, $x=b$,
then the volume of the solid gen. by revolving R about the x-axis is)

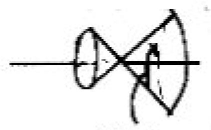
$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Note Still works if $f(x) < 0$ sometimes.





same as before



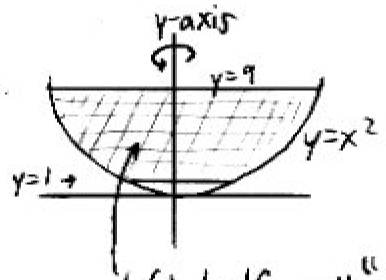
$f(x) < 0$; radius of disk at x is $|f(x)|$

Cross-sec area = $\pi [\underbrace{f(x)}_{\text{OK if } <0}]^2$ because

Can do 1, 5, 7, 11

Ex R is bounded by $y=x^2$, $y=1$, and $y=9$.
(R is revolved about the y-axis.)
Find V.

① Sketch R



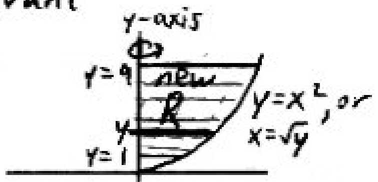
left half will "overlap" (what the right half "does")

(Normally, we worry when the axis of revol. cuts through the region, but we can handle this.)

Imagine revolving doors at a hotel. We're going to end up w/ twice what we want. It's implied that we're going around 1 full revol. 360° Double-counting. We must cut out ½ the figure. $\sqrt{y}-(-\sqrt{y})$ is the diameter.

We would "compensate" "1 ½" if we had [I|II], rotate 360°. This is how we'd rotate. You may as well "just" rotate one of the regions.

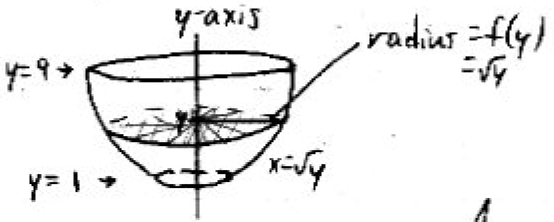
Want



Solve $y=x^2$ for x
$x^2=y$
$x=\pm\sqrt{y}$

Scan ↑, so dy

②, ③

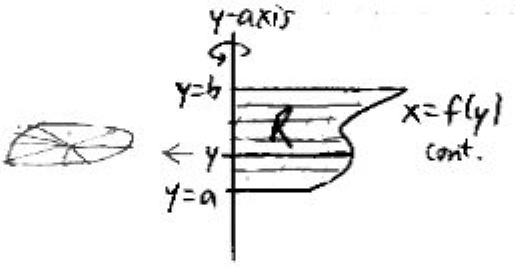


$$\underset{\text{at } y}{\text{Area}} = \pi(\sqrt{y})^2 = \pi y$$

④

$$\int_1^9 \pi y\,dy = \pi\left[\frac{y^2}{2}\right]_1^9 = \pi\left(\left[\frac{(9)^2}{2}\right]-\left[\frac{(1)^2}{2}\right]\right) = 40\pi \text{ cubic units}$$

In general



$$V=\int_a^b \pi\,[f(y)]^2\,dy$$

↑ Some books use $\int_c^d$

Can do HI except 3

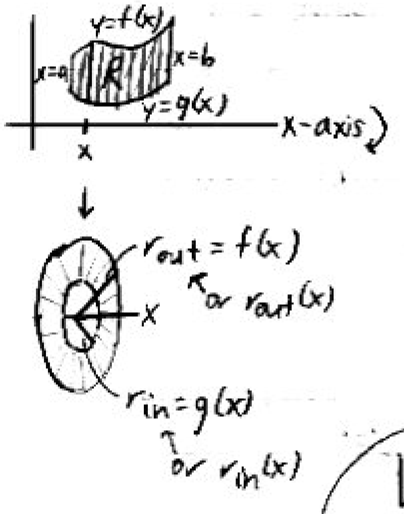
Donut

Ⓑ Washer Method

You can factor out a π, but, You can't take out "2"

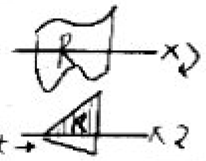
If we take this line seg, rotate it about x-axis

nature of overlap unclear



Note

In trouble:



weird self-overlapping

OK: cut →

(← this takes care of whatever this does)

The cross-sec at x is a washer of area

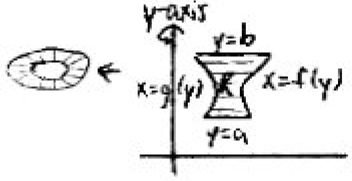
$= $ "big area" $-$ "area of hole"

$= \pi r_{out}^2 - \pi r_{in}^2$

h-o-l-e not w-h-o-l-e

Remember → many misplace ²

$$V=\int_a^b \left(\pi r_{out}^2-\pi r_{in}^2\right)dx$$
$$=\pi\int_a^b \left([f(x)]^2-[g(x)]^2\right)dx$$

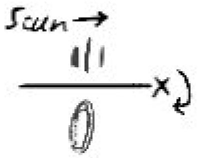


$$V=\int_a^b \left(\pi r_{out}^2-\pi r_{in}^2\right)dy$$
$$=\pi\int_a^b \left([f(y)]^2-[g(y)]^2\right)dy$$

Ex (#14) R is bounded by $x=y^3$, $x^2+y=0$. —x↺
(R is revolved about the x-axis.)
Find V.

① Sketch R

Scan "dx way" or "dy way"
x is changing

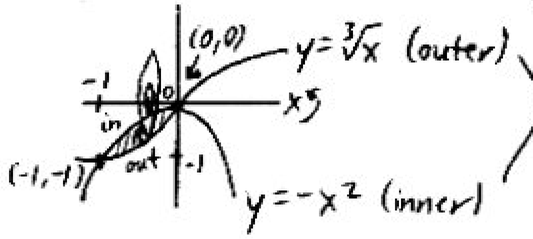


dx

So, solve for y in terms of x.

$$x=y^3 \Rightarrow y=\sqrt[3]{x}$$
$$x^2+y=0 \Rightarrow \underbrace{y=-x^2}_{\text{funcs. of } x}$$

It's not a matter of top vs. bottom.



relative to X-axis (axis of revol.)

Intersection Pts. (x)

$$\sqrt[3]{x}=-x^2$$
$$(\sqrt[3]{x})^3=(-x^2)^3$$
$$x=-x^6$$
$$x^6+x=0$$
$$x(x^5+1)=0$$
$$x=0 \quad x=-1$$
$$y=0 \quad y=-1$$

How do you solve a poly. eq. What do you want on one side? 0.
No DeMoivre's Thm here.
What are the 5 5th roots of -1?
Only 1 real root.

Skip ②: Sketch solid

③, ④ $V=\int_{-1}^{0}\left(\pi r_{out}^2-\pi r_{in}^2\right)dx$

$$=\pi\int_{-1}^{0}\left([\sqrt[3]{x}]^2-[x^2]^2\right)dx$$
$$=\pi\int_{-1}^{0}\left(x^{2/3}-x^4\right)dx$$
$$\vdots$$
$$=\frac{2\pi}{5}$$

Up to 23

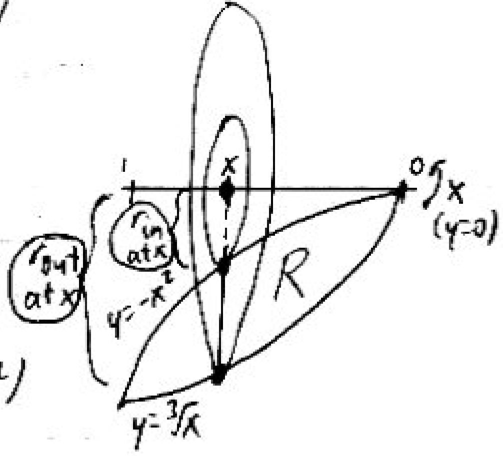
Radii, heights, lengths:

top − bottom, or right − left

They're never negative!

r_{out} (at x) $= y_{top}-y_{bot}$
$= 0-\sqrt[3]{x}$
$= -\sqrt[3]{x}$

r_{in} (at x) $= y_{top}-y_{bot}$
$= 0-(-x^2)$
$= x^2$

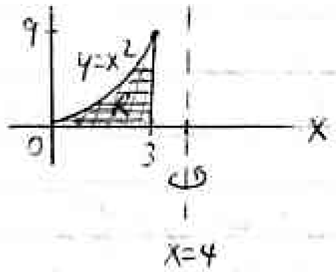


<u>Ex</u> (Weird axis)

R is bounded by $y = x^2$, $x = 0$, and $x = 3$.
R is revolved about $x = 4$.
Find V.

<u>① Sketch R</u>

$x=4$: hor or vert?



②

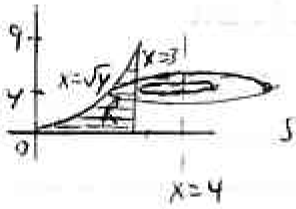
Volcano / bullhorn

③, ④

If you're not sure which way to do it, draw a cross-sec. $x=4 \parallel y$

Is the outer radius $\sqrt{y}$? No

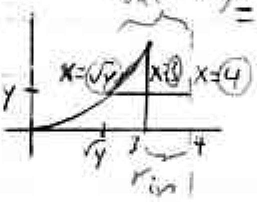
For linear measures like height, radius right-left or top-bot



scan (dy)

$y = x^2$
$\Rightarrow x = \pm\sqrt{y}$

$r_{out} = x_{right} - x_{left} = 4 - \sqrt{y}$

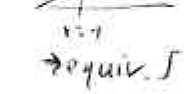


$r_{in} = 4 - 3 = 1$

$$V = \int_0^9 (\pi r_{out}^2 - \pi r_{in}^2)\, dy$$
$$V = \pi \int_0^9 \left([4 - \sqrt{y}]^2 - [1]^2 \right) dy$$

$$= \frac{63\pi}{2}$$

Could translate



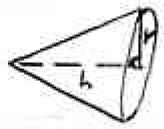
$\rightarrow$ equiv. $\int$

If you forget, can get them thru calc.

Ⓒ Famous Formulas

Ex (#37) HW Find the volume of a right circular cone of altitude h and base radius r.

Could △ of y



What would I need to rotate → x

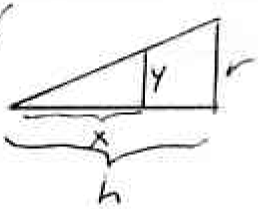
What's the slope?

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{r}{h}$$

$$y = mx + b$$
$$y = \frac{r}{h}x + \cancel{0}$$

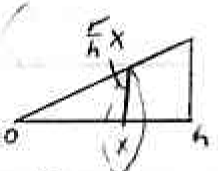
commonly used in calculus

or similar triangles



$$\frac{y}{x} = \frac{r}{h}$$
$$y = \frac{r}{h}x$$

$$V = \int_0^h \pi \underbrace{\left(\frac{r}{h}x\right)}_{\text{radius of cross-sec at } x}{}^2\, dx$$



Write all this out.

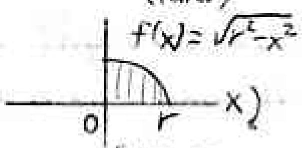
(You) r, h constants

$$= \frac{1}{3}\underbrace{\pi r^2}_{\text{base area}} h \qquad \left(\frac{1}{3}V_{cyl}\right)$$

what do we rotate — x)

$\underline{Ex}$ (#38) HW Sphere of radius r, Find V.

(later)



scan $\rightarrow$ dx

D × 2

(symmetry)

Circle of radius r
w/center $(0,0)$

$$x^2 + y^2 = r^2$$

Solve for y

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

W also works
S × 2

(You)

$$\boxed{V = \frac{4}{3}\pi r^3}$$

$\underline{\text{Note}}$ $D_r(V) = 4\pi r^2$ (surface area formula for a sphere!)

Like: $D_r\left(\underbrace{\pi r^2}_{\text{area of circle}}\right) = \underbrace{2\pi r}_{\text{Circumference of circle}}$

Not a coincidence!

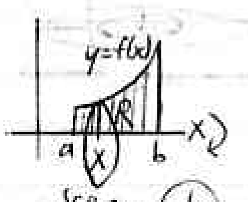
(D) →

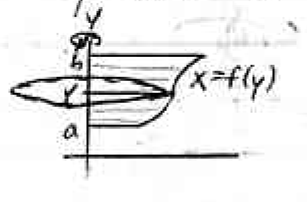
(E) Summary

Axis of revolution

x-axis ⟶ x | y-axis ↑ y

Disk

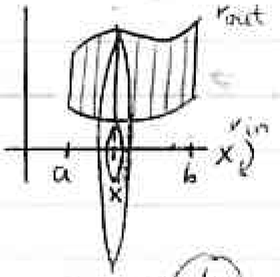


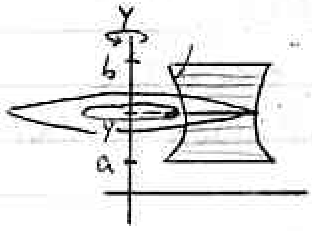


Scan ↑ dy

$$V=\int_a^b \pi[f(x)]^2\,dx \qquad V=\int_a^b \pi[f(y)]^2\,dy$$

Washer





↑ dy

→ dx

not a matter at top, bot.

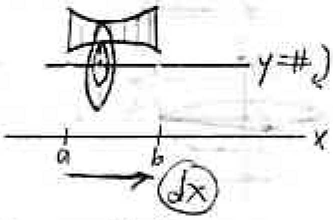
$$V=\int_a^b \Big(\underbrace{\pi r_{out}^2}_{\text{big area}} - \underbrace{\pi r_{in}^2}_{\text{area of hole}}\Big)\, dx \text{ or } dy$$

funcs. of x or y

CONT. →

Weird Axis

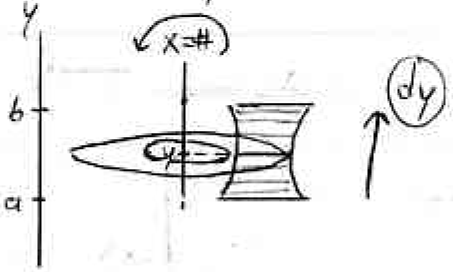
|| to x-axis



$$V = \int_a^b \left(\pi r_{out}^2 - \pi r_{in}^2\right) dx$$

funcs. of x, "top-bot"

|| to y-axis



$$V = \int_a^b \left(\pi r_{out}^2 - \pi r_{in}^2\right) dy$$

funcs. of y "right-left"

If R bounded by axis of revol. ⇒ Disk

(D) HW Tips

x as func of y- still freaky?

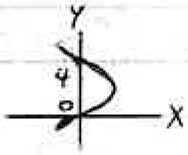
(#11) Graph $x = \overbrace{4y - y^2}^{\text{quadratic } f(y)}$

leading coeff. = $-1 < 0$

⊃

find y-ints:

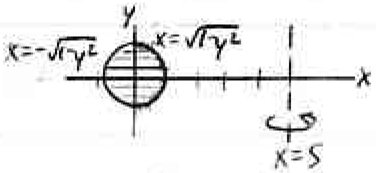
$$0 = 4y - y^2$$
$$0 = y(4-y)$$
$$y = 0,\ y = 4$$



(#33) R is bounded by $x^2 + y^2 = 1$
R is revolved about $x = 5$.

Think Homer

initially surrounded by pawns



r_{in} = right-left = $5 - \sqrt{1-y^2}$

r_{out} = right-left = $5 - (-\sqrt{1-y^2})$

$$x^2 + y^2 = 1$$
$$x^2 = 1 - y^2$$
$$x = \pm\sqrt{1-y^2}$$

Get a donut/torus.
(Toroidal (wraparound) Chess:
Queen? Diff. initial setup,
[N] Knight weird!!)

← (E)