As if we weren't haunted enough by algebra...

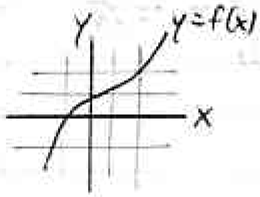
CH. 7: LOG, EXP. FUNCTIONS

They remember...

7.1: INVERSE FUNCTIONS (Skip)

If the graph of f passes VLT and HLT

Ex



$\Rightarrow$ f is one-to-one (1-1) func.

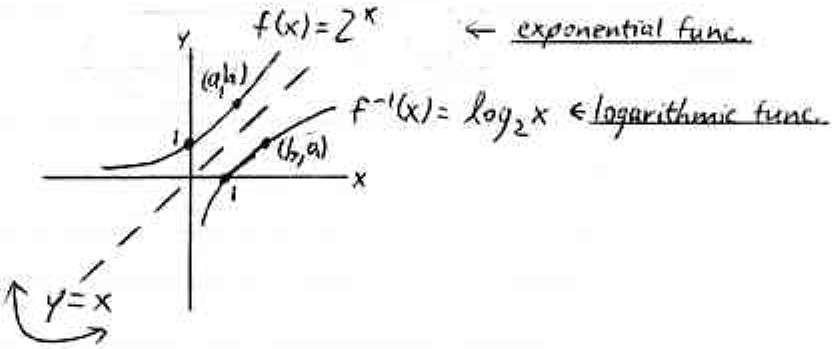
$\Rightarrow$ f has an inverse function, f^{-1} (not "$\frac{1}{f}$")

(such that $f^{-1}(f(x)) = x$, $\forall x$ in Dom(f)
and $f(f^{-1}(x)) = x$, " (f^{-1}))

undo each other

Like $2^x, 3^x, e^x$

Ex



$x \leftrightarrow y$

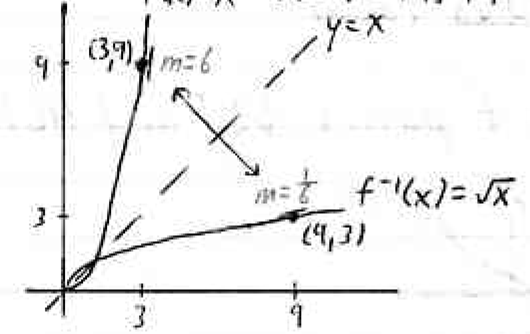
{set of x-coords (pick up for one)} = { y the other}

The domain (inputs "x") of one is the range (outputs "y") of the other.

why are we imposing this restriction?

$f(x)=x^2 \ (x \geq 0) \Rightarrow f$ is 1-1

Ex



$6, \frac{1}{6}$ are reciprocals.

$$\left(\underbrace{(f^{-1})'(9)}_{\frac{1}{6}} = \frac{1}{f'(3) \leftarrow 6} \right)$$

$$\left(\begin{array}{ll} f(x)=x^2 & f^{-1}(x)=\sqrt{x} \text{ or } x^{1/2} \\ f'(x)=2x & (f^{-1})'(x)=\frac{1}{2}x^{-1/2} \text{ or } \frac{1}{2\sqrt{x}} \\ f'(3)=2(3) & (f^{-1})'(9)=\frac{1}{2\sqrt{9}} \\ \quad =6 & \quad =\frac{1}{6} \end{array}\right)$$

reciprocals (same sign)

unless Ovr.DUE x^3 vs. $\sqrt[3]{x}$

$f \nearrow$ here, $f^{-1} \nearrow$ there

Not a coincidence...

7.2: THE NATURAL LOG FUNCTION, $\ln x$

Ⓐ Review (Power Rule for $\int$)

$\underline{\text{Ex}}$ $\int x^2\,dx = \frac{x^3}{3} + C$

OK for any real value of n except

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \underline{\text{if}}\ n \neq -1$$

Oh, +C!!

$$\text{NO: } \int x^{-1}\,dx = \cancel{\frac{x^0}{0} + C}$$

What is $\int x^{-1}\,dx$, or $\int \frac{1}{x}\,dx$?

We only take logs of "+" #s

We will define $\ln x$ so that

$$\int \frac{1}{x}\,dx = \ln x + C \quad \text{on } (0,\infty)$$

Ⓑ $\ln x$

We define

McGraw-Hill Survey: You like intro $\ln x$ as $\int$ or $\log_e x$?

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{on } (0,\infty)$$

(Remarkably, this is consistent w/our Algebra I treatment of $\ln x$ as $\underbrace{\log_e x}$.)

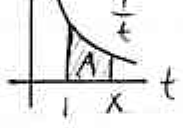
we'll see in 7.3 when we define e.

We'll see in L7-6b $\ln x = \log_e x$, inv. of e^x

Riemann sums converge to signed areas $\int_a^b$

For Ex $\ln 2$ = area from 1 to 2

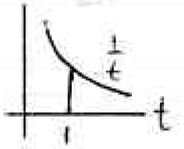
If $x > 1$



$\ln x = A$

$\ln x > 0$

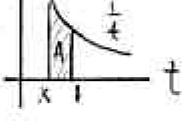
If $x = 1$



$\ln 1 = 0$

If I switch the limits of $\int$, a "$-$" sign pops out

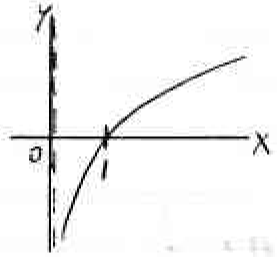
If $0 < x < 1$



$$\ln x = \int_1^x \frac{1}{t}\,dt = \underbrace{-\int_x^1 \frac{1}{t}\,dt}_{A} = -A$$

$\ln x < 0$

Graph $f(x) = \ln x$

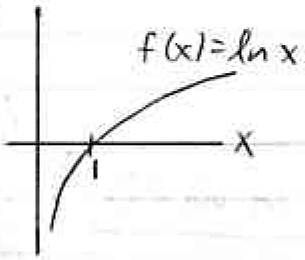


$\underline{\text{© } D_x, \int}$

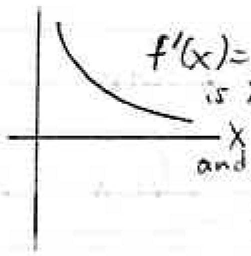
$$D_x(\ln x) = D_x\left(\int_1^x \tfrac{1}{t}\,dt\right)$$
$$= \tfrac{1}{x} \quad \text{by FTC, I}$$

$$D_x(\ln x) = \tfrac{1}{x} \quad \text{on } (0,\infty)$$
$$\Rightarrow \int \tfrac{1}{x}\,dx = \ln x + C \quad \text{on } (0,\infty)$$

Note



$\Rightarrow$



$f'(x) = \frac{1}{x}$ on $(0,\infty)$, is > 0, so $f \nearrow$ and is $\searrow$, so f CD. graph

More generally,

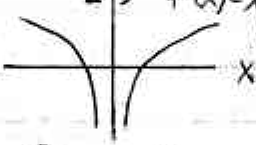
$$D_x(\ln|x|) = \tfrac{1}{x} \quad \text{on any interval excluding } 0$$
$$\Rightarrow \int \tfrac{1}{x}\,dx = \ln|x| + C$$

(Unless it's specified/known that $x > 0$, use there.)

What kind of func? Nice sym in both.

$\frac{1}{x}$ on its whole domain

Here, $\frac{1}{x} < 0$ + if f' ⊖, f is ...?



$\Rightarrow$

(Notice: $\searrow$ $\nearrow$ ⊖ ⊕)

(Deriv. of even func. is odd. Ex $D_x(x^4) = 4x^3$.)

when we learned derivs, what rule greatly expand our powers?

① Chain Rule

If $u = g(x)$, diff'e

What do we need?

Yyyee!

then $D_x(\ln u) = \frac{1}{u}\underbrace{D_x u}_{\text{tail}}$ (where $u > 0$)

$D_x(\ln |u|) = \underbrace{\frac{1}{u} D_x u}$, also (where $u \neq 0$)

can ignore ($|\ |$) or $\frac{D_x u}{u}$

← Think: $D(\ln ☆) = \frac{1}{☆} D(☆)$

ln(blah)

Ex If $f(x) = \ln(3x^2+1)$,

then $f'(x) = \frac{1}{3x^2+1} \cdot \underbrace{D_x(3x^2+1)}_{=6x}$

$= \frac{6x}{3x^2+1}$

Do 1,3

Ex If $f(x) = 3\ln|\sin(x^2)|$,

then $f'(x) = 3\left(\frac{1}{\sin(x^2)}\right) D_x[\sin(x^2)]$

$= \frac{3}{\sin(x^2)} \cdot \cos(x^2) \cdot \underbrace{D_x(x^2)}_{=2x}$

$= 6x\,\frac{\cos(x^2)}{\sin(x^2)}$

Do 1,3,5

$= 6x\cot(x^2)$

Logs are

Ⓔ Log Laws (come from Laws of Exponents)

If $A, B > 0$, then

From laws of exps. $\log_2 8 = 3$ logs are exps. $x^2 x^3 = x^5$

Product Rule

$$\ln(AB) = \ln A + \ln B$$

log of product = sum of logs

Quotient Rule

$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

quotient diff.

Power Rule

$A > 0$, so never need $|A|$

$$\ln A^p = p \ln A$$

Smackdown

Basketball

$^{p}\ln A$

WARNINGS: $(\ln x)^2 \neq 2 \ln x$

$\ln 3x^2 \neq 2 \ln 3x$

YES: $\ln (3x)^2 = 2 \ln 3x \quad (x > 0)$

(Actually, $2 \ln |3x|$)

This is not $\ln 3$ times x

Read $\ln 3x + 7$ as $\ln(3x) + 7$

Not a proof - it's a demonstration

Test $A = e^2$, $B = e^3$ to see if your guess at a Rule may be right.

Exs (p. 387)

Don't confuse 3 w/ $\sqrt[3]{\ }$

Ex If $f(x) = \ln[(x^3)(\sqrt{4x+7})]$, find $f'(x)$.

Old Way $f'(x) = \frac{1}{\text{blah}} \cdot D_x(\text{blah})$

$$\left(f'(x) = \frac{1}{(x^3)(\sqrt{4x+7})} \cdot D_x[(x^3)(\sqrt{4x+7})] \right)$$

$3x^2\sqrt{4x+7}$
$+ x^3(\frac{1}{2}(4x+7)^{-\frac{1}{2}}(4))$

Have fun! ($\rightarrow$ Ugly or Should Simplify)

Using Log Laws

Expand $f(x) = \ln[(x^3)(\sqrt{4x+7})]$

$$\begin{aligned} f(x) &= \ln x^3 + \ln\sqrt{4x+7} \\ &= \ln x^3 + \ln(4x+7)^{1/2} \\ &= 3\ln x + \tfrac{1}{2}\ln(4x+7) \end{aligned}$$

Algebra is your friend!

$\leftarrow$ Much easier to D_x.

$$f'(x) = 3\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{4x+7}\right)\underbrace{D_x(4x+7)}_{=4}$$

$$\left(= \frac{3}{x} + \frac{1}{\cancel{2}}\left(\frac{1}{4x+7}\right)(\cancel{4}2) \right)$$

$$= \frac{3}{x} + \frac{2}{4x+7}$$

$\underline{Ex}$ $f(x) = x \underbrace{(\ln x)^3}_{\neq 3\ln x}$

Product Rule

$$f'(x) = \underbrace{D_x(x)}_{=1} \cdot (\ln x)^3 + x \cdot \underbrace{D_x[(\ln x)^3]}$$

1st use
General) Power Rule

$$= 3(\ln x)^2 \underbrace{D_x(\ln x)}_{=\frac{1}{x}}$$

$$= (\ln x)^3 + \cancel{x}\left(\frac{3(\ln x)^2}{\cancel{x}}\right)$$

up to 35

$$= (\ln x)^3 + 3(\ln x)^2$$

Ⓕ <u>Logarithmic Differentiation</u> "Log Diff"

Used to find $D_x[\underbrace{f(x)}_{\text{has products, quotients, or powers}}]$

<u>Ex</u> Find $D_x\left[\dfrac{(x+1)^2(7x-2)^3}{\sqrt{x}}\right]$

① <u>Let $y=f(x)$</u>

$$y=\frac{(x+1)^2(7x-2)^3}{\sqrt{x}}$$

② <u>Take ln of both sides</u>

$$\ln y=\ln\left[\frac{(x+1)^2(7x-2)^3}{\sqrt{x}}\right]$$ (We voluntarily bring in ln!)

③ <u>Use log laws to expand the right side</u>

Breaking down products, quotients, powers.

$$\ln y=\ln(x+1)^2+\ln(7x-2)^3-\ln\underbrace{\sqrt{x}}_{x^{1/2}}$$

$$\ln y=2\ln(x+1)+3\ln(7x-2)-\tfrac{1}{2}\ln x$$

(Don't need $|x+1|$)

Log Diff uses Imp Diff

④ <u>Use implicit diff'n</u>

<u>Note</u> $D_x(\ln y)=\dfrac{1}{y}\cdot\underbrace{y'}_{\substack{\text{tail}\\ =D_xy \text{ or } \frac{dy}{dx}}}=\dfrac{y'}{y}$

(Now, we D_x both sides.)

$$\frac{y'}{y} = 2\left(\frac{1}{x+1}\right)\underbrace{D_x(x+1)}_{=1} + 3\left(\frac{1}{7x-2}\right)\underbrace{D_x(7x-2)}_{=7}$$

$$-\frac{1}{2}\left(\frac{1}{x}\right)$$

$$\frac{y'}{y} = \frac{2}{x+1} + \frac{21}{7x-2} - \frac{1}{2x}$$

We want y'

<u>⑤ Multiply both sides by y</u>

$$y' = (\quad\quad\quad\quad)y$$

<u>⑥ Write y in terms of x</u>

If you do simplify, you get cancellations. Book.

$$y' = \underbrace{\left(\frac{2}{x+1} + \frac{21}{7x-2} - \frac{1}{2x}\right)}\left(\frac{(x+1)^2(7x-2)^3}{\sqrt{x}}\right)$$

Often, this undoes the advantage that log diff gave you in the 1st place.

<u>⑦ Simplify (Optional)</u> $\Rightarrow$ 1 fraction, cancel

<u>Note</u> $y = f(x)$
$\ln y = \ln[f(x)]$

Expansion works well w/ $|\cdot|$

Log Diff works even if $f(x) < 0$ sometimes!
<u>Idea</u> $D_x(\ln|y|)$ is still $\frac{y'}{y}$. Right side works out.
can ignore

at x-int.

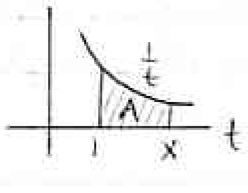
Don't use Log Diff to find f' where $f(x) = 0$.
(can't find slope of tan line to an x-int.)

7.3: THE NATURAL EXPONENTIAL FUNCTION, e^x

Ⓐ e

(Algebra books throw e at you, then define $\ln x = \log_e x$.)

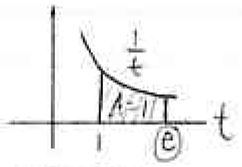
We defined $\ln x = \int_1^x \frac{1}{t}\,dt \quad (x>0)$



If $x>1$, $\ln x = A$

e 1st used by Euler

e is defined to be the sol'n of $\ln x = 1$



$\ln e = 1$

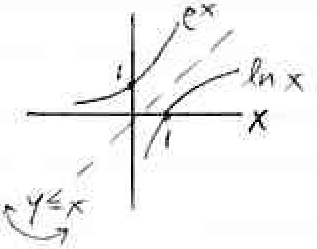
$$\boxed{\begin{array}{l} e \text{ is irrational} \\ e \approx 2.718 \end{array}}$$

Not as famous as big bro π. Both transcendental #s: not roots of any nice poly w/nice coeffs.
rational

Ⓑ e^x

Book does roundabout def'n of e^x.

$$\boxed{\left.\begin{array}{l} \text{If } f(x) = \ln x \quad (\leftarrow 1\text{-}1), \\ \text{then } f^{-1}(x) = e^x \end{array}\right\} \Rightarrow \ln x = \log_e x}$$ Why? →



ln, e as a base undo each other

Why? $\ln e^x = x \underbrace{\ln e}_{=1}$

(ln x, e^x are inverse funcs. $\Rightarrow \ln x = \log_e x$)

$= x$

$$\boxed{\begin{aligned} \ln e^x &= x \quad \text{(for all real } x) \\ e^{\ln x} &= x \quad \text{(for all } x > 0) \end{aligned}}$$

ln (only +)

Inverse props.

What's the 2nd deriv of e^x? 100th

Helps in Calc II.

(C) $D_x(e^x) = e^x$

Proof

$$\ln e^x = x$$

$$D_x(\ln e^x) = D_x(x)$$

$$\frac{1}{e^x} D_x(e^x) = 1$$

"Cool!" they thought so

Solve for

$$D_x(e^x) = e^x$$

Ex Find $D_x(3xe^x)$

$= 3 D_x(xe^x)$

Product Rule

$= 3[(1)(e^x) + (x)(e^x)]$

$= 3[e^x + xe^x]$

or

$3e^x[1+x]$

(D) Chain Rule

If $u = \underset{\text{diff'c}}{g}(x)$,

Deriv. of e^{blah} = $e^{\text{blah}} D(\text{blah})$

then $\boxed{D_x e^u = e^u \underbrace{D_x u}_{\text{tail}}}$ ← Think: $D(e^{A}) = e^{A} D(A)$

$\underline{\text{Ex}}$ $D_x(e^{7x}) = e^{7x} \cdot \underbrace{D_x(7x)}_{=7}$ In general, $\boxed{D_x(e^{\overset{\text{const.}\#}{k}x}) = ke^{kx}}$

Tail in front (butthead)

$= 7e^{7x}$

Quotient Rule? Recip!

$\underline{\text{Ex}}$ $D_x\left(\frac{1}{e^{4x}}\right) = D_x(e^{-4x})$

$= -4e^{-4x}$

We're differentiating a what, overall?

$\underline{\text{Ex}}$ $D_x(1+e^{2x^4})^7 = 7(1+e^{2x^4})^6 \underbrace{D_x(1+e^{2x^4})}_{\text{tail}}$ by Gen. Power Rule

$= e^{2x^4} \underbrace{D_x(2x^4)}_{\text{tail} = 8x^3}$

$= 8x^3 e^{2x^4}$

$= 56x^3 e^{2x^4}(1+e^{2x^4})^6$

What's the deriv of e^{blah}?

$\underline{\text{Ex}}$ $D_x[e^{\tan(3x)}] = e^{\tan(3x)} \cdot \underbrace{D_x[\tan(3x)]}$

$= \sec^2(3x) \cdot \underbrace{D_x(3x)}_{=3}$

$= 3\sec^2(3x)$

$\sec^2(3x) \cdot e^{\tan(3x)}$ helps?

$= 3e^{\tan(3x)} \sec^2(3x)$

Reverse these 2 funcs in the composition
$f(g(\underbrace{h(x)}_{3x}))$
e^x
$\tan x$

Overall, we want the deriv. of a what?

<u>Ex</u> $D_x[\tan(e^{3x})] = \sec^2(e^{3x}) \cdot \underbrace{D_x(e^{3x})}_{=3e^{3x}}$

$= 3e^{3x}\sec^2(e^{3x})$

<u>Ex</u> $D_x[\sin^5(e^{-2x})] = D_x[\sin(e^{-2x})]^5$

$= 5[\sin(e^{-2x})]^4 \cdot \underbrace{D_x[\sin(e^{-2x})]}$

$= [\cos(e^{-2x})][D_x(e^{-2x})]$

$= [\cos(e^{-2x})][-2e^{-2x}]$

$= -10e^{-2x}\cos(e^{-2x})\sin^4(e^{-2x})$

<u>Ex</u> $D_x[\ln(e^{4x})] \overset{\text{Think!}}{=} D_x[4x]$

$= 4$

Suggestion:

(Write D_x for tails, except: $D_x(e^{\underset{\text{const. real \#}}{k}x}) = ke^{kx}$)

7.4: (INTEGRATION)

(A) Exs. with ln

(on any interval excluding 0)

$$D_x(\ln|x|) = \frac{1}{x} \quad \text{(where } x \neq 0\text{)}$$
$$\int \frac{1}{x}\,dx \text{ or } \int \frac{dx}{x} = \ln|x| + C$$

SKIP

If $u = \underset{\text{diff'e}}{g}(x)$,

$$D_x(\ln|u|) = \frac{1}{u} D_x u \quad \text{(where } u \neq 0\text{)}$$
$$\int \frac{1}{u}\,du \text{ or } \int \frac{du}{u} = \ln|u| + C$$

maybe after u-sub

Ex (a) Evaluate $\int \frac{x^3}{x^4+1}\,dx$

Let $u = x^4 + 1$
$du = 4x^3\,dx$

$$= \frac{1}{4}\int \frac{4x^3}{u}\,\underbrace{dx}_{= du}$$
$$= \frac{1}{4}\int \frac{du}{u}$$
$$= \frac{1}{4}\ln|u| + C$$

(Technically, $\frac{1}{4}(\ln|u| + \hat{C})$)

Doesn't the $\frac{1}{4}$ hit C, also?

Go back to x

Do we really need "| |"?

$$= \frac{1}{4}\ln|\underbrace{x^4+1}_{>0 \text{ for all real } x}| + C$$

Go from $|\cdots| \Rightarrow (\cdots)$ when you can.

If you want, bring up $\frac{1}{4} \to \sqrt[4]{\ }$

$$= \frac{1}{4}\ln(x^4+1) + C$$

or $\ln\sqrt[4]{x^4+1} + C$

ⓑ Evaluate $\int_0^1 \frac{x^3}{x^4+1}\,dx$

Method 1 (Use ⓐ: Do indefinite $\int$ 1st.) Good if you like going back to x.

$$\int_0^1 \underbrace{\frac{x^3}{x^4+1}}_{\substack{f(x)\\ \text{cont. on}\\ [0,1]}} dx = \underbrace{\left[\tfrac{1}{4}\ln(x^4+1)\right]_0^1}_{F(x),\text{ an AD}}$$

$$= \left[\tfrac{1}{4}\ln((1)^4+1)\right] - \left[\tfrac{1}{4}\ln((0)^4+1)\right]$$

or Quotient Rule $\ln(\frac{2}{1}) = \ln 2$

$$= \tfrac{1}{4}\left[\ln(2) - \cancel{\ln(1)}^{\,0}\right]$$

$$= \tfrac{1}{4}\ln 2 \text{ or } \ln\sqrt[4]{2}$$

Method 2 ("from scratch") Good if you like doing a solution in "one pass" w/ little side work.

$$\int_0^1 \frac{x^3}{x^4+1}\,dx \quad \text{"KILL"}$$

Let $u = x^4+1$

$du = 4x^3\,dx$

Solve for KILL $\Rightarrow \frac{1}{4}du = x^3\,dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 2$

$= \frac{1}{4}\int_1^2 \frac{du}{u}$ can drop: $u>0$ on $[1,2]$

$= \frac{1}{4}\left[\ln|u|\right]_1^2$

$= \frac{1}{4}\left[\ln 2 - \cancel{\ln 1}^{\,0}\right]$

$= \frac{1}{4}\ln 2$

Ex $\int \frac{(\ln x)^3}{x}\,dx$

Let $u = \ln x$
$du = \frac{1}{x}\,dx$

$= \int u^3\,du$
$= \frac{u^4}{4} + C$
$= \frac{(\ln x)^4}{4} + C$ (Don't smack down 4.)

Ex $\int \ln x\,dx = x\ln x - x + C$ (Math 151)

Use $\int$ by Parts (9.1) - closest thing we have to a Product Rule for $\int$s.

B Exs. with e

$$\boxed{\begin{aligned} D_x(e^x) &= e^x \\ \int e^x\,dx &= e^x + C \end{aligned}}$$

SKIP

If $u = g(x)$, diff'e

then $D_x(e^u) = e^u \cdot D_x u$
$\int e^u\,du = e^u + C$
maybe after u-sub

Ex $\int e^{7x}\,dx = \frac{1}{7}\int 7e^{7x}\,dx$

Let $u = 7x$
$du = 7\,dx$

$= \frac{1}{7}\int e^u\,du$
$= \frac{1}{7}e^u + C$
$= \frac{1}{7}e^{7x} + C$

Notice: $D_x(e^{7x}) = 7e^{7x}$ — we mult. by 7

reverse: $\int e^{7x}\,dx = \frac{e^{7x}}{7} + C$ — we $\div$ by 7 — Don't forget!!

You can also Guess-and-✓ (use sparingly in Calc I).
Find D_x(Guess). If off by a sign or constant factor, adjust accordingly.

In general, $D_x(e^{kx}) = ke^{kx}$ (const. #)
$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

$\underline{Ex}$ $\int \frac{(e^{-x}+4)^6}{e^x}\,dx$

Let $u = e^{-x}+4$
$du = -e^{-x}\,dx$
$= -\frac{1}{e^x}\,dx$

$= -\int u^6\,du$
$= -\frac{u^7}{7} + C$
1-23 odd except 7, 9
$= -\frac{(e^{-x}+4)^7}{7} + C$

Ⓒ Exs. with Trig Funcs.

Guess-and-✓ good, especially to get sign right.

$D_x(\sin x) = \cos x$

$\int \sin x\,dx = -\cos x + C$
$\int \cos x\,dx = \sin x + C$ } from Basic Trig D_x Rules

$\underline{NOW:}$

SURPRISE: We'll be able to $\int$ all 6 elem. trig funcs. (Basically, $\int \frac{du}{u}$)

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$
$du = -\sin x\,dx$

$= -\int \frac{du}{u}$
$= -\ln|u| + C$
$= -\ln|\cos x| + C$
or $\ln(|\cos x|)^{-1} + C$
$\ln|\sec x| + C$

Know results and how we get them!

Even cleaner than $\int \tan x\,dx$ No sign problems.

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$

Let $u = \sin x$
$du = \cos x\,dx$

$$= \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sin x| + C$$

mult. by 1 creatively
$\sec x \cdot \sec x = \sec^2 x$
$\sec x \cdot \tan x$ D of somebody!

$$\int \sec x\,dx = \int \sec x \cdot \underbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}_{=1 \text{ (creatively)}}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

MAGIC!

Let $u = \sec x + \tan x$
$du = (\sec x \tan x + \sec^2 x)\,dx$

$$= \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \int \csc x \cdot \frac{\csc x - \cot x}{\csc x - \cot x}\,dx$$

$= u$
$du = (-\csc x \cot x + \csc^2 x)\,dx$

$\vdots$

$$= \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\csc x - \cot x| + C$$

Ex $\int e^x \tan(2e^x)\,dx$

Let $u = 2e^x$
$du = 2e^x\,dx$

$= \frac{1}{2}\int \underbrace{2e^x \tan(2e^x)\,dx}_{du}$

$= \frac{1}{2}\int \tan u\,du$
$= \frac{1}{2}\ln|\sec u| + C$
$= \frac{1}{2}\ln|\sec(2e^x)| + C$

Don't split. $\int$(product) $\neq$ product of $\int$s.

Ex (#36) $\int \csc x\,(1-\csc x)\,dx$
$= \int(\csc x \underbrace{- \csc^2 x}_{\text{whose deriv. is this?}})\,dx$ $\quad D_x(\cot x) = -\csc^2 x$

$= \ln|\csc x - \cot x| + \cot x + C$

7.5: BEYOND e

a^x, $\log_a x$

$1^x = 1$, not exp'l

Assume $a > 0$, $a \neq 1$ "nice base"
(a = real #)

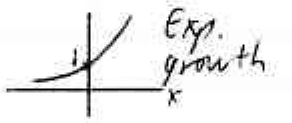
$\underline{(A)\ D_x(a^x) = a^x \ln a}$

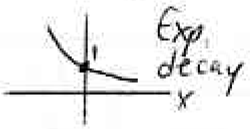
NOT x!!

Graph a^x

$a > 1$ | $0 < a < 1$

J vs. L





<u>Ex</u> $D_x(2^x) = 2^x \ln 2$ <u>NOT</u> $x2^{x-1}$

<u>Ex</u> $D_x(10^x) = 10^x \ln 10$

NOT xe^{x-1} <u>Ex</u> $D_x(e^x) = e^x \cancel{\ln e}^{\,1}$ (Good ✓)

SKIP

Huh? Nicer proof later - log diff

$$\underline{\text{Proof}}\ D_x(a^x) = D_x\left[\left(\overbrace{e^{\ln a}}^{=a}\right)^x\right]$$

$$= D_x\left(e^{x \ln a}\right)$$

$$= e^{x \ln a}(\ln a)$$

$$= a^x \ln a$$

copy, ln a, tail

$$\boxed{\underline{\text{Chain Rule}}\quad D_x(a^u) = (a^u \ln a) \cdot \underbrace{D_x u}_{\text{tail}}}$$

<u>Ex</u> $D_x(5^x) = 5^x \ln 5$

<u>Ex</u> $D_x(5^{x^3}) = (5^{x^3} \ln 5)\underbrace{D_x(x^3)}_{=3x^2}$

Con do 1, 3

$$= 3x^2 5^{x^3} \ln 5$$

$D_x(a^x) = a^x \ln a$ times

(B) $\int a^x\,dx = \frac{a^x}{\ln a} + C$

(Instead of mult. by $\ln 2$, we $\div$ by $\ln 2$)

<u>Ex</u> $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$

<u>Ex</u> $\int e^x\,dx = \frac{e^x}{\cancel{\ln e}_{1}} + C$

<u>Ex</u> (u-sub)

$\int 3^{2x+1}\,dx$

Let $u = 2x+1$

$du = 2dx$

$= \frac{1}{2}\int 3^u(2dx)$

$= \frac{1}{2}\int 3^u du$

technically, $\frac{1}{2}C$ $\quad = \frac{1}{2}\left[\frac{3^u}{\ln 3}\right] + C$

$= \frac{1}{2}\left[\frac{3^{2x+1}}{\ln 3}\right] + C$

or $\ln 9 \rightarrow$ $\quad = \frac{3^{2x+1}}{2\ln 3} + C$

<u>Ex</u> $\int 7^x \sin(7^x)\,dx$

Why? $\quad$ Let $u = 7^x$

$du = 7^x \ln 7\,dx$

$= \frac{1}{\ln 7}\int \underbrace{(7^x \ln 7) \sin(7^x)\,dx}_{du}$

$= \frac{1}{\ln 7}\int \sin u\,du$ $\quad$ Guess-and-✓ Trig

$= \frac{1}{\ln 7}(-\cos u) + C$

$\left(= -\frac{1}{\ln 7}\cos u + C\right)$

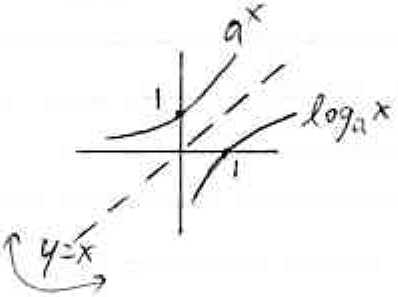
$= -\frac{1}{\ln 7}\cos(7^x) + C$

© $\log_a x$

If $f(x) = a^x$,
then $f^{-1}(x) = \log_a x$

If $a > 1$

What letter does it look like? Exp. growth



Ex $\log_2 8 = 3$ because $2^3 = 8$

Ex $\log_2 9 = ?$

$$\log_2 9 = \frac{\ln 9}{\ln 2}$$

$$\approx 3.17$$

can use logs of any base, as long as you're consistent. We like ln b/c $\frac{d}{dx}(\ln x) = \frac{1}{x}$ makes sense

$$\boxed{\begin{array}{c}\underline{\text{Change-of-Base Property}} \\ \log_a x = \dfrac{\ln x}{\ln a}\end{array}}$$

$y = \log_a x \leftrightarrow x = a^y$, $\ln x = \ln a^y$

Proof p. 411

Ex $\log_2 x = \frac{\ln x}{\ln 2}$ or $\overbrace{\left(\frac{1}{\ln 2}\right)}^{\#} \ln x$

$\log_a x$ and $\log_b x$ differ by a constant factor.
Algorithm analysis: "a log is a log"

Ex Running time as a function of size of data set. $\log_2 x$ treated the same as $\log_{10} x$ even though the constant factor may be significant

① $\underline{D_x(\log_a x)}$

$\underline{Ex}$ $D_x(\log_2 x) = D_x\left(\frac{\ln x}{\ln 2}\right)$ by Change-of-Base

$= \frac{1}{\ln 2} D_x(\ln x)$

$= \underbrace{\frac{1}{\ln 2}}_{\text{"fudge factor"}} \cdot \frac{1}{x}$ or $\frac{1}{x \ln 2}$

SKIP (Book's approach)

SKIP

In general...

① $\underline{D_x(\log_a x) = \overbrace{\frac{1}{\ln a}}^{\text{"fudge factor" } =1 \text{ if } a=e} \cdot \frac{1}{x} \text{ or } \frac{1}{x \ln a}}$

$D_x(\log_a |x|) = (\text{same})$

$\underline{\text{Chain Rule}}$

$D_x(\log_a u) = \overbrace{\frac{1}{\ln a}}^{=1 \text{ if } a=e} \cdot \overbrace{\frac{1}{u} \underbrace{D_x u}_{\text{tail}}}^{\text{Think "ln" now}}$

$D_x(\log_a |u|) = (\text{same})$

or $D_x(\underbrace{\log_3 4}_{c} + \log_3 x)$

$D(\ln(\text{blah})) = \frac{1}{\text{blah}} D(\text{blah})$

Show that: $\underbrace{\frac{1}{\ln 3}}_{\text{fudge}} \cdot \frac{1}{4x} \cdot \underbrace{4}_{\text{tail}}$

formulas: easy to forget/mess up, just remember Change-of-base.

If you forget the fudge factor

$\underline{Ex}$ $D_x[\log_3(4x)] = D_x\left[\frac{\ln(4x)}{\ln 3}\right]$

$= \frac{1}{\ln 3} D_x[\ln(4x)]$

$= \frac{1}{\ln 3} \cdot \frac{1}{\cancel{4}x} \cdot \underbrace{\cancel{4}}_{\text{tail}}$

$= \frac{1}{x \ln 3}$

$\underline{Ex}$ Find $D_x[\log(x^2+3x)]$

Implied base = 10
"common log"
Richter, pH

$$D_x[\log_{10}(x^2+3x)] = D_x\left[\frac{\ln(x^2+3x)}{\ln 10}\right]$$

(by Change-of-Base)

$$= \frac{1}{\ln 10} D_x[\ln(x^2+3x)]$$

$$= \frac{1}{\ln 10} \cdot \frac{1}{x^2+3x} \cdot \underbrace{D_x(x^2+3x)}_{=2x+3}$$

Need
$(\ln 10)(x^2+3x)$
or put on right
trig, too

$$= \frac{2x+3}{(x^2+3x)\ln 10}$$

use (),
← order factors so there is no confusion. We tend to put ln, trig at right end.

SKIP

<u>Short Way</u> (formula)

once you take care of the fudge factor, you can pretend we have ln.

$$D_x(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} D_x u$$

$$D_x[\log_{10}(x^2+3x)] = \frac{1}{\ln 10} \cdot \frac{1}{x^2+3x} \cdot \underbrace{D_x(x^2+3x)}_{=2x+3}$$

$$= \frac{2x+3}{(\ln 10)(x^2+3x)}$$

<u>Ex</u> $D_x(\log_4 x^5) = D_x(5 \log_4 x)$

Log laws work!

$$= 5 \cdot D_x(\log_4 x)$$

$$= 5 \cdot D_x\left[\frac{\ln x}{\ln 4}\right]$$

$$= \frac{5}{\ln 4} \cdot \frac{1}{x}$$

$$= \frac{5}{x \ln 4}$$

Could 5

(or $D_x(\log_4 x^5) = D_x\left(\frac{\ln x^5}{\ln 4}\right) = \frac{1}{\ln 4} \cdot D_x(5 \ln x) = \frac{5}{\ln 4} \cdot \frac{1}{x} = \frac{5}{x \ln 4}$)

Up to 17

Ⓔ Log Diff

const^x

Ex $D_x\,[(\sqrt{2})^{x}] = (\sqrt{2})^x \ln(\sqrt{2})$ exponent 7.5

x^{const}

Ex $D_x\,(x^{\sqrt{2}}) = (\sqrt{2})x^{\sqrt{2}-1}$ (by Power Rule) base

What if... $D_x\,[f(x)]^{g(x)}$?

Ex Find $D_x[(\sin x)^x]$

Don't be all proud of yourself

NOT $x(\sin x)^{x-1}\cdot$ tail

Use Log Diff !!

① Let $y = (\sin x)^x$

② Take ln of both sides

$$\ln y = \underbrace{\ln(\sin x)^x}$$

③ Use log laws to expand

$$\ln y = x\ln(\sin x)$$

④ Use Implicit Diff

$$\frac{y'}{y} = (1)[\ln(\sin x)] + (x)D_x[\ln(\sin x)]$$

by Product Rule

$$\frac{y'}{y} = \ln(\sin x) + (x)\left(\frac{1}{\sin x}\right)\underbrace{(\cos x)}_{\text{tail}}$$

$$\frac{y'}{y} = \ln(\sin x) + x\cot x$$

(5) Mult. both sides by y

Unlike w/Imp Diff we know what y is.

$$y' = [\quad\quad\quad]y$$

(6) Write y in terms of x

$$y' = [\ln(\sin x) + x\cot x][(\sin x)^x]$$

What if you forget $D_x(2^x)$?

New pf? More elegant...

Takes a min...

Ex
$$y = 2^x$$
$$\ln y = \ln 2^x$$
$$D_x(\ln y = x\ln 2)D_x$$
$$\frac{y'}{y} = \ln 2$$
$$y' = y\ln 2$$
$$y' = 2^x \ln 2$$ (In case you forget...)

Use in Calc II
Used in Cont. Comp. Interest
How is e relevant to bank acct?

(F) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (Used in Calc II, Cont. Compound Interest)

(pp. 413-4)

CH.7 REVIEW

Review Chs. 3, 5 — ① Ex u-subs

D_x Ex Product Rule e.g.

7.2 ln

$$D_x(\ln x) = \frac{1}{x}$$

$$D_x(\ln|3x+1|) = \frac{1}{3x+1}\underbrace{D_x(3x+1)}_{=3}$$

can ignore

$$D[\ln(\text{blah})] = \frac{1}{\text{blah}} D(\text{blah})$$

$$D_x\left(\ln\left[\frac{x(x+1)^3}{\sqrt{x}}\right]\right)$$

Use log laws

Log Diff

Find $D_x\left[\frac{x(x+1)^3}{\sqrt{x}}\right]$

Should use ln – otherwise, if you use log, you get all those fudge factors when you D_x

Introduce ln

$$y = [\quad]$$

$$\ln y = \underbrace{\ln[\quad]}_{\text{Use log laws}}$$

D_x $\qquad$ D_x

$$\frac{y'}{y} = (\quad)$$

$$y' = (\quad)y$$

[plug in]

7.3 e

$$D_x(e^x) = e^x$$

Butthead

$$D_x(e^{2x}) = 2e^{2x}$$

$$D(e^{\text{blah}}) = e^{\text{blah}} D(\text{blah})$$

7.4 $\int$ (D_x to ✓)

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{(\ln x)^3}{x}\,dx$$

$$u = \ln x$$
$$du = \frac{1}{x}\,dx$$

Guess-and-✓

$$\int e^x\,dx = e^x + C$$
$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$$

$\int \tan x\,dx$, etc. (Know results and "proofs.")

7.5 Beyond e

$$D_x(2^x) = 2^x \ln 2$$
$$D_x(2^{3x}) = \underbrace{(2^{3x}}_{\text{copy}}\ \underbrace{\ln 2)}_{\ln a}\ \underbrace{\overbrace{D_x(3x)}^{=3}}_{\text{tail}} \quad \leftarrow 3 \text{ pieces}$$

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

$D_x(\log_a A) = D_x\left(\frac{\ln A}{\ln a}\right)$

Exs

$$D_x(\log_3 x) = D_x\left(\frac{\ln x}{\ln 3}\right) = \frac{1}{\ln 3}\cdot\frac{1}{x} = \frac{1}{x\ln 3}$$

$$D_x[\log_3(5x+1)] = D_x\left[\frac{\ln(5x+1)}{\ln 3}\right] = \frac{1}{\ln 3}\cdot\frac{1}{5x+1}\cdot 5 = \frac{5}{(5x+1)\ln 3}$$

$$D_x\,[f(x)]^{g(x)}$$

Use Log Diff. (Log Laws)

(Logs help us bring down exponents.)

Limit def'n of e

$$\left(1+\frac{1}{n}\right)^n \xrightarrow[\text{as } n\to\infty]{} e$$