# 6.5: ARC LENGTH and SURFACES OF REVOL.

## Ⓐ Arc Length

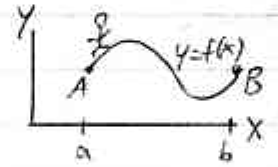
weird funcs where $f'$ exist but not cont.

Edwards Gelbaum 36

$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Ch. 13 generalize to more general graphs: Cart., Polar

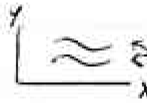



Assume $f$ is smooth ($f'$ is cont.) on $[a,b]$.

The arc length of the graph of $f$ from $A$ to $B$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

to unbelieving arc length

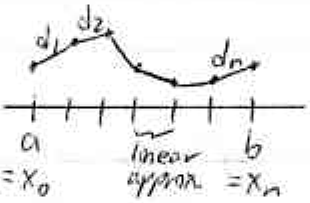



$f'$ is the key.

### Idea of Proof (pp. 333-4)

Sample (n+1) points based on a partition $P$ of $[a,b]$.

These are pieces of secant lines going through the graph.




piecewise-linear approx. to $f$

Total length $= \text{"}L_P\text{"} = \sum_{k=1}^{n} d_k$

$L = \underbrace{\lim_{\|P\| \to 0} L_P}_{} = \underbrace{\int_a^b}_{\text{limit of sums}} \underbrace{\sqrt{1 + [f'(x)]^2}\,dx}_{\text{See Ⓑ}}$

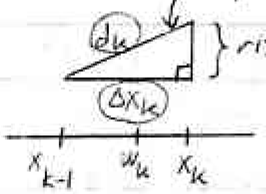
Note on Rigorous Proof

Approx. $L$ by $L_P$, the sum of lengths of secant line segments.

Find $d_k$

Idea from differentials:

By MVT, slope $= f'(w_k)$ for some $w_k$ "inside"




$\Big\}$ rise $=$ (slope)(run) $= f'(w_k)\,\Delta x_k$

Note (cont.)

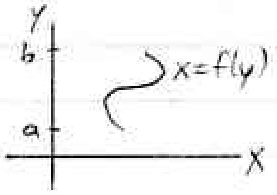
By Pyth. Thm.,

$$d_k = \sqrt{(\Delta x_k)^2 + [f'(w_k)\Delta x_k]^2}$$
$$= \sqrt{(1+[f'(w_k)]^2)(\Delta x_k)^2}$$
$$= \sqrt{1+[\underbrace{f'}_{\text{cont.}}(w_k)]^2}\;\Delta x_k$$

$$L_P = \sum_{k=1}^{n} d_k$$

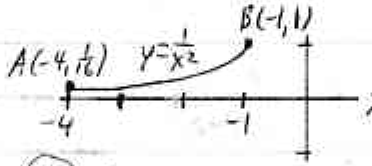
$$L = \lim_{\|P\| \to 0} L_P = \underbrace{\int_a^b}_{\text{limit of sums}} \sqrt{1+[f'(x)]^2}\,dx$$

Also




$$\boxed{L = \int_a^b \sqrt{1+[f'(y)]^2}\,dy}$$

Ex (#4)




Set up an integral for L

If fails VLT, can't

$\int_1^4$ same # by sym/conc

(dx) $\int$ wrt x

$$L = \int_{-4}^{-1} \sqrt{1+[f'(x)]^2}\,dx$$

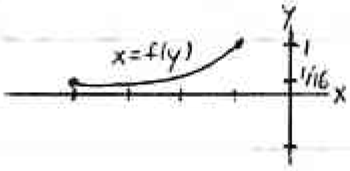
$$f(x) = \frac{1}{x^2} = x^{-2}$$
$$f'(x) = -2x^{-3}$$

Trap Rule gives different result from straight piecewise linear approx

has nothing to do w/ area under original curve

$$= \int_{-4}^{-1} \sqrt{1+[-2x^{-3}]^2}\,dx$$
$$= \int_{-4}^{-1} \sqrt{1+4x^{-6}}\,dx$$

Numerical approx. (Trapezoidal, Simpson) (dy) or $\int$ wrt y




$y = \frac{1}{x^2}$

$x^2 = \frac{1}{y}$

$x = \cancel{\pm} - \sqrt{\frac{1}{y}}$

$= \underbrace{-y^{-\frac{1}{2}}}_{f(y)}$

$f'(y) = \frac{1}{2} y^{-\frac{3}{2}}$

Sign doesn't matter: pops out of deriv.; we square deriv. Also, sym → same L

$$L = \int_{1/16}^{1} \sqrt{1 + [f'(y)]^2}\, dy$$

$$= \int_{1/16}^{1} \sqrt{1 + [\tfrac{1}{2} y^{-\frac{3}{2}}]^2}\, dy$$

Do 1,3,7

## (B) ds Notation

Good review really appreciated this!


$$L = \int_a^b \underbrace{\sqrt{1 + [f'(x)]^2}\, dx}_{= ds,\text{ the differential of arc length}}$$


$$(ds)^2 = (dx)^2 + (dy)^2$$

**Memory** $L = \int_{x=a}^{x=b} ds$, where

ds > 0

$$ds \underset{\text{pos}}{=} \sqrt{(dx)^2 + (dy)^2}$$

"Factor out" $(dx)^2$.

$6x+8 = 2(3x+4)$ we divided by 2.

$$= \sqrt{\left[\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}\right](dx)^2}$$

dx > 0

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (dx > 0)$$

$$\approx \sqrt{1 + [f'(x)]^2}\, dx$$

Idea: $dx \to 0$

$dy = dx$ approx. $f'(x)$

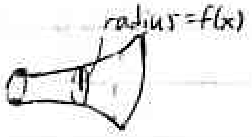
Do 1,3,7

## © Surfaces of Revol.

$f(x) \geq 0$ Need for radius idea


Assume $f$ is smooth, $f(x) \geq 0$ on $[a, b]$.

lateral? um...we have a curve




(For $V$, we $\int$ cross-sec. areas $\underbrace{\text{wrt}}_{\text{with respect to}}$ $x$ or $y$.)

$S$ = surface area

bong Instead of the area of a ⊙, we deal w/...

We $\overset{\int}{\text{integrate}}$ circumferences ($2\pi \cdot$ radius) wrt arc length.

Shorthand

Bad: Book omits!

$$\boxed{S = \int_{x=a}^{x=b} \underbrace{2\pi \underbrace{f(x)}_{\text{radius}}}_{\text{circum.}} \, ds} \qquad ds = \sqrt{1+[f'(x)]^2}\,dx$$

Why?

Sample points

min L bet. a, b

cylinder: could do dx $ds = \sqrt{1+(0)^2}\,dx$


not including these disks

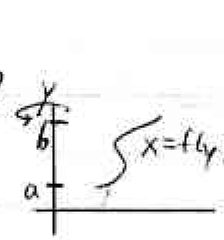
This frustum of a cone has [lateral] surface area $2\pi \underset{\text{average of } r_1, r_2}{r} L$. (Cylinder: $2\pi r h$)

do cyl 1st What do you think r is?

we have to deal w/ slant height issue even if we let $\|P\| \to 0$. This wasn't true with volume.

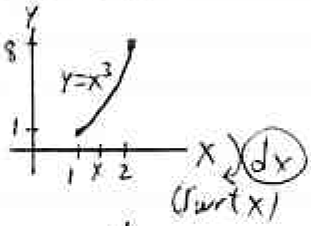
Because of "L", we $\overset{\int}{\text{integrate}}$ wrt arc length.

Also



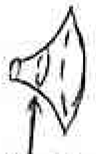

$$\boxed{S = \int_{y=a}^{y=b} 2\pi f(y) \, \underbrace{ds}_{\sqrt{1+[f'(y)]^2}\,dy}}$$

## Ex (#30)




$$y = \underbrace{x^3}_{f(x)}$$

$$f'(x) = 3x^2$$



$$\begin{pmatrix} c = 2\pi r \\ = 2\pi f(x) \\ \text{or } 2\pi x^3 \end{pmatrix}$$

$$\begin{aligned} S &= \int_{x=1}^{x=2} 2\pi f(x)\, ds \\ &= 2\pi \int_1^2 f(x)\sqrt{1+[f'(x)]^2}\, dx \\ &= 2\pi \int_1^2 x^3 \sqrt{1+[3x^2]^2}\, dx \\ &= 2\pi \int_1^2 x^3 \underbrace{\sqrt{1+9x^4}}_{u=1+9x^4}\, dx \end{aligned}$$

$$du = 36x^3\, dx$$

$$\Rightarrow \tfrac{1}{36} du = x^3\, dx$$

$$x = 1 \Rightarrow u = 1 + 9(1)^4 = 10$$

$$x = 2 \Rightarrow u = 1 + 9(2)^4 = 145$$

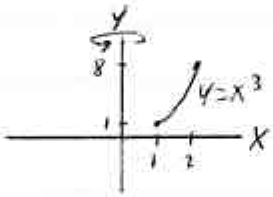
$$= 2\pi \int_{10}^{145} \sqrt{u} \cdot \tfrac{1}{36}\, du$$

$$\left( = \frac{\pi}{18} \int_{u=10}^{u=145} \sqrt{u}\, du \right)$$

$$\vdots$$

$$\approx 199.48$$

## Ex




$$y = x^3$$

$$\left(\sqrt[3]{y} = x\right)$$

$$x = \underbrace{\sqrt[3]{y} \text{ or } y^{1/3}}_{f(y)}$$

$$f'(y) = \tfrac{1}{3} y^{-\frac{2}{3}}$$

$$\begin{aligned} S &= \int_{y=1}^{y=8} 2\pi f(y)\, ds \\ &= 2\pi \int_1^8 f(y)\sqrt{1+[f'(y)]^2}\, dy \\ &= 2\pi \int_1^8 y^{\frac{1}{3}} \sqrt{1+\underbrace{[\tfrac{1}{3} y^{-\frac{2}{3}}]^2}_{\frac{1}{9}y^{-\frac{4}{3}}}}\, dy \\ &\approx 71.41 \end{aligned}$$

Numerical methods